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Why focusing in representation.
From a semiotic to a purely mathematical approach.

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Abstract. Focusing on representation might be considered something that has to be done to favor learning, but that, at the end, might be external to mathematical activity itself. In this paper I call to mind that, on the contrary, representation is not only something that mathematicians also do, but that is in fact at the heart of the mathematical activity. For that, I start from what we regard as some compelling although broad suggestions of historical nature, then briefly look at some psychological aspects of representations as stated in Raymond Duval's semiotic representation theory, and finally present how intimately related are the pedagogical issues of the matter to the abstract contemporary mathematical activity.

Historical approaching

Thinking about the importance of representation to Mathematics, perhaps the very first example that comes to mind is that of Decimal system.

Of course, there is many a reason for that, and it is an interesting naïve exercise to try to imagine what the development of Arithmetic and Mathematics at large would have become without Decimal system: It might be argued that one can make more calculations than what is usually expected, say, with Roman numerals; but it would be easier to assert that those computations would be harder to perform to everybody and hence, from a pedagogical perspective, less convenient. Anyhow, mankind has chosen Decimal system.

An interesting additional example of the importance of representation comes out by considering the struggling of mathematicians in the 16th century to turn the *cosa* (their rendering for Latin *res,* coming in turn from the *al-jabr*'s name *ˇsay'* for the unknown magnitude) in something more manageable: Regiomanto's *census* and *demptis* (1464), abbreviated by Luca Pacioli (1494) and also Stevin (1585) and Klau (1593); abstracted by Viète (1593) and finally turned in Descartes' "*x*" (1637). There is more than a point here: one, of course, is the accessibility of calculations, another one is the connection of the process barely suggested above and the mathematical constructions that are behind it (Complex numbers, e.g.).

In the same vein one might want to consider the comparatively burdensome verbose Aristote's and Megaric-Stoic Logic, take in account the switch made by Leibniz's target "*Calculemus*" (1666) – as opposite to merely arguing – and then turn to the more accessible notations built in the second half of the 19th century, and up to common usage Russell-Whitehead's version (1910). Again, the point here would not only be accessibility of the matter and proficiency in the calculations but also theoretical development.

More directly related to the first case, perhaps, and more interesting for our topic is the following: up to the beginning of the 20th century, mathematicians would write (and think) a function from A to B as $f(A) \subset B$. Now, Saunders Mac Lane claimed that the first person who wrote $f(A) \rightarrow B$, instead, was probably Witold Hurewicz around 1940, and that the arrow expressed well a central interest of topology, and thus prompted the idea of a *functor*, "Thus, a notation, (the arrow) led to a concept (category)". (Mac Lane[1], 1971, p. 9). Category theory is, so to say, an all embracing theory in mathematics, and involves a far reaching idea of representation, as we will see.

The pedagogical facts

It is natural to expect that history (even from a rather broad view, as in some of the examples above) would illuminate a subject's importance. On the other hand, it is also natural to assume that, if mankind has had to struggle with some particular aspects of its scientific endeavor, there is a chance that every individual should experience certain pains when learning them – although perhaps using more paved routes, and with more abstract constrains.

Anyhow, the beauty of the whole undertakings shown by historical approaches somehow shadows when confronted with a teaching situation. As for representation in mathematics, for example, it is so appealing to consider the possibility of a second degree equation having real roots and how many of these should be, by means of drawing the Cartesian graph of the quadratic function involved, but it is far from easy to get kids to grasp this matter.

Duval's approach.

The well known Raymond Duval's semiotic theory is a quite useful tool to look into the importance of focusing into representation. For the sake of the discussion, I will try to remind some essentials of it and of the evidences given. In doing so, I'll attempt not to go into the details of definitions and the like, and, at the same time, not to oversimplify. Thus, I will surely fail; but, on the other hand, I hope that I may be able to show that his centering in representation is very close to the heart of mathematics[2].

Duval's approach is of psychological foundation:

He takes the notion of representation register developed firstly for little kids beliefs and explanations on natural phenomena in the 20's, followed by Piaget's evocation of absent objects (30's to 50's), to the ideas developed by works on acquisition of mathematical knowledge and over learning systems (80's), and turns it into a specific tool for teaching and learning – especially, in Mathematics.

[1] Mac Lane was one of the creators of the theory and also one of the leading mathematicians of the 20th century.

[2] In any case, this does not imply that he has taken his ideas from the mathematics that I'll talk about later, rather, that going deeply into mathematic thinking he found by himself paths that are at the core of contemporary mathematical issues.

He is concerned with the ‘fundamental cognitive actions’: conceptualization, reasoning, problem resolution, text comprehension; Mathematics is a privileged field for studying such activities.

The question posed is in which way information coming from the exterior can get into the internal system, i.e., what description made up with the help of symbols susceptible of being used by the system allows to capture the information coming from outside.

For that, he starts by distinguishing between mental and semiotic representations: A *mental representation* is all set of images and concepts that an individual may have about an object or a situation, and what is related to them. A *semiotic representation* is a production made by the use of signs belonging to a system of representations which has its own constraints of meaning and functioning.

So one would ask if these kinds of representations fulfill the same functions in intellectual development and cognitive activities; it turns out that semiotic representations do comply with the functions that mental representations satisfy, but also with other functions that are not satisfied by these. This is illustrated by the fact that the main semiotic organization is natural language, but some fundamental cognitive activities require of systems of expression and representation that are different from it.

In fact, a semiotic representation is not just a means for an individual to exteriorize his/her inner representations – to make them visible or accesible to others – but it is essential to the way under which information is discovered and taken into account in a representation system. Knowledge progress is always related to creation and develpoment or new specific semiotic systems that coexist with natural languages (which is evident in formation of scientific knowledge in general).

A distinction is made between *semiosis*, that is, aprehension and production of a semiotic representation, and *noesis* the cognitive act of the conceptual aprehension of an object. The distinction been made, the fundamental quest would be what is the relationship between them, if any – they might be independent. It is better for us to translate this question to the mathematical realm.

Duval on Mathematics.

Certainly, Mathematics – or, more precisely, a mathematical theory – is in itself a semiotic system; thus, in Mathematics, semiosis is not only necessary for communication, but it is also indispensable for the activity itself. Also, Duval claims, mathematical objects are not accessible by perception: they need semiotic registers, semiotic systems.

Duval’s main result is of particular importance for the mathematical activity, and hence for its learning and teaching: “There is no noesis without semiosis, it is semiosis that determines the possibility of exercising noesis” [3]. (Duval, 1995, p. 4)

[3] “Il n'y a pas de noésis sans sémiosis, c'est la sémiosis qui détermine les conditions de possibilité d'exercice de la noesis”.

But, of course, a representation in Mathematics does not play the same role than representations do in every other realm. In fact, as we now, mathematical activity is characterized for always being proceeding from one step the following, and symbolical and other properly mathematical representation systems allow to do this progressing *in* them – they are not just immobile pictures.

With a similar view, Duval turns his attention to two other cognitive activities inherent to any representation – other than being a perceptible marc or set of marcs identifiable as depiction of something in a determined system: these are *to transform* the representations according with the rules proper to the system (so that other representations may be obtained in the system), and *to convert* the representations produced in a system into another (which may help making explicit other meanings of what is represented).

Already from an immediate pedagogical perspective, the aforementioned activities are, of course, quite important: for a big percentage of students, comprehension of contents is limited to the representation used. More importantly, Duval goes on to saying that there is no comprehension in Mathematics if distinction is not made between the object and its representation.

Now, common teaching of Mathematics shows the persistence of representations that do not come from a unique semiotic system. Duval claims that, in fact, in Mathematics, at least two semiotic systems are required to produce the representation of an object, a situation or a process. In general terms, a plurality of semiotic systems allows diversifying the representations of one object, which augments cognitive capacities of the person and hence his/her mental representations; but, as it is well known in Mathematics, confusing the representations will always imply loss of comprehension.

The passing of one system to another requires simultaneous movilization from one system to another, that is often far from evident to students[4]. That passage is spontaneous when the systems are *congruent*, that is: when there is semantic correspondence between the significant units that constitute them, conversion of a signficant unit in the source representation in a unique significant unit in the target representation, and equal possible order of aprehension of the significant units in both representations.

Anyhow, there is many a time when there is no congruence between different representations of the same object. Thus, for improving learning, there is a need of doing specific work in a diversity of representation systems, of making use of the properties of each one and also of making comparisons by puting the system in correspondance and examine their mutual 'translation'. If the teacher just centers in contents, coordination of systems is hardly achieved; on the other hand, when learning in the diversity described, students show changes in initiative and performance, and there is a raise of interest on the task.

If we go deeper, Duval claims, there is no noesis without a plurality – albeit potential – of semiotic systems and their coordination by the individual.

[4] Linear Algebra is but a quite illuminating example of this, and of the pedagogical difficulties that this conveys to students, especially if the instructor is not aware of them.

From history to abstract mathematics

It is well know that former mathematicians were also philosophers (at least since Thales, say). They would ponder about the nature (essence, substance, reality) of mathematical objects.

We don't claim that this is not the case today: realism, e. g, is well alive in the mathematical community. Anyhow, perspective has changed, and from a quite specific mathematical point of view, that we briefly describe in the following.

As algebraists and other mathematicians have shown, even if you construct it, you don't really work with an object: you always end up with an (in a general way) equivalent class of isomorphic objects, each one of which is replaceable by anyone else of the class. Hence, you always work with a 'presentation' of 'the' object. This shows that the problem of representation is at the core of Mathematics itself. Let's take a closer look at this:

Suppose that you work with Integers number. You may start (following Brahmagupta, 628) considering that you have a business, and that at the end of a day you want to know how much came in and how much came out: the day ended up in fortune (his word) of debt. This in turn will 'naturally' take you to consider pairs of natural numbers: (3,5) is a way of representing that that day you sold 3 and spent 5 in the process. Now, since selling 3 and spending 5 is, as a result, equivalent to selling 6 and spending 8, you go on to consider equivalence classes of pairs: (3,5) and (6,8) are representative of a class of results of a business day, one that may be represented by a distinguished member, (0,2), that allows to have a better picture of the day. Also, if you want to recapitulate what happened in a lapse of some days, you will have to add up the fortunes and debts of each one of them: (3,5) + (14,6) = (17,11), equivalent to (6,0) shows that you overcame your initial bad luck.

As it happens, because both of practical and intellectual requirements, you will develop that system of yours, and look for properties: commutativity and the like.

Later on, your drive will take you to consider the properties themselves, from a broader perspective: The system is a ring – there are others. You have unique factorization – there are other such structures. You have long division – still, there are others.

The purely mathematical question is, then, what is the fundamental feature of the Integer numbers, what are the properties that characterize their system, so that no other similar ring would satisfy them.

Now, if you come about an answer to that question, you will not be able to avoid the following: if you construct another object with the very same properties, you will have to accept that it will be 'the same' one: as in the case of pairs, the whole system that you are working with is but a representative of an equivalence class – perhaps you chose a convenient one. This reference to equivalent classes may not seem entirely natural, but, in the end, when you are working, you just look for the properties, not for the nature of the elements.

The aforementioned is somehow easier to grasp in the case of Real numbers: you may construct them[5] in several ways: taking into account just the customary ways of building them from Natural numbers, you would end up with 2x1x1x5 = 10 equivalent objects[6]. Each one of them is characterized for being a complete totally ordered field[7].

Summing up: (in a quite specific way) one always works not really with objects but with equivalent classes of objects; it is the properties that count. (Thinking of adding the real numbers 2 and 3, say, to go back to the elements themselves[8]) would be really cumbersome and useless.

Mathematical representation for Mathematics

It is clear that what we were discussing has a lot to do with Duval's *congruence* of semiotic systems. But let's go on searching for analogous of his *transformation* and *conversion* in the mathematical realm.

In any theory, other than the objects themselves, you always have specific '*arrows'* between them: if you are working with sets, there is the functions; in Linear Algebra, you have linear transformations between vector spaces; then, continuous functions, differentiable functions, group homomorphisms, etc. These arrows, in their own way, 'respect' the object's structure: some specific procedures that you can do with elements in the source object can be carried out with the 'images' of these elements in the target object; this is the general notion of a (homo)morphism.

There are this kind of arrows from the object to itself, and also from the object to other objects. There are arrows of these that are reversible (as in bijective functions), let's call them isomorphisms. Thus, Duval's psychological concepts are genuinely close to the mathematical way of thinking.

Moreover, sometimes you want to know about a property of some object and find it easier to get by taking advantage of that 'respecting' of specific arrows. This may be explicitely stated or not: for instance, when one thinks about an *n*-dimensional vector space, one just thinks of the n-tuples (of real numbers, say); the cyclic group on *n* elements we have in mind is always the same[9]. In both cases we have used an isomorphism – for instance, between any *n*-dimensional space and the corresponding *n*-tuples. The heart of the matter is that you not only 'represent', one by one, the elements of an object in another object, but that you can work in the second one as if it were the first: in psychological terms, a pair of congruent registers.

[5] A pedagogical matter, Hilbert would say.
[6] 30 for Complex numbers
[7] A very precise expression: notice that the Complex numbers system is a field, is totally ordered, and is complete.
[8] For instance, to take the corresponding Cauchy sequences of Rational numbers, or the corresponding Dedekind cuts.
[9] In considering the elementary cyclic structure of, say, the days of the week, or of a watch, such groups are present, although in an implicit way.

Interestingly enough, when mathematicians study, for example, groups in general – abstract ones, say – they try to *represent* (their word) them as permutation groups or as matrix groups (or linear groups), which are better known[10].

In fact, one may see that it is arrows that receive the more attention: say, continuous functions, or differentiable functions, or linear transformations. This switch from objects to arrows is very deep, and, of course, it starts as a tool for studying the objects.

Now, sometimes such a resource takes you farther than expected, and you end up in another kind of structure.

One of the firsts of such situation occurred to topologists some time ago: They knew that they could continuously deform, say, a bowl into a plate; but they didn't know whether that could be done between a tire and a ball. This was a difficult topic, one that could not be solved inside the theory of topological spaces. Instead, they had to build aspecific and '*fat*' arrow (in fact, a *functor*) from topological spaces to groups, an arrow that converted topological spaces into groups, and continuous maps into group homomorphisms. The properties of this functor implied that you could not continuously deform a tire into a ball, because that would imply certain conditions in the target objects that couldn't occur.

This in turn opened the door to think, in a quite broad way, the posibility of *representing* objects of one class in objects of another sort – arrows between them included.

The higly abstract theory that deals with this, the Theory of Categories, allows to take the idea of representation much farther, as suggested before: it is not only that the elements of an object (say, a vector space) may be in an appropriate correspondence with elements of another of the same kind (so that to be able to look into the target for answers for the source), but also that all the objects of some kind (say, all topological spaces) may be put in a convenient correspondence with the objects of another kind (all of the groups, e.g.) so that making possible to get information about the sources by looking at the targets.

The problem has turned into a purely mathematical one, especially, perhaps, since the 1960's, when some mathematicians[11] succeeded in *representing* categories as categories of familiar concrete structures and their arrows[12] – a problem that is a remote analogous of thinking, for instance, groups as matrix groups.

As it may be suspected, these last pursuits not only aim at purely descriptive results, but also at, so to say, uncovering unknown facts about these objects – and about the objects inside them.

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[10] The general idea being older, this way of studying started formally around 1890, with G. F. Frobenius.
[11] Notably, Czech algebraists.
[12] More precisely, the question is to embed arbitrary *small* categories in a *faithful* and *full* way – that is, in a reversible way – in more familiar categories;

Conclusion

Given the broadness of the subject and the space constraint, I use the word conclusion just as a means to imply that this paper is ending. Nevertheless, something may be added, even if only as a faint clue for further reflection.

A far shot from direct pedagogical concerns, the last part of our commentary still shows several important points in connection with the idea of representing mathematical objects:

Firstly, representing is a natural mathematical idea, one that one may encounter, for example, in any higher level class of Mathematics: even if the instructor has no pedagogical intentions, the theory that he/she and his/her students deal with has, in its own way, a natural idea of representation, one that allows better calculations, comprehension, make inquires.

Secondly, if you try to look deeper in the nature of mathematical objects, you will find that, as it were, they will avoid your direct grasping of them, and you will surely have to accept that you don't really work with elements: mathematics nature is more abstract, as we now, and we have to admit that we always work with accessible representations of elusive objects.

The bottom line is that representing is intrinsic to mathematical activity, and so, whenever we try to represent an object, we are not taking some extra-mathematical action, but, on the contrary, even if that representation does not belong to a mathematical theory, the endeavor of representing objects and working with the representations has the right to be considered as of mathematical nature.

At any level, it is not possible to learn Mathematics without some kind of representation, and working with more than one of these and making connections between them improve the outcomes. After all, this is the way in which possibly every person who have dealt with Mathematics has been doing, all over History.

References

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