The Educational Potential of Friendly Graphing Software

The case of GRAPES

Abraham Arcavi

INTRODUCTION

The availability of easy-to-use graphing software highlights the role of the graphical representations of functions and relations. There may be intrinsic educational value in the traditional activities of substituting and calculating values in symbolic expressions in order to plot individual points in a Cartesian plane in order to obtain a graph. However, for many expressions, these “hand-made” procedures can be very time consuming and cumbersome, thus rendering them unusable for most educational purposes. In contrast, the immediacy of obtaining a graph for a certain function or relation opens up new opportunities for many activities and issues to be learned. For example:

- Students may be able to model and study graphical representations of problems involving complicated algebraic expressions, impossible to obtain otherwise.

- Traditionally, the graph was an end point of many mathematical problems: on the basis of a given symbolic expression and by means of certain analytic tools (e.g. derivatives of a function) students are required to deduce the main characteristics of a graph, and proceeded to sketch it. These sketches can be easily checked with a graphing software, but, the possibility to instantaneously obtain a graph may put into question such traditional problems altogether. Instead, graphing software may be used for reversing these problems, namely, given a graph can we find the symbolic expression which generated it? Once we conjecture a certain expression for a given graph, the graphing software serves as a means for checking our conjectured expression. In the case that our conjecture is wrong or approximate, we can use the graphing software to revise, adjust and refine our proposals until we succeed. This process directs attention to the role of the coefficients in a symbolic expression, and provides a sense of how they influence the shape of a graph. Such a sense may be only phenomenological at first, but it may be further studied analytically.

- Graphing software may help visualizing families of functions and relations, making more transparent the roles of parameters.

- Working with graphs bring to the fore the issue of scaling, for example, drawing attention to the fact that the parts of a graph observable in the display of the graphical window depends on the axes range, which one is free to stipulate. Sometimes one may think of a graph as linear only because its scale produces such illusion. Sometimes one may be surprised that the software is not producing any graph at all, and then realizing that it may be outside the range selected for the axes. And so on.

- Graphing software may produce unexpected results, which forces one to engage on interpretation, using all the knowledge at one's disposal. Sometimes, surprising results are due to mathematical phenomena of which we may not be aware at first sight, in other occasions, they are the result of our wrong input, and yet other times they may be due to the limitations of the technology. In either case, unpacking the reasons for a surprise has learning potential, as it may require explanations based on checking and coordinating different representation, and making connections between different kinds of knowledge.

- Graphs can serve as the basis for the solution to problems, traditionally solved by other means. Graphs can be operated upon (e.g. added, subtracted), can be translated and rotated. Graphs may be the source for symbolic insights. In sum, graphing software may support a way of mathematical reasoning not developed before, and which may help different students to learn in different ways.

The brief descriptions above include only some of the educational possibilities afforded by a graphing software. In the following, we bring examples of problems, with accompanying brief commentaries, to illustrate the points above. Three of the problems were discussed in interviews with two Japanese teachers, one of them an experienced user of computerized environments in education (who made written comments to most of the problems after the interview), and the other, an experienced teacher who had not used technology before. The reports from their experiences and opinions during the interviews is attached.

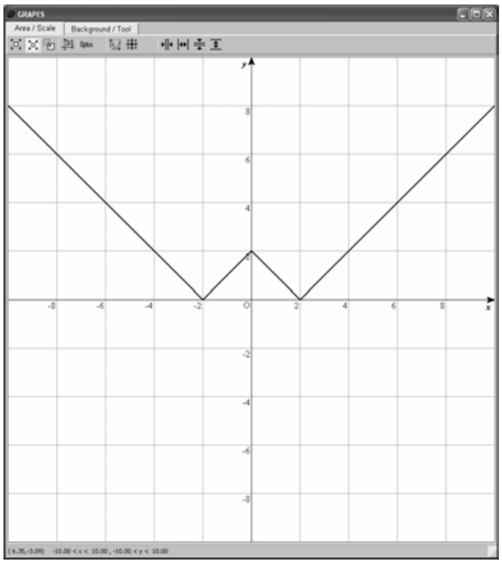
Note that each of the problems below are presented are out of the context of a curriculum sequence. The way of approaching these problems may be highly contingent on their location within an instructional sequence and on the previous knowledge, experiences and emphasis of what the solvers have learned. Nevertheless, as the problems illustrate possible ways for the use of technology, they can be presented either as is (within a context or outside it) or adapted to different classroom or curriculum materials. The problems may also serve as examples to inspire and trigger the design of tasks and inquiry projects by teachers and curriculum developers.

The following problems were designed with and for Grapes, a graphing software developed at the Center for Research on International Cooperation in Educational Development (CRICED), University of Tsukuba, Japan.. The problems make use of only a few of the many utilities of the software, and they can be used and implemented with other graphing software.

At the end of this report, we append a brief annotated bibliography which brings more detailed analyses, discussions and examples of the potential of graphing software, including some research results about their implementation with students.

SAMPLE PROBLEMS

1- Reproduce in your screen the following graph:



Commentary

In order to reproduce the graph, one needs to engineer a symbolic expression, to input it and to check for the desired result.

A possible approach would be to split the domain of the graph into four sub-domains and establish a symbolic expression for each, in this case, four linear functions. Thus the following four inputs will produce the graph:

$y = x-2$ ($x>2$); $y = -x+2$ ($0<x<2$); $y = x+2$ ($-2<x<0$); $y = -x-2$ ($x<-2$)

Another possible approach is to subdivide the domain in only two sub-domains and make use of the absolute value function:

$y = |x-2|$ ($x>0$); $y = |x+2|$ ($x<0$)

Yet another approach is to use produce only one expression for the whole domain:

$y = ||x|-2|$

When using absolute value, working with such problems may draw attention to relationships between the application of the absolute value to any symbolic expression and its effect on the graph.

Also, problems of this type may promote independent investigations by the students, who by experimenting with the absolute value may come up with many interesting graphs of their own, and challenge each other to reproduce them by finding their corresponding symbolic expressions.

2- Create different parallelograms using equations of intersecting straight lines, such that the vertices are at (0,0), (2,2) and (4,0). How many can parallelograms you find?

Commentary

This task requires to envision the fourth vertex of a parallelogram when the three others are given, to produce the linear equations for its sides (given two vertices) and to check that the graphical result obtained is the parallelogram required. Note that there are three such parallelograms (with the fourth vertex at (6,2), (2,-2) or (-2,2)), and one can discuss the fact that there are no more.

3- What do the lines of the family y = ax + a have in common? Explain your finding symbolically and graphically.

<u>*Commentary*</u>

There are several ways one may approach this problem. One possibility is to use the graphing software to draw many graphs (for different values of *a*) and to observe the common property that all of them intersect at (-1,0).

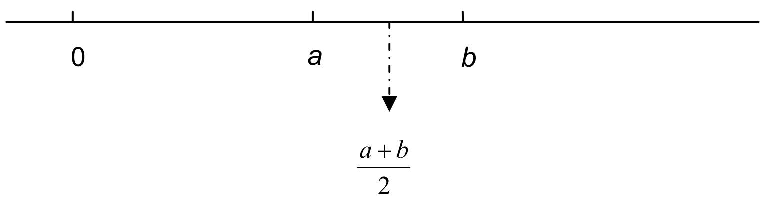
This can be done by inputting the general equation *y = ax + a* and using the software option to generate the corresponding graphs for the different values of the parameter *a*. Another possibility is to begin analytically, by transforming $y = ax + a \Rightarrow y = a(x+1)$, and observing that regardless of the value of *a*, when *x = -1, y = 0*. The graphing software in this case, can serve as the visualization of the analytic finding. Yet another possibility is to reason from the graphical meaning of the concepts of slope and y-intercept (in this case *a* is both). Assuming that *a>0*, since it is the y-intercept of the all the lines, we may generically mark the segment from (0,0) to (0,a) on the upper part of the *y*-axis. Since the slope is also *a*, and slope is “rise over run”, the run in this case should be 1. Drawing the generic example yields that all lines of this family with *a>0* must go through (-1,0). The same argument can be made for *a<0* (it obviously holds for *a=0*).

4- The average of two numbers *a* and *b* is calculated by the formula $\frac{a+b}{2}$. Consider *a>0* and *b>0*, and note that sometimes their average is just the same as their (positive) difference (for example, the average between 10 and 30 is 20, and 30-20=10, but the average between 10 and 16 is 13, but 16-13≠3). Explore with Grapes (or other means), for which pairs of numbers their average is the same as their (positive) difference.

<u>*Commentary*</u>

Traditionally, this problem can be approached analytically by stating the equation $\frac{a+b}{2} = a - b$, (assuming *a>b*) and solving to obtain the simple relationship $a = 3b$. Namely, whenever the larger number is three times the smaller, their average and their difference are the same. This problem can solved by requesting the software to plot the relationship $\frac{a+b}{2} = a - b$. The graph is a line, from which one has to read the relationship between the two numbers. Combining these two approaches may be a good way to compare and contrast representations.

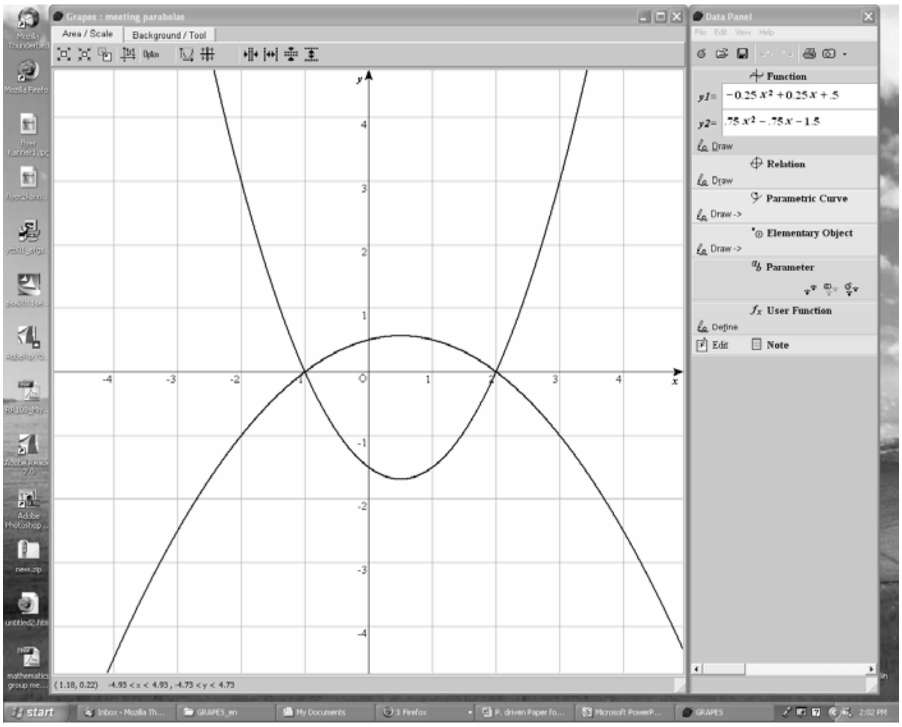
One may claim that both solution approaches above produce the answer, but they may fail to convey an intuitive explanation for why this could be the case. Maybe, a third solution, which makes use of the number line can speak to our intuition. Consider the following number-line diagram:



The solution to the problem requires that the length of the segment $\overline{ab}$ be equal to the length of the segment whose extremes are 0 and $\frac{a+b}{2}$. These two segments already have a part in common (the segment from *a* to $\frac{a+b}{2}$), therefore the other two remaining parts of each have to be equal, namely, the segment from 0 to *a* should be of the same length as the segment from $\frac{a+b}{2}$ to *b*. If this is so, the segment from 0 to *b* has been subdivided in three equal parts, implying that *b* is three times *a*.

Further problems can be propsoed and investigated, for example, find when the average of two numbers is equal to their sum, their product or their quotient. The two latter cases produce interesting graphs which may result in fruitful classroom discussions around their interpretation (e.g. the meaning of the asymptotes, etc).

5- Reproduce with Grapes the following screen



Commentary

This problem requires the finding of the parameters of the two quadratic functions in order to obtain their graphs. It indirectly draws attention to the fact that two points do not fix a single parabola. For a discussion of possible ways of solution, see interview report below.

6- Use Grapes to graph $x^2y + y^3 - x^2 - y^2 = 0$. Explain your findings.

<u>Commentary</u>

This problem was deliberately designed in order to produce a surprise. The visible graph obtained is the line corresponding to *y=1*, contrary to the expectations, and in sharp contrast with *y=ax+b*, the prototypical symbolic expression for a linear function. Any surprising or unexpected result can be a good source for investigation and learning. In this case, one may, for example, resort to algebraic technique to factor the expression $x^2y+y^3-x^2-y^2=0$, to obtain $(x^2+y^2)(y-1)=0$, from which one deduces that the graphs has two parts, one of which corresponds to the line seen on screen, and the other corresponds to the point (0,0). Thus, the graph for the expression includes a point and a line. What would be the educational potential of such a problem? Firstly, students may realize (if they did not know it already) that it is possible to construct a single symbolic expression for two “separate” graphical entities, a point and a line. Secondly, the algebraic knowledge for transforming expressions served here as the basis for finding an explanation to a surprising graphical result. Thirdly, one realizes that the software does not always display all the graphical information, as the point (0,0) will not show on screen (In Grapes, it is possible to make appropriate scale changes to visualize the point.) The solution to this problem may have an implicit “warning lesson”: always look critically at the result of a graph produced by a graphing software.

See the report on the interviews with the teachers for more comments on this task.

7- Use Grapes to graph $x^2y+y^3-6x^2-6y^2=0$. Explain your findings.

8- Use Grapes to graph $y=x-9.95$. Describe your findings.

9- Use Grapes to graph $x^2+y^2=48.5$. Describe your findings.

<u>Commentary</u>

These problems are a follow up to the previous, in which the result of the graphing are surprising. If one keeps the default scales (-5 to 5 in both axes), the graph will not show at all (question 7), or only tiny bits of it will appear in the corners of the screen. These problems are aimed to raise the awareness to the issue of scaling, and to the fact that it is up to the user to determine the portions of the graph to be seen. Students can also be requested to solve an “inverse” problem, that is to select, in several differnet ways, the scales for a certain graph of a function, such that the graph <u>does not</u> show on the screen.

10- Produce different quadratic functions (of the form $y=ax^2+bx+c$) such that their graphs go through (0,0). (Check it with Grapes).

<u>Commentary</u>

The aim of this problem is to explore the different possibilities for the parameters that would produce graphs that go through (0,0), and to summarize the conclusions. For $a \neq 0$, the cases to consider, to describe (graphically) and to propose conclusions are

$b=0$ and $c=0$, $b \neq 0$ and $c=0$, $b=0$ and $c \neq 0$, and $b \neq 0$ and $c \neq 0$.

This problem also provides an opportunity to coordinate between representations, for example, by considering the roots of a quadratic equation both symbolically and graphically. This problem ac be solved also using another general expression for a quadratic function.

11- Explore with Grapes (in two different ways) the following problem: Of all rectangles of perimeter 4, which one has the largest area?

Commentary

One way to solve the problem is to write a symbolic expression for the area function, namely $y = x(2 - x)$, to draw its graph and to find and interpret (geometrically) its maximum value. Another possible way, is to graph the relation $xy = 1$, in which each point on its graph (when *x*>0) can represent a rectangle of area 1. The graph of the relationship $x + y = 2$, in the open interval (0,2) represents all rectangles of perimeter 4. The relative position of the two graphs can be read as follows: All the rectangles of perimeter 4 , except for the one for which *x* = *y* = 1 have area less than 1.

One can produce also a geometrical proof, which does not make use of graphing software of this kind.

12- Use Grapes to find the set of values which solve $|x - 2| > |2x - 1|$.

- Find other ways to solve it, and discuss the advantages and disadvantages of them.
- Find the value of *x* for which the difference is largest. Explain the ways you used.

Commentary

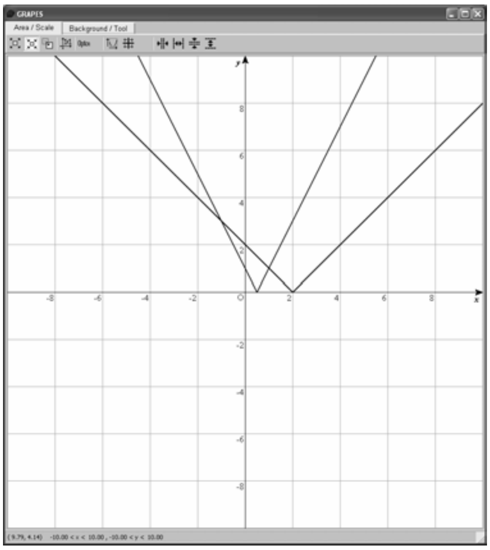
In most of the traditional algebra texts in which this type of problems was proposed in the past, the solutions process consisted of considering all possible cases, namely,

- $x - 2 > 0$ and $2x - 1 > 0$
- $x - 2 < 0$ and $2x - 1 > 0$
- $x - 2 > 0$ and $2x - 1 < 0$
- $x - 2 < 0$ and $2x - 1 < 0$

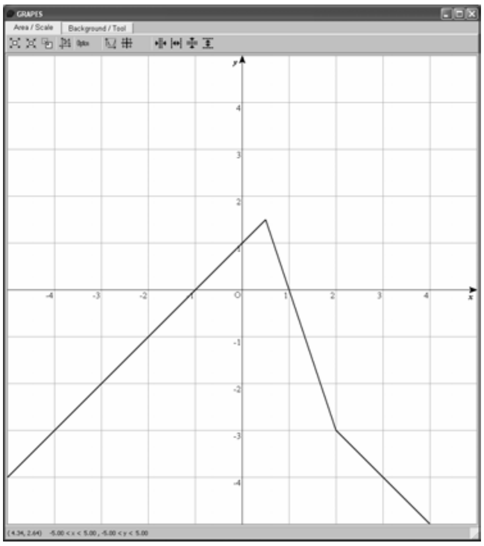
For each of these cases, the equation should be solved by removing the absolute value, taking caring of reversing the positive and negative signs when needed.

The process involves the use of many logical connectives, is long and error prone.

One way to use a graphing software is to consider each side of this inequation as symbolic expressions of functions, request their graphs and compare the domains for which the values of the first function are grater than the values of the second. These can be read from the graphs by a quick look at them.



The graphs can also provide an answer to the second question, by looking at the largest distance between them. However, there is a second way to approach the question. One may request the graph of the difference between the two functions, $y = |x-2| - |2x-1|$, and look in it for the maximum value, as shown in the following graph.



Beyond the location of its maximum value, it may be interesting to discuss the rest of the features of such resulting graph.

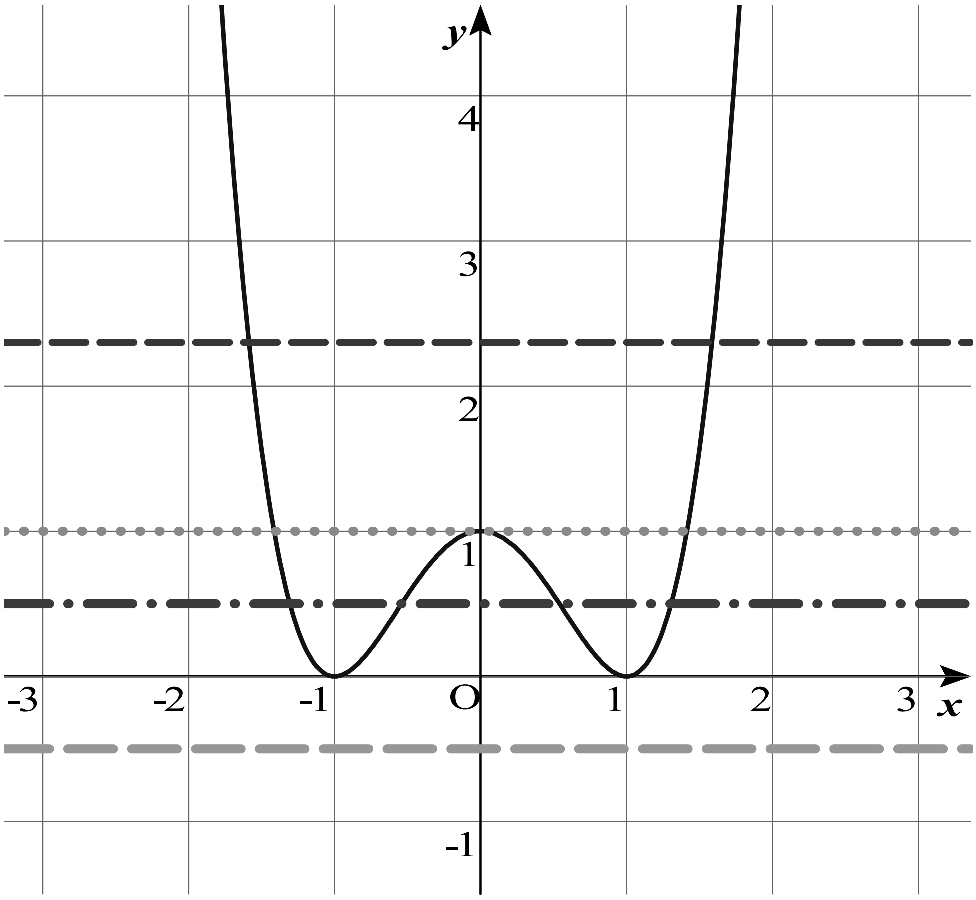
13- Draw the graph of the function $y = x^4 - 2x^2 + 1$. Draw a straight line, such that:

a) it does not intersect with the graph of *y*;
b) has exactly two points of intersection with the graph of *y*;
c) has exactly three points of intersection with the graph of *y*;
d) has exactly four points of intersection with the graph of *y*.
e) has only one point of intersection with the graph of *y*;

How would you propose to check your answers.

Commentary

A straightforward answer to the first four items, a)-d), is to draw graphs of linear functions of the form $y = a$ (namely, horizontal lines) at appropriate locations, for example, $y = -0.5$, $y = 2.3$, $y = 1$ and $y = 0.5$ respectively, as shown in the figure below.



In fact, many students interpret “straight” lines as line in horizontal or vertical positions only. Item e) has no solution with an horizontal line (this can be visualized in the graph, and proven analytically).

In order to find a linear function with only one point of intersection with the given quartic function, one possibility would be to proceed analytically to find the equation of a tangent to the graph in a pre-selected point, such that that tangent line is not intersecting the graph elsewhere. For example, if we choose the point (2,9), the equation of its tangent is $y = 24x - 39$.

One may attempt to find linear functions with non-horizontal graphs for all the other items, and the graphing software supports an immediate visual feedback for what we may find. Many times, such a feedback should be carefully checked.

Since the expression of the given function is a quartic, analytic checking would involve solving very complicated equations (even in this case in which the symbolic expression can be used in its simpler form $y = (x^2 - 1)^2$). Graphical checking may provide useful (even if not always accurate) feedback.

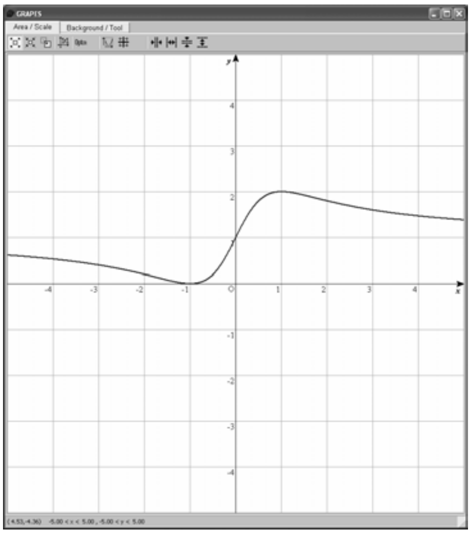
Consider, for example, checking how many points of intersection does this function have with the line $y = x - 1.05$. Sometimes using the zoom may be helpful. These situations can be used in order to discuss the rigor (or lack of it) of the graphical representation, yet appreciating its intuitive and heuristic value for global visualization and sense-making.

14- Draw the graph of the function $y = \dfrac{x^2 + 2x + 1}{x^2 + 1}$. Use your knowledge of algebra to explain as many characteristics of the graph as you can.

Commentary

Questions like this may support the verbalization the many connections between the symbolic and graphical representations of functions. Such activities may support the development of the habit of explaining a-posteriori graphical features using arguments anchored in the symbolic expression and vice versa.

The graph of this function is shown in the figure below.



The following are some of the connections one may help students to notice:

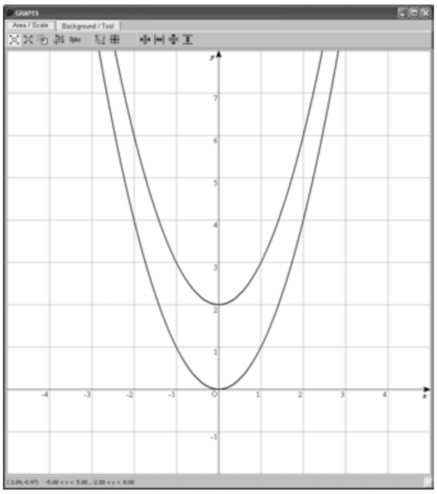
- The graph seems to be entirely above the *x*-axis. It must be so, because except for *x* = -1, the expression is always positive, since both the numerator (which is in fact $y = (x+1)^2$) and the denominator are positive.
- The intersections with the axes can be easily calculated and it can be shown that there are no others than those shown on screen.
- Would *y* = 1 be an asymptote, as the graph seems to suggest? A quick qualitative analysis of the expression confirms that for large absolute values of *x*, the values of $(x+1)^2$ and $x^2 + 1$ become very close, thus their quotient tends to 1.
- In the graph, a segment of it looks like a straight line. When inspecting the expression, for small values of *x*, x^2 becomes negligible, and thus the graph of the function almost coincides with the graph of *y* = 2*x* +1.

15- Given $y_1 = x^2$ and $y_2 = x^2 + 2$, predict the shapes of the functions $y_2 - y_1$ and y_2 / y_1.

Check your prediction using GRAPES and explain your findings.

Commentary

From the algebraic expression, it is clear that difference $y_2 - y_1$ is constant. What may be of educational importance in this case is that simple operations between functions may have results which are obvious analytically and yet insightful graphically. When drawing the graphs of y_2 and y_1, an optical illusion may lead us to believe that their vertical distance (that is their difference) diminishes. See figure below.



In this example, the importance of the connections between representations is precisely to dispel misleading graphical illusions.

The quotient between the functions and the analysis of its graph leads to a discussion similar to that of problem 14.

An interview with two mathematics teachers working through tasks with GRAPES. Summary Report

An hour and a half interview session was conducted by Takeshi Miyakawa, Masami Isoda and Abraham Arcavi with two experienced mathematics teachers (both Master Degree students in Mathematics Education at Tsukuba University) on May 15, 2005 (8:30-10:00a.m.). The object of the interview was to gather first impressions about:

a) the ways the teachers make use of Grapes to solve ad-hoc designed problems,
b) their opinions about the mathematical/didactical potential of the use of Grapes.

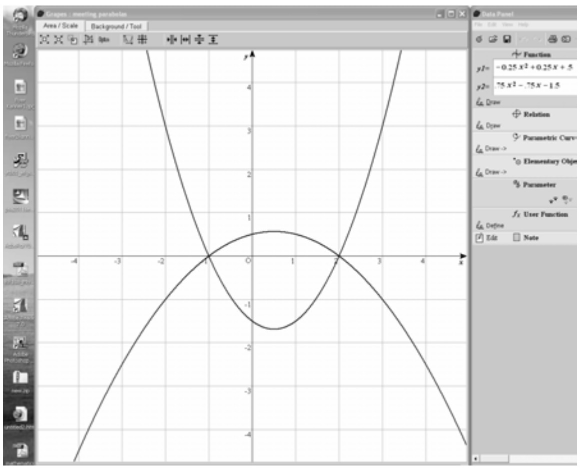
The interviews did not follow a pre-prepared script, it rather consisted of a free conversation which took place after the teachers solved the problem, and around the two issues above.

One of the teachers is very knowledgeable of Grapes and other tools as well (Teacher A) whereas the other (Teacher B) does not have experience with any technological tool for mathematics teaching. At the beginning of the session, Teacher B was briefly introduced to the main two features of Grapes (inputting the symbolic expression for a function or a relation and requesting their graphs), and he received minimal technical assistance along the way as needed. The two teachers worked simultaneously, but separately, on the following three problems (in this order). The conversations took place after they finished each problem, and were specifically related to them. At the end , more general issues regarding the potential of using technology (such as this) in classrooms were discussed.

a) Create a parallelogram using equations of intersecting straight lines, such that the vertices are at (0,0), (2,2) and (4,0).
b) Create another one.

Use Grapes to graph $x^2y + y^3 - x^2 - y^2 = 0$*. Explain your findings.*

Reproduce with Grapes the following screen



Main observations

Role of previous experience with the software (and with technology in general)
As expected, the solutions and the solution processes of the two teachers were highly dependent on their previous experience with technology in general, and specifically with Grapes. The following are some of the ways in which it is possible to attribute the previous experiences (or lack of them) to differences in the teachers actions.

- Teacher A approached the problems mostly using Grapes only, he made almost no use of paper and pencil. Teacher B made ample use of paper and pencil, sometimes before using Grapes, sometimes in parallel to it, and sometimes after using Grapes.

- In the first problem, Teacher A was very concerned with drawing the parallelogram (with fourth vertex at (6,2)), and his emphasis was in establishing in Grapes the domain for the linear function, in order not to have the graphs of the whole lines, but just the segments that determine the parallelogram he envisioned. We hypothesize that his knowledge of the software capabilities led him to do so. Teacher B instead started with drawing on paper and pencil the given points, and to consider all the possibilities for the location of the fourth point, before approaching Grapes to input the lines for all the possibilities. He was not concerned with the domain, and when all the graphs were drawn, he was satisfied when the three parallelograms appeared embedded in the figures with full linear graphs.

- Regarding the first problem, Teacher A compared this software to geometrical constructions software such as Cabri. He said that if this task was to be given with Cabri, one could have completed it without applying any knowledge of graphs and algebra, just by making use of the "macro" for the construction of parallel lines to given lines through requested points. The same could have been done with paper and pencil. Therefore, he sees value in doing this task with Grapes, because it forces one to make use of one's knowledge of functions (and possibly analytic geometry), since there is no other way to draw a picture on the screen.

- In the third problem, Teacher A envisioned the objective of the task as locating and drawing two members of a certain family of parabolas of the form $y = a(x-2)(x+1)$ and used the software capability of dynamically changing the parameter *a* in order to find the two graphs sought. Teacher B wrote three symbolic forms for the parabola: $y = ax^2 + bx + c$, $y = a(x-p)^2 + n$, and $y = a(x-x_1)(x+x_2)$. And decided that the third was the most appropriate to use in order to solve this problem. Whereas he used the same formula as Teacher A, he did it in a different way, he looked at it as an equation in *a* to be satisfied by (3,3), a point he observed as belonging to the graph of the parabola facing upwards (namely, 3=a(3-2)(3+1)) and (3,-1) as a point on the other.

- Teacher A, apparently due to his awareness of the software features, was very concerned with the use of scale and scaling, and was aware different scales influence what we see or do not seen the screen. Thus when he was surprised by the graph obtained in the second problem, his first reaction was to change scales in order to see whether the graph was linear everywhere. Teacher B, reacted differently to the surprise. He turned to paper and pencil to work the algebraic form and to look in it for the explanation. Finally both of them played with scales to see if and when the isolated point (0,0) would be drawn by Grapes.

Views on the potential of the software

- Both teachers were more willing to talk about their solution processes than about the potential of the software for classroom use.

- After probing them, they noted that the following issues are important pedagogically/mathematically: the possibility of scaling (and thus looking at the same graph in different scales), the possibility of meeting surprises and investigating their origins (according to them, had the second task requested to analyze the expected graph before graphing it, there would have been no surprise and possibly less learning than in the present version in which you turn to the symbols to explain an unexpected graph – this brings closer the relationship between graphs and their equations than it is the case without the technology), the possibility of dynamic change of parameters to better appreciate a family of functions.

- The task both teachers seemed to like best was the second due to the surprise.

- They were hesitant about whether bringing tasks like this to the classroom. Teacher A claimed that it is difficult to discuss isolated tasks without paying attention to the whole curriculum, and also the level of the students is a variable to take into account.

- When asked why do they think teachers in Japan do not use technology in their classes, Teacher B said that teachers have little experience using technology themselves, and that they may not see the benefits of using it. Teacher A said that in secondary school much is determined by the university entrance examinations, and as they do not include problems to be solved using technology, it is not incorporated to the classrooms.

- When asked to think about tasks which can be good to be solved using Grapes, Teacher A, who teaches classes to graduate students about Grapes, said that he will ask this question to his students!.

A few suggestions for further reading

Books

Romberg, T.A., Fennema, E. and Carpenter, T.P. (Eds.), 1993, *Integrating Research on the Graphical Representations of Functions.* Hillsdale, NJ: Erlbaum.

All the chapters of this book are highly recommended for those willing to learn about a range of issues related to the uses of technology for graphical representations of functions. The different chapters discuss rationale, approaches, criticism, examples, curriculum perspectives and research results.

Articles

Arcavi, A. and Hadas, N., 2000, "Computer mediated learning: An example of an approach" *International Journal of Computers for Mathematical Learning*, 5, pp. 25-45.

This article describes how geometrical phenomena can be modeled, interpreted and studied using graphical representations of functions in a dynamic geometry environment.

Dugdale, S., 1992, "Visualizing polynomial functions: New insights from an old method in a new medium", *Journal of Computers in Mathematics* and *Science Teaching*, 11(2), pp.123-141.

Describes and illustrates easy manipulation and exploration of graphical representations of polynomial functions. Reduces emphasis on memorized rules in favor of qualitative understanding of functional behavior, visualization of functional relationships, and graphical investigation of mathematical concepts.

Dugdale, S., 1982, "Green Globs: a microcomputer application for graphing of equations", *Mathematics Teacher*, 75, pp. 208-214

Green Globs is one of the most successful computerized games, in which students are challenged by the uses of graphs of functions. A further description of this game can be found in Dugdale's chapter in the book listed above.

Yerushalmy, M. and Gafni, R., 1992, "Syntactic manipulations and semantic interpretations in algebra: The effect of graphic representation" *Learning and Instruction* 2, pp. 303-319.

This study examines the effect of graphic representation of algebraic functions on performance of tasks involving expressions' transformations.

Yerushalmy, M. and Gilead, S., 1997, "Solving equations in a technological environment: Seeing and manipulating" *Mathematics Teacher* 90(2), pp. 156-163.

This article describes the uses of graphing software to solve algebraic equations in middle school and the effects it had on students.

http://www-groups.dcs.st-and.ac.uk/~history/Curves/Curves.html

This web site includes a compendium of the most interesting curves, their equations and history. It may serve a source of examples to explore.