Designing Mathematics Conjecturing Activities to Foster Thinking and Constructing Actively

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Aims

Based on the perspective that

\textit{a good lesson must provide opportunities for learner’s to think and construct actively.}

The aims of this address are

- to present a framework of designing conjecturing (FDC) with examples;
- to show that conjecturing is an avenue towards all phases of mathematics learning - conceptualizing, procedural operating, problem solving and proving, and
- to argue that conjecturing is to encourage thinking and constructing actively, hence to drive innovation.
• **Rationale:**

**A Good Lesson Must Provide Opportunities for Learners to Think and Construct Actively.**

Why is *thinking and constructing actively* very crucial for APEC’s learners?

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§ **Data of TIMSS-2003 Presents A Dilemma**

Dilemma between Students’ Achievement and Self-Confidence of Mathematics (TIMSS-2003)

<table>
<thead>
<tr>
<th>Achievement</th>
<th>High</th>
<th>Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>Self-Confidence</td>
<td></td>
<td>Malaysia, Australia, U.S., Indonesia, Chile, Philippines</td>
</tr>
<tr>
<td></td>
<td>High</td>
<td>Singapore, Korea, Hong Kong, Japan, Taiwan</td>
</tr>
<tr>
<td></td>
<td>Low</td>
<td></td>
</tr>
</tbody>
</table>
### Students’ Self-Confidence in Learning Math-Grade 4 (TIMSS 2003):

<table>
<thead>
<tr>
<th>Countries</th>
<th>High SCM</th>
<th>Medium SCM</th>
<th>Low SCM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>% of students</td>
<td>Avg. Achievement</td>
<td>% of students</td>
</tr>
<tr>
<td>Singapore</td>
<td>49</td>
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</tr>
<tr>
<td>Hong Kong</td>
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<td>601</td>
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<tr>
<td>Japan</td>
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<td>600</td>
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<tr>
<td>Taiwan</td>
<td>41</td>
<td>591</td>
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</tr>
<tr>
<td>International Avg.</td>
<td>55</td>
<td>522</td>
<td>33</td>
</tr>
</tbody>
</table>

### Students’ Self-Confidence in Learning Math-Grade 4 (TIMSS 2003):

<table>
<thead>
<tr>
<th>Countries</th>
<th>High SCM</th>
<th>Medium SCM</th>
<th>Low SCM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>% of students</td>
<td>Avg. Achievement</td>
<td>% of students</td>
</tr>
<tr>
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<td>64</td>
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<tr>
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<td>Philippines</td>
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<td>International Avg.</td>
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<td>522</td>
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</tr>
</tbody>
</table>
### Students’ Self-Confidence in Learning Math-Grade 8 (TIMSS 2003):

<table>
<thead>
<tr>
<th>Countries</th>
<th>High SCM</th>
<th>Medium SCM</th>
<th>Low SCM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>% of students</td>
<td>Avg. Achievement</td>
<td>% of students</td>
</tr>
<tr>
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<td>Japan</td>
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<tr>
<td>International Avg.</td>
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<td>504</td>
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</table>

### Students’ Self-Confidence in Learning Math-Grade 8 (TIMSS 2003):

<table>
<thead>
<tr>
<th>Countries</th>
<th>High SCM</th>
<th>Medium SCM</th>
<th>Low SCM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>% of students</td>
<td>Avg. Achievement</td>
<td>% of students</td>
</tr>
<tr>
<td>Malaysia</td>
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<td>546</td>
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<td>U.S.</td>
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<td>534</td>
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<tr>
<td>Indonesia</td>
<td>27</td>
<td>420</td>
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<td>Chile</td>
<td>35</td>
<td>427</td>
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<tr>
<td>Philippines</td>
<td>29</td>
<td>405</td>
<td>59</td>
</tr>
<tr>
<td>International Avg.</td>
<td>40</td>
<td>504</td>
<td>38</td>
</tr>
</tbody>
</table>
A Conjecture on the Phenomenon of High Achievement and Low Self-Confidence in Mathematics

Competitive examination system drives passive and rote learning.

Resolution?

A Good Lesson Must Provide Opportunities for Learners to Think and Construct Actively.
I. Thinking: Experiential and Behavioral Point of View

1. Experiential/Phenomenological point of view

Thinking consists in
- envisaging, realizing structural features and structural requirements; proceeding in accordance with, and determined by, these requirements; and thereby changing the situation in the direction of structural improvements, which involves:

- that gaps, trouble-regions, disturbances, superficialities, etc., be viewed and dealt with structurally;
- that inner structural relations – fitting or not fitting – be sought among such disturbances and the given situation as a whole and among its various parts;
- that there be operations of structural grouping and segregation, of centering, etc.;
- that operations be viewed and treated in their structural place, role, dynamic meaning, including realization of the changes which thus involves.

~Productive Thinking (Wertheimer, 1961)
2. Behavioral point of view

- Comparison and discrimination (identification of similarities and differences)
- Analysis (looking at parts)
- Induction (generalisation, both empricial and structural)
- Experience (gathering facts or vividly grasping structure)
- Experimentation (seeking to decide between possible hypotheses)
Expressing one variable is a function of another variable

Associating (items together and recognising structural relationships)

Repeating

Trial and error

Learning on the basis of success (with or without appreciating structural significance)

(based on Wertheimer, 1961, pp. 248-51)

3. Learner’s powers

- Discerning similarities and differences
  – to distinguish
  – to discern
  – to make distinctions

- Mental imagery and imagination
  – the power that we are able to be simultaneously present and yet “somewhere else”
• **Generalising and abstracting**
  – generalising and abstraction are the foundations of language

• **Generalising and specialising**
  – two sides of the coin
    • seeing the particular in the general
    • seeing the general through the particular

• **Conjecturing and convincing**
  – making an assertion about a pattern detected
  – justifying it so that others are convinced
    (Mason & Johnston-Wilder, 2004)

Components of Thinking (behavior point of view/learner’s power) & Meta-Cognition (thinking about thinking) will be used to analyze Conjecturing Activities.
Ⅱ. Designing Conjecturing Activity: 
Examples and a Framework of Conjecturing

1. Three Entries of Conjecturing
   Starting with:
   - A False Statement
   - A True Statement
   - A Conjecture of Learners

1-1 False statement as starting point
ex.(1) Using students' misconception
   e.g. - \(a \times b > a\) and \(a \times b > b\)
   - \(4/9 > 2/3\) (if \(a > c\) and \(b > d\), then \(b/a > d/c\))
   - A multiple must be an integer or a half
     ⊗ the additive strategy on ratio task
   - A quadrilateral with one pair of opposite right
     angle is a rectangle
   ⊗ the sum of a multiple of 3 and 6 is a multiple of 9
   ⊗ the square of a given number is even
   ...

**ex.(2) Applying a procedualized refutation model**

**A proceduralized refutation model (PRM)**

<table>
<thead>
<tr>
<th>Teacher’s role</th>
<th>Student’s activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduce a false proposition</td>
<td>8. Showing Image</td>
</tr>
<tr>
<td>Make sure students comprehend the proposition</td>
<td>1. Giving single example</td>
</tr>
<tr>
<td></td>
<td>2. Giving more examples</td>
</tr>
<tr>
<td>Encourage students to exhaust examples</td>
<td>3. Giving more types of examples</td>
</tr>
<tr>
<td></td>
<td>4. Making distinction of supporting and rejected exs.</td>
</tr>
<tr>
<td>Check/demo writing mathematically</td>
<td>5.1. Finding the common properties of supporting exs.</td>
</tr>
<tr>
<td></td>
<td>5.2.</td>
</tr>
<tr>
<td>Encourage students to make conjectures</td>
<td>6. Reconfirming the correctness of given statement</td>
</tr>
<tr>
<td></td>
<td>7-8. Making conjectures and more conjectures</td>
</tr>
</tbody>
</table>

Model of Proceduralized Refuting and Making Conjectures
### Table 1. PRM Item-Thinking Analysis

<table>
<thead>
<tr>
<th>Thinking Item</th>
<th>Primitive image</th>
<th>Specializing</th>
<th>Classifying</th>
<th>Generalizing</th>
<th>Abstracting</th>
<th>Adapting</th>
<th>Formalizing</th>
<th>Extrapolating</th>
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<tbody>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✓</td>
</tr>
</tbody>
</table>

### Table 2. PRM Item-Metacognition Analysis

<table>
<thead>
<tr>
<th>Metacognition</th>
<th>Awareness</th>
<th>Evaluation</th>
<th>Self-regulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
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<td></td>
</tr>
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<td>3</td>
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<td>✓</td>
<td></td>
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<tr>
<td>4</td>
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<tr>
<td>5-1</td>
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</tr>
<tr>
<td>5-2</td>
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<td></td>
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</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td>✓</td>
</tr>
</tbody>
</table>
1-2 True statement as starting point

**ex1. Heron’s Formula as Starting Point:**

\[ A = \sqrt{s(s-a)(s-b)(s-c)} \]

where \( s = \frac{1}{2} (a+b+c) \)

(i) Making your own sense of the formula:

Convincing yourself that \( A \) do represent the area of a triangle with three sides \( a, b, \) and \( c. \)

Observing it's beauty.
(ii) A model of conjecturing: A triad of mathematics thinking

• Symmetry
• Degree of the Form
• Specializing/Extreme Cases

(iii) Application of the Triad

e.g. What can you say about the formula B:

\[ B = \sqrt{(s-a)(s-b)(s-c)(s-d)} \], where \( s = \frac{1}{2}(a+b+c+d) \)
(iv) Your conjecture about B will be:

(v) Convincing yourself and peers about your conjecture.

(vi) Conjecturing the volume of

\[ V = ? \]

\[ V = ? \]
1-3 Starting with students’ own conjecture

(1) Defining activity

ex. Swimming Pool

Conan is going to move to a new home. He has a rectangular swimming pool built in the backyard. When he checked the pool, he said, "Is it really a rectangular swimming pool?" If you were Conan, what places and what properties would you ask the workers to measure so that you can be sure it is rectangular? (It costs NT$1000 to check each item.)

Be sure, the payment is the less the better.

(Lin & Yang, 2002)

(2) Perceiving from an exploration process

ex. Triangle and Tetrahedron

(i) Demo:

Folding out a tetrahedron from a given regular triangle:
(ii) Could you folding out a tetrahedron from a given isosceles triangle?

(iii) Would some kind of isosceles triangles work?

(iv) Could an isosceles right triangle work?

(v) How would you classify triangles?

(vi) According to your classification, which kind of triangle would work?

(vii) Making your conjectures

(viii) Un-folding a tetrahedron, which kind of polygon you can obtain?
(3) Constructing Premise/Conclusion

ex.
If..., then the sum (product) of two numbers is even
If the sum (product) of two numbers is even, then ...
If..., then their product is bigger than each of them
If their product is bigger than each of them, then ...
If..., then the line L bisects the area of the quadrilateral
If a is an intersection point of two diagonal lines of a quadrilateral, then ...

2. A Frame for Designing Conjecturing (FDC)

<table>
<thead>
<tr>
<th>Starting</th>
<th>Learning Strategy/Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>• False Statement</td>
<td>Proceduralized refutation learning model</td>
</tr>
<tr>
<td></td>
<td>A thinking triad</td>
</tr>
<tr>
<td></td>
<td>“What if not” strategy to improve problem posing (Brown &amp; Walter, 1983)</td>
</tr>
<tr>
<td></td>
<td>Specialization/Generalization (Polya, 1962; Mason, Burton, &amp; Stacey, 1985)</td>
</tr>
<tr>
<td></td>
<td>Analogous</td>
</tr>
<tr>
<td></td>
<td>Re-modification: modify-remodify till one makes sense of it</td>
</tr>
<tr>
<td>• True Statement</td>
<td>Defining</td>
</tr>
<tr>
<td></td>
<td>Exploration</td>
</tr>
<tr>
<td></td>
<td>Constructing Premise/Conclusion</td>
</tr>
</tbody>
</table>
III. Mathematics Learning Processes

1. What is Mathematics?
Mathematics viewed as concepts and patterns with their underlying situations

   - Conceptualizing for Conceptual Understanding
   - Procedural Operating for Procedural Fluency
   - Problem Solving for Strategic Competence
   - Proving for Adaptive Reasoning
   - Productive Disposition associates with all phases of learning processes
Mathematical Proficiency
-to learn mathematics successfully

- Conceptual Understanding
- Procedural Fluency
- Strategic Competence
- Adaptive Reasoning
- Productive Disposition

(NRC, 2001)

3. To Show: Conjecturing is the Core of Mathematizing

[Diagram showing relationships between Conjecturing, Conceptualizing, Procedural Operating, Problem Solving, and Proving]
3-1 Conjecturing to Enhance Conceptual Understanding

ex.(1) Using students’ misconceptions as the starting statement in PRM.

ex.(2) Inviting students to make conjecture of fraction addition after they have learned the meaning of fractions. Using the error pattern \( \frac{a}{b} + \frac{c}{d} = \frac{(a+c)}{(b+d)} \) as the starting statement in PRM.

3-2 Conjecturing to Facilitate Procedural Operating

• ex.(1) Using “the sum of a multiple of 3 and a multiple of 6 is a multiple of 9” as the starting statement is PRM.

• ex.(2) Focusing on the Thinking Triad to make conjecture of the volume of a conical shape.
3-3 Conjecturing to Develop Competency of Proving

(1) Learning strategy: Constructing Premise/Conclusion
ex. Refer to 1-3:(3)

(2) Learning strategy: Defining
ex. the Swimming Pool Task

3-4 Conjecturing is a Necessary Process of Problem Solving

• Mathematical Discovery (Polya, 1962)
  - Mathematics thinking as problem solving: the first and foremost duty of the high school in teaching mathematics is to emphasize mathematical…problem solving.
  - Specialising and generalising as an ascent and descent, in an ongoing process of conjecturing.
• Thinking Mathematically (Mason, Burton, & Stacey, 1985)
  Specialising
  Generalising
  Conjecturing
  Convincing

4. Conjecturing Approach

Participating in a conjecturing designed with FDC in which everyone is encouraged
1) to construct extreme and paradigmatic examples,
2) to construct and test with different kind of examples,
3) to organize and classify all kinds of examples,
4) to realize structural features of supporting examples
5) to find counter-examples when realizing a falsehood,
6) to experiment
7) to adapt conceptually
8) to evaluate one’s own doing-thinking
9) to formalize a mathematical statement
10) to image/extrapolate/explore a statement
11) to grasp fundamental principles of mathematics

involves learners in thinking and constructing actively.

Conjecturing Involves Learners in Thinking & Constructing Actively

Participating in a conjecturing atmosphere in which everyone is encouraged to construct extreme and paradigmatic examples, and to try to find counter-examples (through exploring previously unnoticed dimensions-of-possible-variation) involves learners in thinking and constructing actively. This involves learners in, for example, generalising and specialising.

(Mason, J. & Johnstone-Wilder, S., 2004, p.142)

This extract have been extrapolated in the above synthesis.
Conjecturing as a Strategy for Innovation

Since
Conjecturing encourage learners to think and to construct actively.
And
Thinking & constructing actively is the foundation of innovation.

Conjecturing is an adequate learning strategy for innovation.

Features of FDC
- unlike modeling which share the same core status with mathematizing as conjecturing

• FDC is easy to implement

• Some case studies have shown its effectiveness

Inviting all of you to experience FDC’s power!
Let’s Do It Together!

Appendix:

<table>
<thead>
<tr>
<th>Mathematics Conjecturing Task</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Class</strong></td>
</tr>
<tr>
<td>School</td>
</tr>
</tbody>
</table>

Statement A: The multiples of 3 and the multiples of 6 are the multiples of 9. Is the statement correct? Please describe your points.

My answer: [ ] correct  [ ] incorrect  [ ] uncertain

My reason: ____________________________

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<table>
<thead>
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<th>Mathematics Conjecturing Task</th>
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<td><strong>Class</strong></td>
</tr>
<tr>
<td>School</td>
</tr>
</tbody>
</table>

1. Please provide an example for the statement A.

1. _________________________________

2. Please provide some more examples for the statement A.

1. _________________________________
2. _________________________________
3. _________________________________
4. _________________________________
3. Please provide even more kinds of examples till you cannot think of any other kind of examples.

4. From the example you provided in Q2 and Q3, distinguish which examples support the statement A, and which ones reject. Please list numbers/letters as support or rejection.

5.1 What are the common properties among the supporting examples?

5.2 What are the common properties in the rejected examples?

6. Reconsider the statement A.

Do you think the statement is correct or incorrect?

[ ] correct  [ ] incorrect  [ ] uncertain

My reasons or proof:

7. From the above process, please restate the statement A into a correct statement.

Correct statement:

8. Please do your best to give more statements similar to A that you believe to be correct.

1.
2.
3.
4.
5.
References: