# REPORT
# Mathematical Communication through Proof and Representation

APEC - Tsukuba
International Conference on Innovative Teaching Mathematics
8 December 2007 - 15 December 2007

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**Introduction**

Mathematical thinking is always an important element in mathematics curriculum. Mathematical proof is a kind of mathematical thinking, representation, and communication. In this paper, we will discuss the component of mathematics communication in the Hong Kong mathematics curriculum, how mathematics communication can be developed in the classroom through the mathematics tasks in finding the greatest product of multiplication. And include the discussion of teaching approaches is preferred to develop mathematics communication. The mathematical tasks will be used in three levels of classes, primary 3, primary 4 and primary 5.

The paper tried to address the following questions on mathematical communication.

| | |
|---|---|
| Q 1 | How does the Hong Kong curriculum document enhance communication or mathematical communication for students? |
| Q 2 | What are the components of mathematical communication to develop? |
| Q 3 | What kinds of approach will you prefer to develop the communication in classroom? |

**Communication within the Hong Kong mathematics curriculum document**

In the 1983 version of the Hong Kong Primary Mathematics curriculum, there is no specific mention of the term “communication”. However, in the later developed TOC curriculum (Targeted Oriented Curriculum) based on 1983 curriculum, communication is one of the five components (investigation, concepts, problem solving, reasoning and communication).

The term communication in the document usually refers to the generic nature, such as discussion among students, discussion between students etc. It s not refereeing to the mathematical nature such as proof or representation.

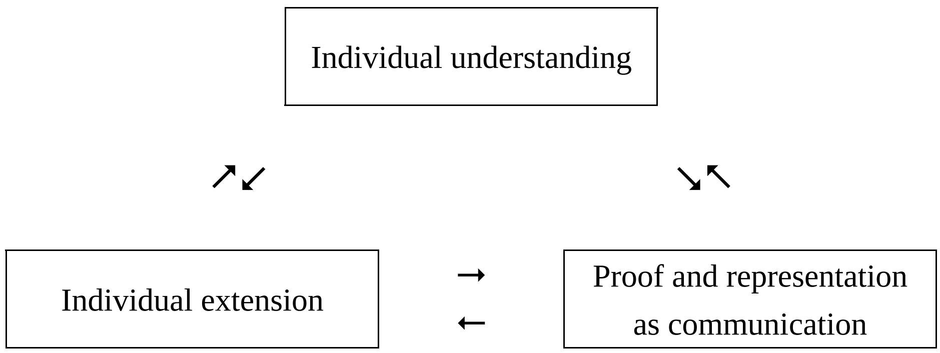
**Components of mathematical communication**

The main components of mathematics communication are representation and proof. There are three kinds of communication of mathematical knowledge.

The first kind of communication is the internal understanding of mathematics of the individual. It is a thinking process so that the mathematics problem and also the path of solving the problem become a kind of coherence thinking process. With the representation of the solution in diagram or in words or in symbol, students obtain the understanding of the process.

The second kind of communication is the explanation of the solution to others. This involves the understanding of other thoughts in understanding the questions and the solutions. We may not understand why other people do not understand the meaning of the problem or the solution path of the problem. One possibility is the representation of the problem and the solution. With a good representation or an acceptable representation, students understand what other means and the content of mathematics.

The third kind of communication is internal extension of the mathematical content of the individual, trying to extend that mathematics content that is convinced by other and extend the difficulty of the problem or posing new problem.




**The mathematics tasks and the process**

The aim of the paper is to discuss how primary school students communicate their thinking through the following problems.

<table>
<tr><td colspan="3">Task A<br><br>Using the numbers to fill the boxes and make the greatest product</td></tr>
<tr><td>Using the set of numbers<br><br>(1, 2, 3), (1,2,5), (2,3, 6)<br><br>□ □<br>× □</td><td>Using the numbers<br><br>(1, 2, 3, 4), (1, 2, 4, 5).<br><br>□ □ □<br>× □</td><td>Using the numbers<br><br>(1, 2, 3, 4, 5),<br><br>(1, 2, 3, 5, 6).<br><br>□ □ □ □<br>× □</td></tr>
</table>

<table>
<tr><td colspan="3">Task B<br><br>Using the numbers to fill the boxes and make the greatest product</td></tr>
<tr><td>Using numbers<br><br>(1, 2, 3, 4)<br><br>(1, 3, 5, 7)<br><br>□ □<br>× □ □</td><td>Using numbers<br><br>(1, 2, 3, 4, 5),<br><br>(1, 2, 4, 5, 7, ).<br><br>□ □ □<br>× □ □</td><td>Using numbers<br><br>(1, 2, 3, 4, 5, 6),<br><br>(4, 5, 6, 7, 8, 9)<br><br>□ □ □<br>× □ □ □</td></tr>
</table>

**Teaching approach for communication**

The most important part of teaching is the selection of appropriate questions. Such tasks allow students to express their solution through different representation. The questions are set in the way that the answer can be observed by the pattern. When students predict their answer, they can confirm the answer through their communication.

Mathematics questions selected are within the mathematics curriculum, so that students can perform mathematical investigations based on their knowledge on arithmetic operations. The beginning questions use some simple number so that students can obtain answers easily.

Encourage students to use generic skills such as listing and observing pattern. Encourage students to explain their answer based on pattern and observation and explain it through different representation.

**Task A, finding the greatest product of multiple digits with one digit**

| Question A1： | Question A2： |
|---|---|
| Using the numbers 1, 2, 3 to fill in the boxes to make the greatest product <br> $\begin{array}{r} \square\ \square \\ \times \quad \square \\ \hline \end{array}$ | Using the numbers 1, 2, 3, 4 to fill in the boxes to make the greatest product <br> $\begin{array}{r} \square\ \square\ \square \\ \times \quad \square \\ \hline \end{array}$ <br> \ |

**Listing and using pattern observation**
Students are asked to list all the possible multiplications and they obtain the following results:

$$\begin{array}{r} 1\ 2 \\ \times\quad 3 \\ \hline 3\ 6 \end{array} \qquad \begin{array}{r} 1\ 3 \\ \times\quad 2 \\ \hline 2\ 6 \end{array} \qquad \begin{array}{r} 2\ 1 \\ \times\quad 3 \\ \hline 6\ 3 \end{array} \qquad \begin{array}{r} 2\ 3 \\ \times\quad 1 \\ \hline 2\ 3 \end{array} \qquad \begin{array}{r} 3\ 2 \\ \times\quad 1 \\ \hline 3\ 2 \end{array} \qquad \begin{array}{r} 3\ 1 \\ \times\quad 2 \\ \hline 6\ 2 \end{array}$$

Next they are asked to find the greatest product by arranging a set of three numbers, for example, using the numbers 2, 3, 5. After some examination, many students easily found that the following arrangements could not produce greatest product. They are

$$\begin{array}{r} 5\ 3 \\ \times\quad 2 \\ \hline 10\ 6 \end{array} \qquad \begin{array}{r} 3\ 5 \\ \times\quad 2 \\ \hline 7\ 0 \end{array}$$

And they concluded that large numbers should be placed at the position P and Q.

$$\begin{array}{r} P\ \square \\ \times \quad Q \\ \hline \end{array}$$

When they come to question A2, using 1, 2, 3, 4 to form a greatest product in the form of □□□×□, many students can predict the greatest product through pattern observation and give the following as answer.

$$\begin{array}{r} 3\ 2\ 1 \\ \times \quad\ \ 4 \\ \hline 12\ 8\ 4 \end{array}$$

**Students' logical thought on the results 321×4 = 1284**

Some students found that there is a dominant effect of the larger number in certain "critical" place value. They try to compare the following multiplication with the large number at the critical place. Hence the number in the second line could not be 1.

At one instance, when he found out that putting number 1 could not give the greatest product, a student reckon that it will not be a possible to get greatest product if the number 2 is put in, and also not the number 3. Finally, they conclude that the largest number 4 should be put at position Q. And they convinced that the following give the largest product.

$$\begin{array}{r} 3\ 2\ 1 \\ \times \quad\ \ 4 \\ \hline 12\ 8\ 4 \end{array}$$

**Explanation of 321×4 = 1284 is the greatest product**

However, this is not "accepted" as explanation and they need to convince others. And they try using comparison of different product for explanation. Though many students agree that the following arrnagment could not give largest product.

$$\begin{array}{r} 4\ \square\ \square \\ \times \quad\ \ 2 \\ \hline \end{array} \quad \begin{array}{r} 3\ \square\ \square \\ \times \quad\ \ 2 \\ \hline \end{array} \quad \begin{array}{r} 4\ \square\ \square \\ \times \quad\ \ 3 \\ \hline \end{array} \quad \begin{array}{r} 2\ \square\ \square \\ \times \quad\ \ 3 \\ \hline \end{array} \quad \begin{array}{r} 3\ \square\ \square \\ \times \quad\ \ 4 \\ \hline \end{array} \quad \begin{array}{r} 2\ \square\ \square \\ \times \quad\ \ 4 \\ \hline \end{array}$$

The following is the representation of multiplication used in many mathematics textbooks and difficult to see the comparison.

$$\begin{array}{rrrr} & 3 & 2 & 1 \\ \times & & & 4 \\ \hline & 12 & 8 & 4 \end{array} \qquad \begin{array}{rrrr} & 4 & 2 & 1 \\ \times & & & 3 \\ \hline & 12 & 6 & 3 \end{array}$$

The teacher tries to introduce multiplication in the following format so that students can compare different layers of number. Once it is written, many students start to figure why the arrangement give the greatest product.

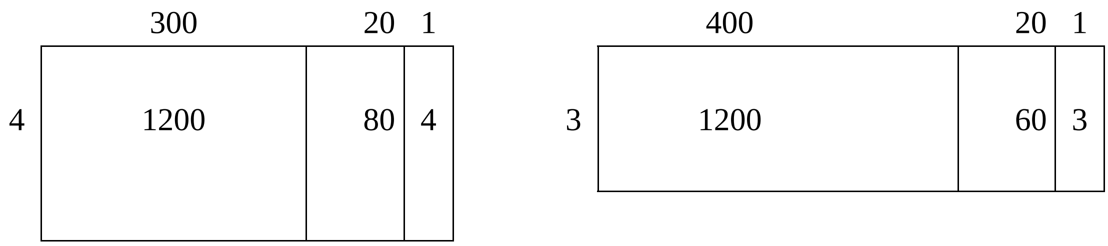
$$\begin{array}{rrrr} & 3 & 2 & 1 \\ \times & & & 4 \\ \hline & 12 & 0 & 0 \\ & & 8 & 0 \\ & & & 4 \\ \hline & 12 & 8 & 4 \end{array} \qquad \begin{array}{rrrr} & 4 & 2 & 1 \\ \times & & & 3 \\ \hline & 12 & 0 & 0 \\ & & 6 & 0 \\ & & & 3 \\ \hline & 12 & 6 & 3 \end{array}$$

$321 \times 4 = 1200 + 80 + 4$ $\qquad$ $421 \times 3 = 1200 + 60 + 3$

By this partition of multiplication, they can compare the sum of product layer by layer and obtain the conclusion that 321×4 is in fact the largest.

**Second explanation**

For primary 5 students, some use the concepts of areas to explain why the arrangement gives greatest product. This is a pictorial representation of the above calculation. In fact they can compare the sum of the three numbers 1200 + 60 + 3 with the sum of the three numbers 1200 + 80 + 4.



**Explanation 3**

However, students also able to use their own representation to show that the product they find is greatest by some logical argument. When students continue to work on this type of questions, some have a new idea on comparison.

| Question A3： |
|---|
| Using the numbers 1, 2, 3, 5, 6 to fill in the boxes to make the greatest product<br>    □ □ □ □<br>×          □ |

Many students already noticed that there are two places that the greatest number should be put in. They are the position P and Q.

```
   P □ □ □
×        Q
---------
```

They then compare the following two products 5321×6 and 6321×5.

Many students notice that there is a common part (321) in the two products, so they can just compare the product of multiplying the number 321×6 and 321×5.

In fact they are using distributive law to do that without noticing it, comparing (5000 + 321) × 6 and (6000 + 321) × 5.

As “5000× 6 = 6000× 5”, students know that they only need to compare the term 321× 6 and 321× 5, and obviously 321× 6 is larger than 321× 5. Hence they come to the conclusion that “5321×6” is the greatest product.

| Product 1： | Product 2： |
|---|---|
| 5 [3] [2] [1]<br>×       6 | 6 [3] [2] [1]<br>×       5 |

**Tasks B  Finding the greatest product of two digits times two digits**

| Question B1： | Question B2： |
|---|---|
| Using the numbers 1, 2, 3, 4 to fill in the boxes to make the greatest product<br>$\begin{array}{r} \square\ \square \\ \times\ \square\ \square \\ \hline \end{array}$ | Using the numbers 1, 3, 5, 7 to fill in the boxes to make the greatest product<br>$\begin{array}{r} \square\ \square \\ \times\ \square\ \square \\ \hline \end{array}$ |

Students have tried out nearly all the possible arrangement of the numbers.

$$\begin{array}{r} 4\ 3 \\ \times\ \ 2\ 1 \\ \hline \end{array} \qquad \begin{array}{r} 4\ 2 \\ \times\ \ 3\ 1 \\ \hline \end{array} \qquad \begin{array}{r} 2\ 4 \\ \times\ \ 3\ 1 \\ \hline \end{array} \qquad \begin{array}{r} 2\ 3 \\ \times\ \ 4\ 1 \\ \hline \end{array}$$

Students observe a pattern of arrangement of the numbers when it gives the greatest product. They observed they need to maximize the "tenth" digits to give a greatest product. After working with the tasks A, students already have the knowledge that large number should be placed at position P and Q.

$$\begin{array}{r} P\ \square \\ \times\ \ Q\ \square \\ \hline \end{array}$$

This comes to the comparison of the values of the two products

$$\begin{array}{r} 4\ 2 \\ \times\ \ 3\ 1 \\ \hline 1\,3\,0\,2 \end{array} \qquad \begin{array}{r} 4\ 1 \\ \times\ \ 3\ 2 \\ \hline 1\,3\,1\,2 \end{array}$$

Obviously the second arrangement $41\times32 = 1312$ gives a larger product.

**Pattern observation**

If there is 4 numbers, we need to "equally" fill in the numbers. Equally means a small number should go with a large number, so the largest number with pair with the smallest.

Students are then asked to notice and predict their result or explain their results. They use the following logic

Divide the numbers into two groups. So the two large numbers 3 and 4 will go into different group. And to compensate the value, a smaller number "1" will go with 4 and the larger number will go with 3 so as to offset the difference. Hence they try the product of the grouping 41 and 32.

For the next question, they list two to three possible arrangement, however, they know that 52×43 will be one of it as all include this product for their later comparison.

**Discussion of why such arrangement gives a larger value**

Students also try to explain why the difference of the two product 41×32 = 1312 and 42×31 = 1302 is 10.

One of the explanations is to obtain the product line by line. This is a kind of represenation which the students need to learn and use in the discussion.

| | | |
|---|---|---|
| 4 2 | 4 1 | |
| × 3 1 | × 3 2 | |
| 1 2 0 0 | 1 2 0 0 | (equal) |
| 4 0 | 8 0 | ( + 40) |
| 6 0 | 3 0 | (–3 0) |
| 2 | 2 | (equal) |
| 1 3 0 2 | 1 3 1 2 | |

Students see that two of the products are the equal (40×30 = 1200 and 1×2 =2) and the other two product are 40×1 = 40 and 40×2 = 80, differ by 40 and 30×1 = 30 and 30×2 = 60. The above result in a difference of sum in 10.

Usually the multiplication in Hong Kong primary school is in the following format. This kind of represenation unable to discuss why the first product is smaller than the second product.

| | |
|---|---|
| 4 2 | 4 1 |
| × 3 1 | × 3 2 |
| 1 2 6 0 | 1 2 3 0 |

$$\begin{array}{r} 42 \\ \hline 1302 \end{array} \qquad \begin{array}{r} 82 \\ \hline 1312 \end{array}$$

However, other students give an explanation on the following product...

> Question B2：
>
> Using the numbers 1, 3, 5, 7 to fill in the boxes to make the greatest product
>
> $$\begin{array}{r} \square\ \square \\ \times\ \square\ \square \\ \hline \end{array}$$

**Representation 2**

$$\begin{array}{r} 7\ \ 3 \\ \times\ \ 5\ \ 1 \\ \hline \end{array} \qquad \begin{array}{r} 7\ \ 1 \\ \times\ \ 5\ \ 3 \\ \hline \end{array}$$

He use the first arrangement 73×51 and subtract 71×53, finding that it is not easy to do so, he compare 73×51 and 71×51, knowing that he need to add in 71×2 later on. As 73×51 - 71×51 = 2×51, he knows the answer is 102. However, he need to subtract the sum 71×2 = 142, so that we know that the difference of the two numbers is 142 – 102 = 40. That is 71×53 is larger than 73×51 by 40.

| | 70 | 3 |
|---|---|---|
| 50 | 3500 | 150 |
| 1 | 70 | 3 |

| | 70 | 1 |
|---|---|---|
| 50 | 3500 | 50 |
| 3 | 210 | 3 |

This is equivalent to the format of multiplication. Some students compare the two formats and discuss the values of the two products.

**Representation 3**

The third representation is similar to representation 2. They know that the comparison

$$\begin{array}{rcc} & 7 & 3 \\ \times & 5 & 1 \\ \hline \end{array} \qquad \begin{array}{rcc} & 7 & 1 \\ \times & 5 & 3 \\ \hline \end{array}$$

So they compare 7×1 + 5×3 and 7×3 + 5×1.

The teacher also uses area approach to calculate the two products and invite students to compare the answers.

**Some argument and counter examples**

One student thinks that the answer should be putting the four numbers into two groups such that the sum of each group is the same. In the problem of 1, 3, 5, 7, he think that the answer is 71×53, as 7+1 = 5+3 = 8.

(i)

$$\begin{array}{rcc} & 7 & 1 \\ \times & 5 & 3 \\ \hline & 3763 & \end{array} \qquad \begin{array}{l} 7 + 1 = 8 \\ 5 + 3 = 8 \end{array}$$

Many students try to figure out whether the argument is valid. Some students bring out the following counter examples

The first example is using the 4 numbers 4, 5, 7, 9. They know that the greatest product is 94×75 = 7050. However, 9 + 4 = 13, and 7 + 5 = 12. The two sums are not equal and the second sum 7 + 5 = 12 is less than the first sum 9 + 4 = 13.

The second counter example is using the set of numbers 1, 7, 8, 9, which tries to use more extreme values. The greatest product is 91×87 = 7917. However the two sums are 9+1= 10, and the second sum is 8+7 = 15. Contrary to the first counter example, the second sum 8+7=15 is larger than the first sum 9 + 1 = 10.

Hence many students, after hearing the argument of the counters examples, agree that the first explanation is not a valid one.

(4, 5, 7, 9)

9 4 9 + 4 = 13

(1, 7, 8, 9)

9 1 9 + 1 = 10

$$\begin{array}{r} \times\ \ 7\ \ 5 \\ \hline 7050 \end{array} \quad 7 + 5 = 12 \qquad \begin{array}{r} \times\ \ 8\ \ 7 \\ \hline 7917 \end{array} \quad 8 + 7 = 15$$

**Using analogy for new question**

| |
|---|
| Question B5： <br> Using the numbers 1, 2, 3, 4, 5 to fill in the boxes to make the greatest product <br> □ □ □ <br> × □ □ |

**Using analogy and assumption**

One student uses the thinking that supposes that the number is 1, 1, 3, 4, 5. Since the answer to 3, 4, 5 is known. They use the following result

| Question ： | Answer ： |
|---|---|
| Using 3, 4, 5to form the greatest product. <br> □ □ <br> × □ | 4 3 <br> × 5 |

| New Question ： | Anaology： |
|---|---|

| Using 1, 2, 3, 4, 5 to form the greatest product. | Using the previous result and fill in 1 and 2. |
|---|---|
| $\begin{array}{r} \square\ \square\ \square \\ \times \quad \square\ \square \\ \hline \end{array}$ | $\begin{array}{r} 4\ 3\ \square \\ \times \quad 5\ \square \\ \hline \end{array}$ |

For the new problem, using the number 1,2 3, 4, 5 to form a greatest product in the form of □□□×□□, students tries to apply the results of the above example. Assume that the two numbers added are the same 1 and 1, then the greatest product will be

$$\begin{array}{r} 4\ 3\ 1 \\ \times \quad 5\ 1 \\ \hline \end{array}$$

If the two new numbers are 1 and 2, then there are two possibilities for greatest product.:

(i) $\begin{array}{r} 4\ 3\ 1 \\ \times \quad 5\ 2 \\ \hline \end{array}$ (ii) $\begin{array}{r} 4\ 3\ 2 \\ \times \quad 5\ 1 \\ \hline \end{array}$

Product (i) will give a larger value than (ii) as 431×2 is larger than 432×1.

After that, they tries to find out all potential large products and confirm their results.

$$\begin{array}{r} 4\ 3\ 1 \\ \times \quad 5\ 2 \\ \hline 22\ 412 \end{array} \qquad \begin{array}{r} 4\ 3\ 2 \\ \times \quad 5\ 1 \\ \hline 22\ 032 \end{array} \qquad \begin{array}{r} 5\ 3\ 2 \\ \times \quad 4\ 1 \\ \hline 21\ 812 \end{array} \qquad \begin{array}{r} 5\ 3\ 1 \\ \times \quad 4\ 2 \\ \hline 22\ 302 \end{array}$$

In another class of students, they have the following discussion.

| New Question ： | Anaology： |
|---|---|

| The greatest product for using the numbers 6, 7, 8, 9 is<br>$\begin{array}{r} 9\ 6 \\ \times\ 8\ 7 \\ \hline \end{array}$ | Using the numbers 4, 5, 6, 7, 8, 9 to fill in the boxes to make the greatest product<br>$\begin{array}{r} \square\ \square\ \square \\ \times\ \square\ \square\ \square \\ \hline \end{array}$ |
|---|---|

Hence the greatest product of 6 numbers 9, 8, 7, 6, 5, 4 is adding the new number 4, and 5.
Many students predict that the greatest product is

$$\begin{array}{r} 9\ 6\ 4 \\ \times\ 8\ 7\ 5 \\ \hline 843500 \end{array}$$

As they only have to consider the two cases.

$$\begin{array}{r} 9\ 6\ \square \\ \times\ 8\ 7\ \square \\ \hline \end{array} \qquad \text{(i)} \begin{array}{r} 9\ 6\ 5 \\ \times\ 8\ 7\ 4 \\ \hline 843410 \end{array} \qquad \text{(ii)} \begin{array}{r} 9\ 6\ 4 \\ \times\ 8\ 7\ 5 \\ \hline 843500 \end{array}$$

The results confirm with their prediction.

**Conclusion**

1
Communication and understanding depends on different representations.

2
Listing is a powerful starting point for communication.

3
Encourage students to use analogy and counter example in their mathematical thinking.

4

In the process of finding the greatest product, students show the following different path of communication.

| | |
|---|---|
| 1 | List all products and Pattern observation |
| 2 | Compare products of all different multiplication<br>Compare products of selected multiplication |
| 3 | Compare products using area concepts |
| 4 | Using previous results and analogy, logical argument |

5

The teaching process can add in simpler question. The following are some examples.

| Question 1： | Question 2： |
|---|---|
| Fill in number 7, and 8, so that the product is greatest.<br>3 ☐<br>× 6 ☐ | Fill in number 7, and 8, so that the product is greatest.<br>3 4 ☐<br>× 6 5 ☐ |

6

The discussion of mathematics communication is summarized in the following framework.

| Framework of three levels of mathematical communication | |
|---|---|
| 1 | Individual understanding of the problem and solution (observe pattern and predict answer) |
| 2 | Explanation and proof through different representations such as area concepts and layers of products |
| 3 | Individual understanding, extend further understanding through changing the solution path or proposing new problem and their solution. (for example, changing the consecutive numbers to non-consecutive, or repeated number) |