APEC Lecture Paper

The Future of Mathematics Teaching in Japan

Developing Lesson to Captivate Children

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Introduction

Over its long history, mathematics education in elementary schools in Japan has risen to a high level in terms of content and methodology. However, participation in various lesson study groups reveals that many aspects that are in need of improvement.

In this regard, I would like to present a number of personal views about the future of mathematics teaching in Japan that stem from my own experience in education.

My perspective focuses on the aim of creating courses that will captivate children.

The following themes will be discussed below.

1. Honing teaching skills
2. Using the open-ended approach to bring about diversity
3. Hands-on math
4. Example of a class that captivates children (Shapes)
5. Example of a class that captivates children (Numbers and Arithmetic)

1. Honing teaching skills

(1) Focusing on class study groups

Worldwide attention is currently being focused on Japanese mathematics education and, more specifically, on lesson study groups.

This interest comes from the fact that teachers in Japan continue to study even after they become teachers.

Practicing teachers open up classes to large numbers of other teachers for observation.

Participating teachers thus realize that there are teaching methods that differ from their own and learn that materials other than textbooks can be used in lessons.

Participating teachers find that what students say and how they think are interesting, and this experience changes their own outlook on teaching. They then strive to create more captivating classes of their own the next day.

Lesson study groups generate such proactive attitudes from participating teachers.

This type of study group has been commonly conducted in Japan for a long time.

However, this method is currently receiving attention from teachers in many advanced nations, such as the United States, and there have increased interest been from developing countries about Japan’s mathematics teaching as well.

Why is that?

One reason is that this is a novel method for the other countries.

Another reason is that there is a need to increase the ability of teachers to boost the scholastic aptitude of students.

Moreover, the type of lessons that evokes the most interest is not the knowledge-transmission type but the creative type, where students are encouraged to make discoveries on their own.

At the beginning of the Meiji era, Japan, too, was a developing country.

At that time, the focus of education was on imparting as much knowledge as possible to children and lessons were of the transmission type.

Students of that period would only recall what was taught by rote learning.

They were assessed in terms of memory and their ability to absorb information.

However, in our current society, it is desirable to design lessons in which children make discoveries by working together and consulting with each other.

Classes must hone children’s ability to gain, use, and transmit knowledge. Such classes will certainly have a strong impact on children and give them the ability to use the knowledge that they have acquired.

(2) In the context of teaching the multiplication table

Let us consider the case of a class in which the multiplication table is being taught. With autumn arrives in Japan, second grade students everywhere eagerly recite the multiplication table in front of their teachers.

What kind of guidance do teachers give when their students learn the multiplication table?

The following sequence of steps is often used when teaching the multiplication table: (1) the meaning of multiplication is given, (2) the structure of the multiplication table is explained, (3) students are made to recite the multiplication table, and (4) the multiplication table is then utilized.

However, students should not just memorize the multiplication table as if it were some kind of song. They should be given activities in which they can discover the beautiful alignment of numbers in the various rows of answers that make up the multiplication table.

For example, the sum of the ones digit and the tens digit of any product in the multiplication table for nine is always equal to 9, for instance 9 x 7 = 63, 6 + 3 = 9.

Moreover, if you take any product from the first half of the row in the multiplication table for nine, and add it to the corresponding product number from the opposite end of the second half, the result will be 90. Take for examples, 9 x 1 = 9 and 9 x 9 = 81, and 9 + 81 = 90 and similarly 9 x 2 = 18 and 9 x 8 = 72, and 18 + 72 = 90.

As children discover such rules, they are bound to go beyond simply reciting the multiplication table and acquire a richer sense of numbers.

Children learn many things such as how to make discoveries and how to look at numbers.

The teacher’s role is to find ways to promote such learning and this skill is what is needed for mathematics education in the future.

2. Using the open-ended approach to bring about diversity

(1) Open-ended problems

Solving problems is the norm in mathematics lessons because learning how to solve problems is considered important.

However, very often even if there are various ways to solve the same problem, there is only one correct answer. Some people find this approach refreshing, and they like it. This characteristic is well conveyed by the expression “cut-and-dried like mathematics”.

On the other hand, others hate the fact that there is only one non-negotiable answer.

What happens when there are various possible answers instead of one single correct answer?

If one asks, “What is 3 + 4?” a kid will shout out “7!” and that will be it.

However, if one asks, “7 is the sum of what and what?” there are many possible answers. Of course, there will be kids who answer, “3 + 4.” Others might give the opposite reply that is, “4 + 3.” Some other children will say, “1 + 6,” while still others will say, “0 + 7.” Children from higher grades who say that 1.6 + 5.4 are also right.

In this way, the idea that fractions may work as well starts to spread very quickly.

The introduction of such problems involving several correct answers in class allows many children to become more involved in the lesson.

Because each of these various answers is correct, the number of children who are evaluated positively will increase.

Problems that have several correct answers, or ends, are called *open-ended problems*, and classes that use such problems are said to be using the *open-ended approach*.

(2) Types of Open-ended problems

The problem given in the above example is an inversely structured problem because of the way the conditions and answers are positioned.

There are many other types of open-ended problems. Problems that allow students to discover rule is one type. An example of this using the multiplication table has been discussed.

Classification problems where students distinguish numbers from various viewpoints are another type of problems.

There are also **numeric-conversion** problems, where one has to think about using a ranking method when teams compete in marathons. The score of either the top performer in a team or the team's average can be used as the group's score.

Furthermore, insufficient-condition problems can sometimes be considered open-ended problems. A common problem goes something like this: the number 36 is formed by *x* number of 10s and *x* number of 1s. The question posed by this problem requires an answer that contains a combination of numbers, so "3 times 10 and 6 times 1" is given. However, thinking carefully reveals that the answer given above is not the only one. Other answers, such as "2 times 10 and 16 times 1" works too. Thinking along these directions is required for calculations that require "Borrow over" such as "36 - 19," because calculating the 1's digits requires thinking about borrowing from the 10's digit. Thus, students become aware that although this particular answer is correct, there are still other possible answers which are equally correct.

By using the techniques described above, various open-ended problems can be created. Future lessons can thus be improved by incorporating such problems with the aim of encouraging diversity in thinking and greater flexibility. In mathematics classes, it is possible to shift away from the notion that only conventional things should be studied.

In this way, children will surely gain experience in seeking answers on their own, and acquire the ability to view things around them from a broader perspective, thus demonstrating diverse problem solving skills not only within the realm of mathematics but also when dealing with the various real-life events they face.

3. Hands-on math

(1) What is hands-on math?

Hands-on math simply means practical and experiential mathematical activities.

Hands-on means *using one's hands*. *Math* is short for *mathematics*.

By nature, mathematics is abstract and logical. However, teaching children mathematics in elementary school needs more concrete images. Children are not capable of studying abstract subjects from the very beginning. It is important to learn by manipulating things with one's hands, thereby gaining a solid understanding of what is being taught.

Hands-on math involves a real-world approach to teaching mathematics.

(2) Teaching tools

Children manipulate things as they count them (1, 2, 3,...), touching them one by one concretely and using their fingers to count can be called hands-on math.

Moreover, in the study of shapes, particularly three-dimensional shapes, the observation of tangible objects or models and the creation of three-dimensional objects out of cardboard are also considered hands-on mathematics.

In the old days, there have been various tools to aid in learning. These are called learning aid tools.

A careful analysis of these tools indicates first how the tools help explain difficult concepts by providing an understanding through observation.

Number lines, marbles, and blocks are examples of such tools. Problems involving addition, such as 8 + 6, can be verified clearly using blocks. This category of tools is called *explanation tools*.

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Rulers and scales to measure lengths and weights of things, compasses to draw circles, set squares to draw shapes, and many other tools are indispensable in the world of mathematics. This category of tools is called *instruments*.

In addition to what has been mentioned above, there are tools with which children make new discoveries in the process of manipulating objects with their hands.

Some examples are *intelligent boards,* such as tangrams and pattern blocks that can be used to create tiling patterns.

Teaching tools that promote such discoveries by children themselves may be called by the coined term *thinking tools*.

Experiential mathematics that seeks to make mathematics into something concrete (concretized mathematics) through the use of these various tools is what we call *hands-on math*.

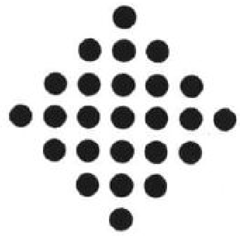
Originally, the word *hands-on* was used in museum education and environmental education. The term *hands-on science* is used in science education to describe learning by touching the natural tangible world through experimentation, observation, etc.

(3) Counting the number of spots

The following is an introduction to a concrete aspect of mathematics.

Counting is the most basic mathematic activity.

If children have physical items at their disposal during such an activity, various methods to assist mathematical thinking can be devised.



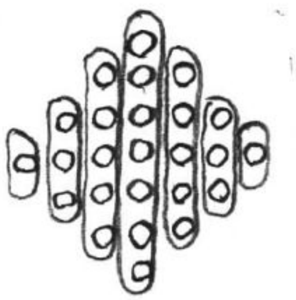
One example of a such material is the tactile directional tiles for the blind, which are a common sight in cities. These yellow tiles can be found in front of traffic lights and on train platforms. The bumps on the tiles are arranged in an interesting manner.

Rearranged and given to children, such tiles can be used in numerous ways for counting. Each child can be given one tile and, by applying colors to it, create their own unique way of counting.

Take the following mathematical expression, for example. A dynamic class discussion can arise through the interpretation of this expression and used to express a way of counting.

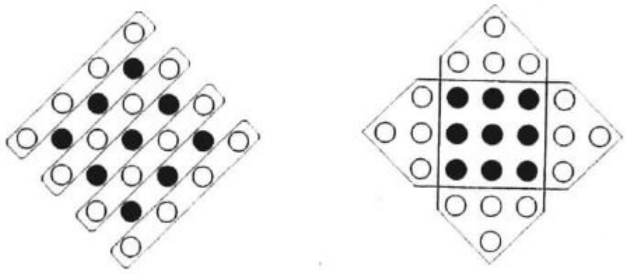
① $1+3+5+7+5+3+1=25$

This method consists of adding up the numbers (represented by the bumps on the tile) line by line. This can be done either horizontally or vertically.



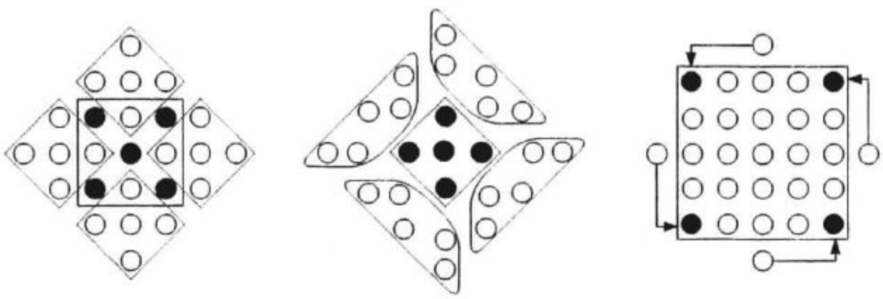
② $4\times4+3\times3=25$

Different children will look at this expression in different ways. For example, some children will see the bumps as dumplings on a stick, with sticks of four dumplings alternating with sticks of three dumplings. Others will see three sticks of three dumplings in the middle of the tile, forming a square, with the remaining dumplings (not on sticks) forming triangles at the corners (one dumpling at the apex of the triangle and the remaining three forming the base).



③ 5×5=25

Some children will see the five spots of a die in five locations on the tile, while others will move the spots from the four corners into a new location, turning the tile into a 5 x 5 square.



(4) Cognitive process

Such activities involve the following type of cognitive process.

1. The child encounters an object. Here, the child expresses his/her eagerness by saying things, such as “That’s interesting” or “I want to try that.”
2. The child takes in the object through his/her five senses.
3. The child perceives the object that was taken in.
4. The child takes action on the object. Trial and error takes place.
5. A new knowledge (**認識**) arises.

This series of activities is considered to be the core of hands-on math.

(5) Benefits of hands-on math

An analysis of hands-on math activities reveals various benefits, some of which are listed below:

1. Children go beyond just thinking with their heads and construct the unexpected situations that enable them to think (**構成＝**constructions).
2. Images are transmitted, and actual verification is carried out (Verification).
3. The boundaries of mathematics are expanded (Development).
4. Children gain creativity (Creativity).
5. Memory is reinforced through experience (Body intelligence).
6. The learning experience is retained as a tactile work (**作品化＝**making into a final product that can be shown).
7. Children work with their classmates (Collaboration).
8. Children become interested in the activity itself (Interest and Curiosity)

The above benefits are concerned with children. From the teacher's perspective, a number of benefits are listed as follows:

1. Teachers change the way they look at teaching materials (Teaching materials view).
2. Teachers develop the ability to discover teaching aids (Teaching aids view)
3. Teachers enjoys and cultivates their minds to create their own teaching aids (**作業＝**Benefit of action of developing something).
4. Teachers aim for lessons that are not limited to textbooks and notes (Teaching Philosophy).
5. Teachers change the way they look at children (Evaluation view).

4. Example of a class that captivates children (Shapes)

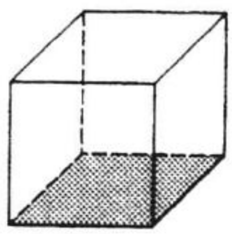
(1) Class using two three-dimensional objects

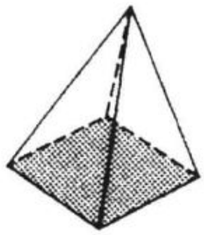
One object has a box-like shape that is a familiar object to children. The box is similar to those in which caramel is sold, and it resembles a die. In mathematical terms it is a called a *cube*.

However, in this case, a cube without a lid is used.

Specifically, this cube comprises five equal-sized squares: one base and four sides.

The other object is three-dimensional and shaped like a pyramid, more specifically, a square pyramid. The base is a square and its sides are isosceles triangles.





Before the lesson, create a number of three-dimensional objects as described above.

An alternative is to let children who have studied squares and isosceles triangles to create these objects themselves.

The first question to ask is "Which of these objects, when taken apart, has the largest number of different shapes?" (3-D Representation Layout)

In actual practice, the teacher will prompt students in the following way.

"Have you ever taken apart boxes that look like this?"

"What kinds of shapes do you think they will have after you take them apart?"

"The first shape that comes to mind is that of a cross, right?"

"This shape consists of five connected squares."

"However, that is not the only shape you will get after taking the box apart."

"Actually, when you try it yourself, you will get many different shapes."

"The shape you get after taking the box apart is called its 3-D Representation Layout.'"

It is essential for elementary school students to learn by manipulating real objects with their hands.

Even though mathematics is an abstract and logical subject, this does not mean that it has to be approached only using notebooks and pencils.

It is important to conjure up concrete images in one's head.

Mathematics that is studied using such ways both in terms of work and experience is called hands-on math.

One method that can be used to arouse the interest of children is to divide the class into two groups: Group A consisting of students who think that the cube will produce the largest number of different shapes when the cube is taken apart and Group B consisting of students who think that the pyramid will produce more shapes. Then have each group take apart their object, reporting to each other the shapes that emerge and they recognize and wait to observe which group lasts longer as they create more shapes.

(2) Starting game-like activities

Have one representative from the two groups to step forward and take apart the object of his/her team and report on the shapes.

The activity can become more interesting by asking each representative to announce the shapes by assigning a nickname to each of the 3-D Representation Layout. On the surface, this simply appears to be a way of making the activity more enjoyable but by assigning nicknames to the 3-D Representation Layout, it will help students' communication later on.

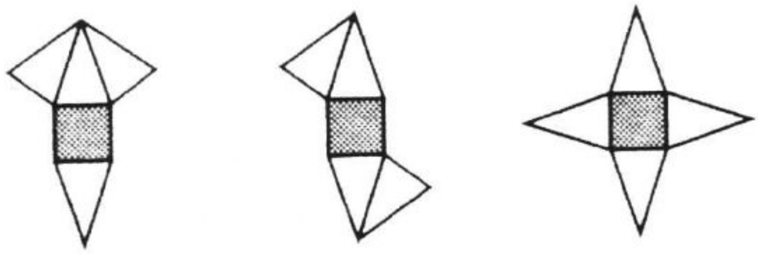
Because this is a group learning activity, the teacher should aim to have each and every student participate fully.

Here, "game-like activities" were introduced.

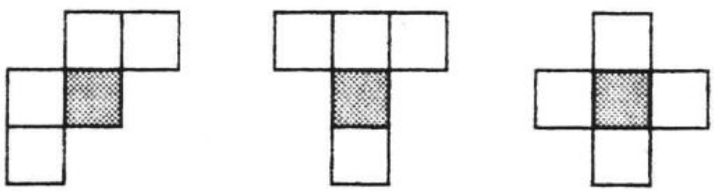
A game setting in which each team vies to win can be established. As the students strived their best to win, they will acquire a mathematical way of thinking.

During the first presentation, the three shapes shown below are produced.

Cube



Pyramid



By the time the presentation reaches this stage, you will hear a voice saying, "Oh!"

The teacher asks, "What?"

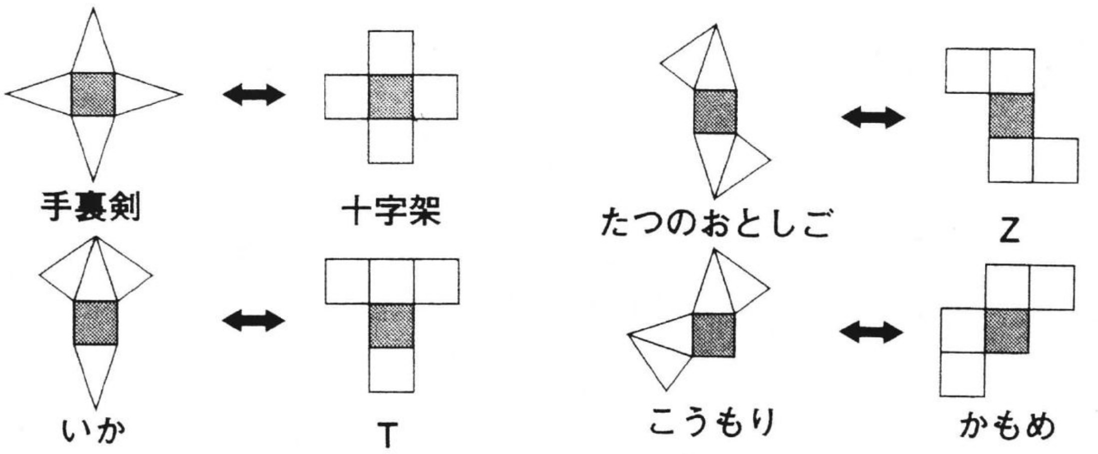
The student will then say something like, "I just noticed something interesting."

When students say such things, the teacher is eager to hear them out.

"Tell us," the teacher prompts. The student will then say, "The shape we nicknamed the 'cross' resembles the 'ninja throwing star,' and the 'T' resembles the 'squid.'"

Meanwhile, everybody else is listening intently. Some of the children exclaim, "She's right!" The children group similar shapes together.

Some of the children who noticed this will then find other similar shapes and group two additional ones together.



The teacher must be careful not to miss these small utterances of the children.

Children who are thinking hard will voice their true feelings.

Therefore, it is important to listen to what they have to say.

Similarly, the teacher must be careful not to miss the small actions of the children as they work during the lesson.

In the process, the teacher will discover students who are unable to express their discoveries verbally but have a good grasp of the material.

The teacher should strive to call out such students to present their result in front of the other students. The teacher should endeavor to praise the student's findings as much as possible.

In other words, the aim is to break the old routine of "neither seeing nor listening to nor speaking to students" and instead deal with the children in the spirit of "always seeing, listening to, and speaking to students."

(3) Things that could not be seen come into view

Once the children are capable of grouping similar shapes together, there will still be some that will not be able to match.

For both the cube and pyramid, there will be one item left.

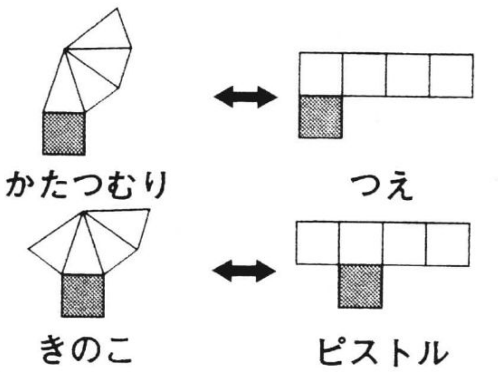
At that point, one of the students exclaims, "Maybe there is a shape that matches that leftover one."

Then all the other students say, “She may be right!”

The children immediately begin thinking about whether there is such a shape.

This time, the children take apart the three-dimensional objects mentally instead of physically.

Then, the teacher asked a child go to the blackboard and draw the corresponding shape.



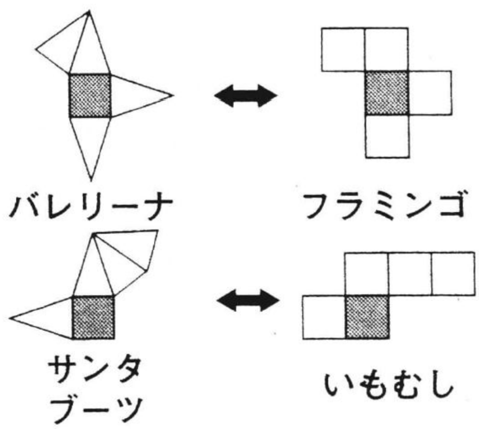
Certainly, it looks as if the grouping is possible.

Then, the students physically take the object apart. If the 3-D Representation Layout could be drawn, then their speculation was correct.

Both teams become entirely absorbed in the task of taking apart their objects.

A student loudly announces, “We did it!”

Both teams successfully created the 3-D Representation Layout that work. Furthermore, even with the 3-D Representation Layouts that neither side had come up with until now, a total of as much as eight were created.



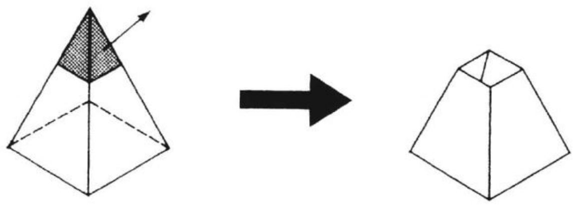
(4) Thinking about *why*?

The teacher continues by saying, "Both teams have eight items, so it's a tie. Why are the 3-D Representation Layouts the same?"

All the students start thinking again.

A number of students raise their hands and attempt with an explanation.

"These two three-dimensional objects look the same. If you cut the top of the pyramid and remove it, it becomes a cube with a slightly tapered top. That's why the 3-D Representation Layouts are the same."

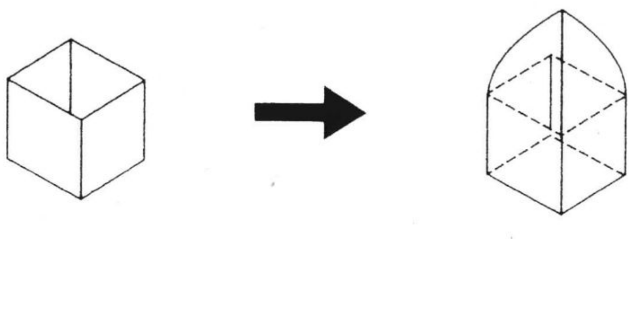


This is a rather good idea. If the pyramid looks like a frustum of a square-based pyramid, its structure can be considered the same as that of a cube consisting of one square base with the four sides consisting of a trapezium.

Another idea is being bring forward too.

"Think of the cube as being made of soft rubber. Then by holding it from the top, press and raise the sides, it will take on the shape of a pyramid for sure," says one student. All the others nod their head in agreement.

This thinking is what is mathematically termed as a topological way of thinking.



At first, the students were asked which of the 3-D Representation Layouts produce the largest number of shapes as part of the game but as they progress through the process of exploring this question, their ability to view 3-D Representation Layouts changed and they discovered new things.

From the ability to perceive similarities, the students found commonalities between shapes that appeared different at first glance.

They were thus able to gain a “topological eye” to complement their eye for shapes.

In the world of mathematics, developing the ability to think about *why* a phenomenon exists is an important task when aiming to raise students’ ability.

This is included in developing the ability to think logically.

Opportunities in this regard abound in learning. Teachers should strive to value such opportunities and make the most out of them so that children will be able to explain things by themselves.

Rather than the knowledge-transmission type of lessons in which the teacher teaches by rote learning, the creative type of lessons in which children are called upon to create things on their own are sought after.

5. Example of a class that captivates children (Numbers and Arithmetic)

(1) Setting the stage for *why*

Children develop a sense of wonderment through the discovery of problematic aspects within problems, and ideally classes should be designed to allow children to pursue these *why* questions.

Here, teaching materials were developed to make children look at two multiplication expressions and marvel at the fact that the answers are the same. They ask why, looking carefully at these expressions and developing an answer by finding the relationships between the numbers and transform them.

Specifically, the following two expressions were presented to the children.

One is the multiplication of the number 4 twelve times and the other is the multiplication of the number 8 eight times.

(A) 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4

(B) 8 x 8 x 8 x 8 x 8 x 8 x 8 x 8

The children were then asked which expression yields the largest result.

However, the answer is not that easy to find even if the calculation is done on paper. Hence the children are allowed to use a calculator to obtain the answer. At this time, they can use the constant calculation function.

The function consists of pressing

4 x x = =…

8 x x = =…

When the correct answers are displayed on the calculator, the two answers are exactly the same.

The answer to both problems is 16777216.

At this time, the question of why the answers are equal surfaces in the minds of the children.

The lesson hence revolves around the time spent by the children trying to answer this question and discussing this problem among them.

The aim is to allow the interaction of the children as a contribution to a more rewarding class. In other words, the teacher should steer the discussion toward mathematical thinking.

For example, the teacher should try to have the children gain an understanding of the numbers 4 and 8, such as realizing that 4 x 4 x 4 = 64 and 8 x 8 = 64 are equal or reduce the numbers like 4 = 2 x 2 and 8 = 2 x 2 x 2.

The structure of this problem uses the fact that $4^{12} = 8^{8}$, in other words $4^{12} = (2^2)^{12}$, $8^8 = (2^3)^8$.

(2) Starting the class

"I will now write two mathematical expressions on the blackboard. As soon as I am done, I will ask you which one has the largest answer. Intuitive guessing is allowed, so raise your hand for the expression you think has the largest answer." The teacher then writes the following two expressions on the blackboard in silence.

The children watch intently as the teacher writes the expressions on the blackboard. They are thinking about the answer to the two addition problems.

(A) 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 +4 + 4 + 4 + 4

(B) 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8

After writing the expressions on the blackboard, the teacher says to the class, "Well then, I'll ask you. First, who thinks that A has the largest answer?"

Very few students raise their hands.

The teacher continues with, "Who thinks that B has the largest answer?"

This time, many students raise their hands.

The majority thinks that B has the largest answer.

The teacher then asks the students, "Why do you think so?"

The students are likely to give many different answers.

The teacher asks one of the students, T, who raised her hand.

"I calculated the answer. I mean its simple addition."

The teacher then asks, "Well then, how did you calculate the answer?" upon which T replies that she used multiplication.

When asked to write out the expression, T writes the following.

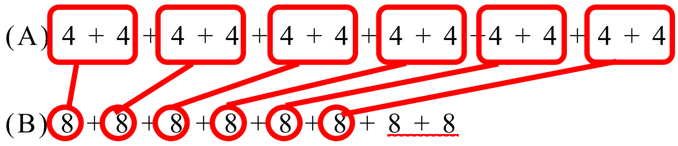
(A) 4 x 12 = 48

(B) 8 x 8 = 64

Most children agree that this is correct.

The teacher asks further, “Any other reason?”

Another student, Y, gives a different reason. Walking over to the blackboard, he attempts to explain by drawing lines between the expressions.



This is quite something. After drawing the lines, I planned on having the other children explain them.

“Can anyone explain the meaning of these lines?”

Asking in this manner, many children understand and many put up their hands.

A third student, N, gives the following explanation.

“Since 4 + 4 and 8 are the same, by pairing two 4 with one 8, we are left with two extra 8 in Expression B. Therefore, B has the larger answer and we know that it is greater by 16.”

This approach also consists of matching equivalent parts between the two expressions. This too is quite an ingenious method. The teacher lavishes praise on the student.

Still other students offer different reasons.

The following reason given by H is a transformation of expression.

“In calculating B, 8 = 4 + 4. So, transforming the expression, we get sixteen 4’s. Because A and B have twelve and sixteen 4’s respectively, the expression with sixteen 4’s has the largest answer.”

Upon hearing this, M says, “In this case, in calculating A, 4 + 4 = 8, so changing the expression accordingly, we get six 8’s. Therefore, Expression A, having six 8’s and Expression B has eight 8’s, we can see that Expression A has the smallest answer.”

This too is a good explanation.

Let us write the expressions in a manner that is easy to understand.

(A) (4 + 4) + (4 + 4) + (4 + 4) + (4 + 4) + (4 + 4) + (4 + 4)
= 8 + 8 + 8 + 8 + 8 + 8

(B) 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8
= (4 + 4) + (4 + 4) + (4 + 4) + (4 + 4)

This comes in very handy for the next problem.

(3) Changing Plus signs to Times signs

Now, this is the main problem of the lesson.

"Okay, I will now change the calculation a little. Which mathematical expression has the largest answer this time?"

While saying this, the teacher replaces all the plus signs to times signs with red chalk.

(A) 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4

(B) 8 x 8 x 8 x 8 x 8 x 8 x 8 x 8

This time around, the students are all shocked and cannot perform the calculation swiftly.

"Well then, which one do you *think* has the largest result? Again, intuitive guessing is allowed, so please raise your hands."

This time also, Expression B gets the majority of raised hands. However, Expression A gets a number of raised hands as well.

The teacher asks, "Why do you think that B is larger? There are also those of you who think that A is larger. Why?"

No clear answers are provided as response. Some of the answers that some students gave include the following.

"Well, I thought that since B had the largest result when we added all the numbers, it would be even larger if we multiplied them."

"I thought it would be A this time because it was B the previous time."

The teacher then proposes, "Let's go ahead and actually calculate the answers."

(4) Calculating the answers

The children start performing the calculations in their notebooks, but because this is multiplication, they quickly realize that the calculations are really difficult.

"That calculation is really hard."

"The numbers keep getting larger, so I wind up making mistakes."

"If only we could use a calculator..."

When such comments start appearing, the teacher says, "OK then, let's use calculator."

At this point, it is fine to just multiply the numbers, but the teacher decides to teach the students an interesting way to do the calculation.

This method is called constant calculation.

"The calculator has a function called constant operation. For example, by pressing the '+' key twice, as in '10++', a series of 10's can be added up infinitely. Pressing '10++5=' causes 15 to be displayed, and then pressing '2=' causes 12 to be displayed. If '100=' is pressed next, 110 is displayed.

This can be used for this question.

First, press 4, then press x twice and lastly press = to multiply 4 infinitely."

"Try and press 4xx==."

"Oh, I got 64!"

"Right! Since the first input is 4, inputting = twice results in 4 being multiplied three times, that is, 4 x 4 x 4= 64."

"OK, class, please try this."

All the students do as they are told in earnest.

After a while,

(A) 4 x x = = … = 16777216

(A) 8 x x = = … = 16777216

"Oh! The answers are the same!"

"Why?"

Similar utterances can be heard throughout the classroom.

(5) Why are the answers the same?

Because the two expressions turn out to have the same answer, the question of *why* is left in everyone's mind.

The class is developed around this discovery.

"You have all found out through your calculations that the answers are the same. I think it would be ingenious if we could say that without having to perform these calculations."

"Oh, the 64 that we got a little while back…"

The teacher does not miss such mutterings.

"You just noticed something interesting!"

Upon hearing this, another child exclaims, "I got it!"

The teacher asks that child to elaborate.

The child goes to the front of the class and encloses parts of the expressions.

These are "4 x 4 x 4" and "8 x 8."

The teacher then asks the class, "Seeing this, do you know what he is trying to do?"

Many children raise their hands.

"Since 4 x 4 x 4 = 64 and 8 x 8 = 64, Expression A becomes

4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4
= 64 x 64 x 64 x 64

and Expression B becomes

8 x 8 x 8 x 8 x 8 x 8 x 8 x 8
= 64 x 64 x 64 x 64

and we can see that these results are identical.

This is a very good answer, and everybody agreed.

The teacher then goes on to ask, “Are there any other explanations?”

At that point, some children make the connection with the addition problem.

“The fact that the answers for the two expressions are the same means that by changing the expressions we may obtain the same expression.”

“Let’s try and change the expressions to make them identical.”

“8 = 4 x 2!”

“So the calculation for B becomes

8 x 8 x 8 x 8 x 8 x 8 x 8 x 8
= (4 x 2) x (2 x 4) x (4 x 2) x (2 x 4) x (4 x 2) x (2 x 4)
= (4 x 4 x 4) x (4 x 4 x 4) x (4 x 4 x 4) x (4 x 4 x 4)

which is the same as A.”

A child whispers, “What about converting everything into 2s?” after seeing the above explanation. The teacher develops on this thought, saying, “Someone just proposed changing everything into 2’s. What do you think?”

“Let’s see, 4 = 2 x 2.”

“Can 8 be changed into all 2’s?”

“8 is 2 x 2 x 2.”

“Well then, let’s use this to change the expressions.”

The teacher asks the child who gave this explanation to explain on the blackboard.

(A) 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4 x 4
= 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2
x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2

(B) 8 x 8 x 8 x 8 x 8 x 8 x 8 x 8
= 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2
x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2

As a result, both expressions have twenty-four 2’s and the answers are the same.

Another alternative suggested by some children are a simple pairing up of numbers from one expression to the other expression.

"The numbers in both expressions can be paired up until the eighth 4, at which point the product of Expression B is 256 times greater than the product of Expression A. Four 4s are left over in Expression A, which is equal to 256. Therefore, both expressions are the same if you do not multiply the leftover 256."

This is a rather difficult explanation, but it can be illustrated by the following expression. The idea is to create a balance:

(8 x 8 x 8 x 8 x 8 x 8 x 8 x 8) ÷ (4 x 4 x 4 x 4 x 4 x 4 x 4 x 4) = 256...Factor by which B is greater than A

(4 x 4 x 4 x 4) = 256...Leftover from A

256 ÷ 256 = 1

The children are thus able to see that the answer can be obtained by various means involving many different things.

(6) Expansion

Further expansion of this problem allows the children to even investigate deeper into the matter and should be attempted if there is enough time.

This can be done by presenting additional problems, such as "If we continue multiplying, at what point will the answers be the same?" or "Can we achieve the same thing with other numbers?"

For example, in this problem, we had $4^{12} = 8^{8}$. So, if each number of each figure is a multiple of 3 and 2, we can know that the answers are the same for the case of $4^{15} = 8^{10}$.

By exploring deeply into such problems, children will become truly fascinated.

6. Conclusion

A number of suggestions have been given regarding the future of mathematics teaching in Japan. The class examples introduced here have all been conducted a number of times, and the fact that they can be fully applied at regular public elementary schools has been verified by a large number of teachers who have observed these classes.

It is hoped that these methods will be applied not only in Japan but also around the world.