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-- *Argumentation and Reflection* --

Mathematics Education and Reflective Experience[Note]

--- The Significance of “Unlearning” in Mathematics Education ---

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Preface

Human knowledge does not resemble a dictionary. Rather than being static, knowledge exhibits flexibility. Knowledge acquired through arithmetic/mathematics, however, is often taken in statically by young students. Knowledge acquired through arithmetic/mathematics has a high level of abstraction and exhibits exceptional universality. We can therefore assume that it will be used broadly and freely in mental activity. Once we assume mathematical knowledge, which exhibits this aforementioned universality and flexibility, is deeply connected with the mental behavior of an individual's mathematical activity, a more in-depth examination of the mental behavior of young students begs us to consider applying the resulting findings toward improving the teaching of arithmetic/mathematics.

It was J. Dewey (1859-1952) who first carefully analyzed the thought process behind acquiring new knowledge based on previously acquired knowledge, abstracted the characteristic mental action (reflective thinking), and suggested its applicability to education. Based on Dewey's theory of experience, in this paper I will cover learning activities associated with the order (size) relation of fractions and present examples of arithmetic/mathematics, such as solving Hippocrates' lunes, in an attempt to more concretely ascertain the reality of mental behavior from a cognitive standpoint. Through such observations, I would like to figure out what is necessary to improve the teaching of arithmetic/mathematics.

One improvement strategy asserts the need for emphasizing introspective behavior in young students, at least in their lessons, and in mathematics education, closely observing subjects to emphasize the mathematical relationships, universal nature, and experience to see into the generality that lies behind them, all in an effort to deepen the understanding of arithmetic/mathematics and transform this understanding into knowledge. In short, what we need is reflective experience. Education sometimes involves questioning yourself about what you learned and then relearning it. We come across the term "unlearn" in thinking about this. The circumstances behind this will be discussed at the end of this paper.

I would also like to mention one of my other impressions. The spirit of the term "unlearn" can be thought of as philosophically equivalent to reflective experience, but to put unlearning into practice in the teaching of arithmetic/mathematics, and to make it sink in, particularly at the compulsory education level, we desperately need improvements in curriculums at the level of the Course of Study (Gakusyushidoyoryo in Japanese).

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Chapter 1 Acquisition of Viable Knowledge

1-1 Flexible Understanding and Viable Knowledge: Thinking Like a Human

1-1-1 What Is Viable Knowledge?

The size of two or more fractions can be decided by reducing them to a common denominator. This technique is quite common. It is even taught extensively at elementary schools. However, when you tell university students to consider a technique other than reducing to a common denominator for the following problem, they will employ a variety of solutions. Here I will pursue my observation based on examples of such reactions.

Problem: Which is larger, 5/13 or 11/19?

In addition to students who will decide which is larger by a rough estimate based on 0.5, there are, of course, those who will convert the fractions into decimal values and compare them while others will make the numerators the same, compare them using 1 as the standard, or try to visually solve the problem by dividing two line segment diagrams equally. (The conversion of fractions into decimal values is allowed.)

A careful look into the reasoning process seen in a number of these reactions follows.

Reasoning 1

The numerator 5 of 5/13 is about half of the denominator 13, while the numerator 11 of 11/19 also is about half of the denominator 19. Therefore, both fractions are approximately 0.5 (1/2). However, with the fraction 5/13, the numerator is 5 and the denominator 13, so even if the numerator were 6, the fraction would be 6/13 which is less than 6/12 (0.5) because the denominator is larger. And since 5 is one less than 6, the fraction 5/13 must be less than 6/13. Therefore, 5/13 < 0.5.

On the other hand, since the numerator of 11/19 is 11 and the denominator 19, we would get close to 0.5 if the numerator were 9, which is about half of the denominator (by division). Since 11 is two greater than 9, 11/19 must be greater than 9/19 (0.5). Therefore, 11/19 > 0.5.

If we put these two relations together, we get 5/13 < 0.5 <11/19, and therefore, 5/13 < 11/19.

Reasoning 2

The numerator 5 of 5/13 is about half of the denominator 13, while the numerator 11 of 11/19 also is about half of the denominator 19. Therefore, both fractions are approximately 0.5 (1/2). If the fraction were 5/10, it would be equal to 1/2. Since the denominator is 13, 5/13 is less than 1/2. On the other hand, 10/20 is also equal to 1/2, so it follows that 11/19, which has a larger numerator than 10/20, is greater than 1/2 because its denominator is smaller.

Therefore, 5/13 < 1/2 < 11/19. Consequently, 5/13 < 11/19.

In one sense, this is estimation based on approximation, but it also involves precise analytical thinking when necessary. This shows just how flexible human thought is.

Application of Previously Acquired Cognitive Frames

The problems that were given dealt with the comparison of fraction sizes. The first thing sensed or thought, or in other words, intuition, is important here. The first thing the students sense more or less is probably something like “a direct comparison is difficult because of the different denominators.” At this stage, they call upon previous experiences to intellectually organize the problem.

Next, they seek out a means for judging size. For example, they could figure out sizes if fractions could be converted into decimal values. In short, they no doubt apply previously acquired cognitive frames, recall previously acquired knowledge, make selections, and at times, even use estimation. The knowledge applied at such times is “the size relation is clearer with decimal values than it is with fractions.”

Alternatively, the thought process behind the aforementioned reasoning is probably something like “I could reduce the fractions to a common denominator, but 13 and 19 are large numbers, and it would also be possible to convert them into decimal values, but that seems tedious, so I’ll use a method that is at least a little more efficient. In this way, students execute their selected solution method.

The behavior of students observed by the professor is the external portion of the activity realized as a result of that.

An Analysis of the Thinking Process

The two reasoning differ in the fact that the former focuses on the denominator and manipulates the value of the numerator, while the latter focuses on the numerator and manipulates the denominator. I would now like to look at what the two have in common to try to derive the knowledge that is usually employed.

Common to both reasoning is the fact that the students must prepare some sort of appropriate standards to judge size. It is just like when we have to make a choice between one thing or another in daily life. In that respect, the action of setting some sort of a standard is simplistic, even in our daily lives, but it is happening naturally.

While it is said that learning the fundamentals/basics in life is important, such information is sometimes vague. As I previously mentioned, the setting of a certain standard (viewpoint) for making a judgment, which is used when pursuing ordinary thoughts, is fundamental/basic viable knowledge, and even in learning the size relation of fractions, it is important to be aware of the fact that you are acquiring such knowledge.

Another feature is of both reasoning 1 and 2 is the attempt to comparatively study of the values of the numerators and denominators and compare the size of the two fractions. This is an extremely mathematical point of view.

1-1-2 The Nature of Knowledge: Evolvability

Knowing something is more than simply being aware that “you should do this” for something in particular. In other words, “you should do this “ has in its background the knowledge that “you have to do this in this case.” Another way to look at this is you likely know how something should be done. In the previously mentioned example, the fact that the judgment of fraction size is possible without reasoning to a common denominator is also knowledge. At the same time, we also know that there are other possible methods. If you think about, there is no rule anywhere saying that the judgment of fraction size is not possible without reducing to a common denominator or that reasoning to a common denominator even has to be performed.

Knowledge Is Not Only “Knowing”

Knowledge is more than merely knowing. A more abstract way of saying this would be knowledge is what prompts the subject (person) into acting according to the situation. Knowledge can be thought of as something having built-in evolvability, in the sense that it spurs new actions.

The late Ryoichiro Sato (8th President of The Japan Society of Mathematical Education) once said the following.[1]

> Generally speaking, the methods and concepts that are taught are alive in the minds of students, and therein lies evolvability. What I am saying is, by one’s own power you simplify, expand, and evolve the self. Power like this is the method by which this is conferred, and for there to be concepts, the methods and concepts themselves must be basic. They must have generality. They must also be employed in and applied to a variety of situations. They must be used. By using, employing, and applying them, a close relationship is forged between concepts and between methods, and those concepts and methods take root deep in the mind, resulting in the potential for self-evolvability.
>
> Thus, knowledge that is taught, i.e. working knowledge, is knowledge that organically

evolves. Perhaps it is this sort of knowledge that people lack today. The education of today is producing knowledgeable people, but it is being criticized for not producing people of ability, and perhaps this is based in the lack of preparation in this area.

Generally, when you talk about mathematical knowledge, concepts, facts, or formulas come to mind. Mathematical facts can be conveyed through language, and students can probably learn that. However, it is wrong to readily think that mathematics learned this way is transformed in their knowledge. Furthermore, it must be noted that this is nothing more than ready-made knowledge prepared by adults. If it does not have any impact on the future action of students, then it cannot necessarily be called viable knowledge.

1-1-3 The Knowledge Surrounding Knowledge

Activity (behavior) tends to be thought of as occurring as a direct reaction to sensory stimuli, but that is actually not true. The role of sensory function is to instantly convey to the brain a wide range of information based on stimuli from an object. This wide range of information is organized within the brain to form a mental image of the object. This is known as a representation. This representation is collated and compared with previously acquired schema (thought patterns) that are related to it and then selected, thereby activating an optimal schema. Since the reaction is extremely immediate, it only appears to be a direct reaction to sensory stimuli.

Disposition

What this boils down to is we react to representations that are mentally formed based on the stimuli received from objects. Consequently, there should be a more methodical plan in this representation process at the time of teaching. For problem solving in mathematics learning, instead of demanding immediate solutions, we need activities where, for example, various solutions methods are considered or close examination/interpretations are carried out for what was obtained as a result.

The knowledge acquired in this manner includes the disposition of wondering “What should I do now?” or “Maybe I should do this next.” This is the nonverbal side of the knowledge possessed by each and every student. That means there are portions of such knowledge that can be shared between students, but there are also potions in which understanding differs.

We should realize that when you know something, the world you know expands, while at the same time, the world you know nothing about also expands. We must keep in mind the fact that the knowledge that comprises a student is oriented toward the future and possesses evolvability into the unknown world, or more plainly stated, there is knowledge surrounding knowledge.

1-2 Presentiments of the Future and Mathematical Activity: Transcending the Sensory Domain

1-2-1 Regulation of Cognitive Conflict

There have been countless research studies on the teaching of the initial concept of fractions and the calculation of fractions. However, the expected outcome from such learning has not always been obtained. There are even some young students who think that 1/2 + 1/3 = 2/5 is true or that the fraction pairs (4/5, 14/15) and (1/3, 15/35) respectively represent the same value. Considering the fact that the rate of correct answers is near 100% for the calculation of positive and negative numbers, we can clearly see that the problem is with fractions.

In the next chapter, I will carefully track the initial teaching of fractions, in particular the magnificence of mathematical activity that ventures into unknown worlds and its aspects, by using examples of learning related to the understanding of order relations (size relations).

Cognitive Conflict

The understanding of things is accompanied by difficulties. Excessive difficulty discourages thinking, but a moderate degree of difficulty brings with it a sense of accomplishment and leads to even deeper understanding. D. E. Berlyne defined moderate difficulty through the context of surprise (huh?), doubt (I wonder), perplexity (which one?), frustration (what should I do?), and contradiction (what’s going on?). Cognitive (conceptual) conflict can be thought of in this context.

The previously acquired cognitive frame that lies behind the belief that 1/2 + 1/3 = 2/5 is likely the additive structure that supports integer sets. Cognitive conflict was likely present in the understanding that 1/2 is greater than 1/3, even though 2 is less than 3, as was presented earlier in the comparison of the two fractions. Fractions have a numerical relationship that differs from the numbers the students had previously encountered. However, if they do not perceive this, their only means for solving fractional problems is counting. The aforementioned reactions of younger students can be thought of as deriving from this perceptual deficit. In other words, understanding the concept of rational numbers requires logical reasoning that transcends addition-based solutions.

The real problem is not one of “form” (i.e. the fraction x/y), but rather it is a matter of entering into a world of numbers that have a different relational structure that what was known thus far.

Accommodation

J. Piaget (1896-1980) carefully analyzed the development process of the individual, thereby ascertaining that there is a stage in which an individual endless wavers between attempts at organization and the effects of illusions. (I would like to point out that even adults are sometimes confronted by such circumstances.) For the individual, this suggests the start of accommodation aimed at establishing the logical multiplication of relations.

Unique regulation of cognitive mechanisms, such as realization or creative behavior, is left entirely to the mental activity of the student. Generally, young students learn through textbooks in which text and symbols are printed or through the verbal instruction of an instructor explaining such texts. Model images and symbols are contained within, but there is a limit even in the stimulation of knowledge activity (activity in which mathematical knowledge is acquired) in people (young students) by linguistic interpretation and symbolic description alone. Thomas R. Post offered the following comment on this limit.[2]

> This limitation reflects a very severe misunderstanding of the nature of human learning. It is further exacerbated by the fact that most texts focus on operations with fractions rather than on such fundamental concepts as partitioning and order and equivalence.

He also stated that in the new numeric relation to be learned here, there is the lurking problem of shifting from an additive structure to a multiplicative structure and that this is truly a momentous shift within the mathematical domain.

Mathematically, we consider the ordered pair of integers (a, b), define equality, size, and the four arithmetic operations, and call the ordered pair on which operation by these definitions are performed rational numbers. The definition of the size relation by the ordered pair here is a concept unlike the size relation of the natural numbers students have studied up to this point.

In the size comparison of 1/2 and 1/3 mentioned on the previous page, the knee-jerk reaction is to focus only on the denominator in an attempt to understand the relational structure in which 1/2 is greater than 1/3, even though 2 is less than 3. Even in this reaction, however, we must not overlook the lurking mathematical perception as an exceedingly primitive and simplistic relative view of rational numbers when considering that the numerators were already recognized as being the same.

R. L. Wilder had this to say.[3]

> One difficulty, no doubt, is that mathematics requires such refined and complicated symbolic techniques for its expression. The teacher who becomes so utterly engrossed in mastering techniques for operating with symbols that he forgets their conceptual background is fated to lose the interest of his pupils and will also leave them with some of the misconceptions of mathematics that are so current today.

Even without waiting for such things to be pointed out, one starts to wonder whether the learning of young students is being overly constrained by language and numeric symbols. J. S. Bruner once pointed out that there are three stages in the representation process seen in cognitive development and asserted that we should stress the process of enactive representation for forming rich concepts. However, the conceptual background of symbols that Wilder talked about can be thought of as indicating this. Perhaps the learning of mathematics is seen as an event removed from practical action or realistic thinking where special operations governed by symbols and numeric values take place.

It would seem that young students are being forced into symbolic reflex behavior, including the learning of fractions. There is a need for a setting wherein mathematical activity that stimulates the understanding of the basic concepts of division, order, and equality (equivalence) for fractions can take place.

1-2-2 Mathematical Activity That Helps Concept Formation

The problem of accommodation for gaining an understanding of things, or in other words, what sort of method of understanding was used (what sort of mental regulation was attempted), has a major impact on the arithmetic/mathematical understanding of students.

I will start with a textbook (arithmetic) problem about an equality relation below.

Problem 1. Compare the size of fractions that have the same denominators.

(2/9　5/9),　(4/5　3/5)

2. Compare the size of fractions that have the same numerators.

(1/2　1/3),　(2/5　2/9)

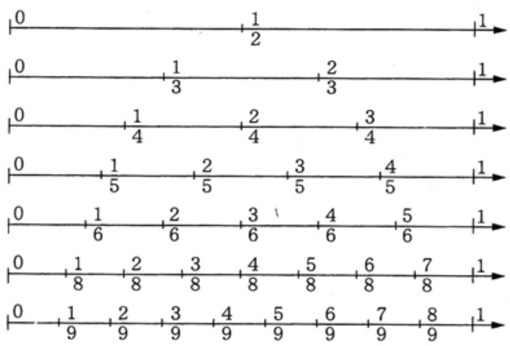


Figure 1

Obviously, the aim of Problems 1 and 2 is the discovery of the following rules.

Among fractions with same denominator, the fraction with the larger numerator is greater. *a)*

Among fractions with same numerator, the fraction with the smaller denominator is greater. *b)*

Let's call rules *a)* and *b)*, which were linguistically represented above, a formula (linguistic knowledge) for judging the size of fractions. If anything, the aim is to essentially gain complete structural insight into this rule. This will be further touched on in the next chapter. We can also learn the following here.

> Even among equal fractions, there are many that are the same size even though they have different numerators and denominators. *c)*

Do you suppose children will be able to understand the universal nature of the size relation of fractions through the verbal description and transmission of this information? And do you suppose they will be able to grasp the structural characteristic of all rational number sets by partial inspection? The generality of *a)* and *b)* is not thought to be recognizable simply with a passing look at Figure 1. Of course, the problem is more than a debate of the size of (2/9, 5/9) or (1/2, 1/3). Furthermore, we can think of how the generality of *a)* and *b)*, for example, is recognized as follows by calling attention to the fact that the intention is not only to see the vicinity of Figure 1.

1-2-3 Observing

Fraction line diagram no. 1

Now let's consider the previously discussed Problems 1 and 2 using the following fraction line diagram (Figure 2). The fraction to be compared can be seen as follows in the diagram.

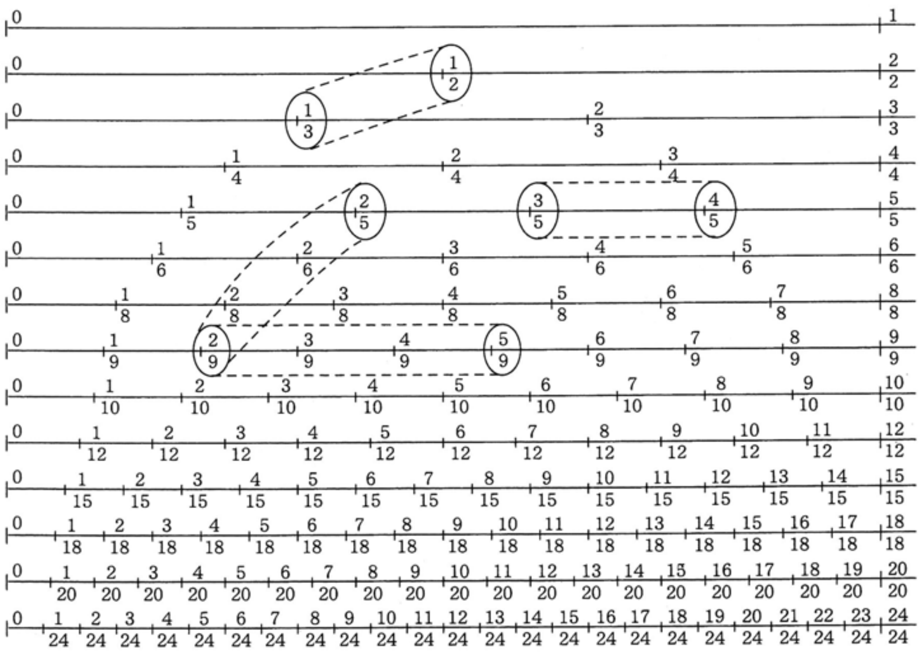


Figure 2

The line diagram shown here only represents a small portion of fractions (rational numbers). It is clear that the figure that was shown in the textbook problem is an even smaller portion. In symbols, 3/9 and 4/9 come between the only two fractions (2/9 and 5/9) that appear, and similarly, when trying to compare 2/5 and 2/9), the fact that 2/6, (2/7), and 2/8 come in between is indicated.

The size of 2/9 and 5/9 or 1/2 and 1/3 can be answered because a figure is presented to the side. Furthermore, it is a figure that is understandable simply by looking at it. In short, it can be processed at a sensory level (sensible domain). However, we are left worrying whether only a partial understanding was achieved, even though an answer was given, and whether the essential meaning has been lost.

Experience is not simply saying that one did this or that in past. Rather, experience must be perceived as a more active concept. Thinking about whether there is something more to be gained than an answer is an issue for teachers who aim to improve their instruction. Let's take a look at this using the following figure.

In the case of the rule "among fractions with same numerator, the fraction with the smaller denominator is greater" in Problem 2, we notice a pattern like *b)* when saying how the figure is seen. In other words, $b_1)$ is a generalization at minimum. We will call this the primary generalization of Problem 2. However, this is not the only time when the rule "among fractions with same numerator, the fraction with the smaller denominator is greater" applies. It also applies in the case of $b_2)$, $b_3)$, and $b_4)$. Once we realize this, the secondary generalization can be said to be made.

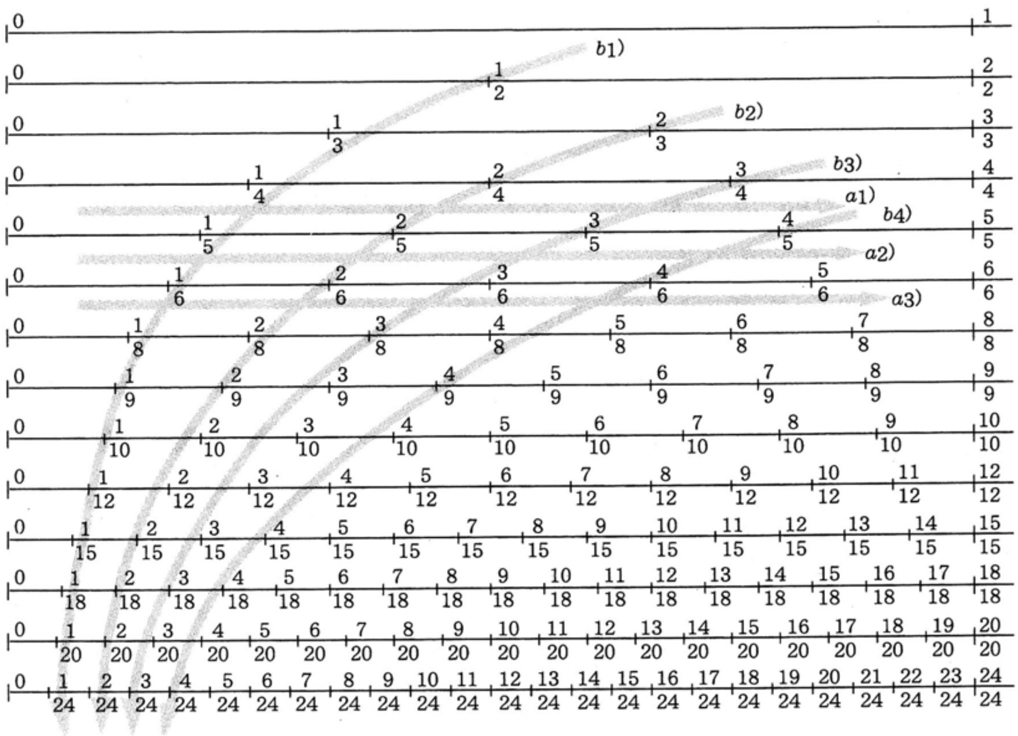


Figure 3

Now, an overall structural understanding of the same rule being established for all fractions not noted here is prompted. This is the tertiary generalization. The fun part of learning and thinking about mathematics lies in this sort of sudden opening up of the field of view. Furthermore, this is the meaning of *a)*, *b)*, and *c)*.

I would also like to emphasize that there are cases where you view the subject with a new viewpoint (inner activity rather than physical activity oriented toward the outside world) in this manner when engaged in mathematical activity.

Chapter 2 Reflective Experience and Mathematical Activity

2-1 Mathematical Activity and Insight: Observing That Which Keeps Out of Sight

2-1-1 The Act of Seeing Into the True Nature of Events

Knowledge is organized within the individual (student) through the careful observation of a fact and through actions and mental activity that sees into the background that forms that fact. I believe that is what mathematical activity (action) is. Human beings have the ability to engage in action within the mind and action transcending facts.

More is expected from young students than simply copying 1/2, 1/3 or 2/5, 2/9. Simply getting a result (answer) can never reach *b)*. Since this is a mental phenomenon, it is merely something predictive, but I would like to illustrate the representation process for the size relation of fractions based on R. B. Davis's argument concerning representation.[4] Davis's argument concerns the problem of representing the meaningful knowledge that the individual acquires.

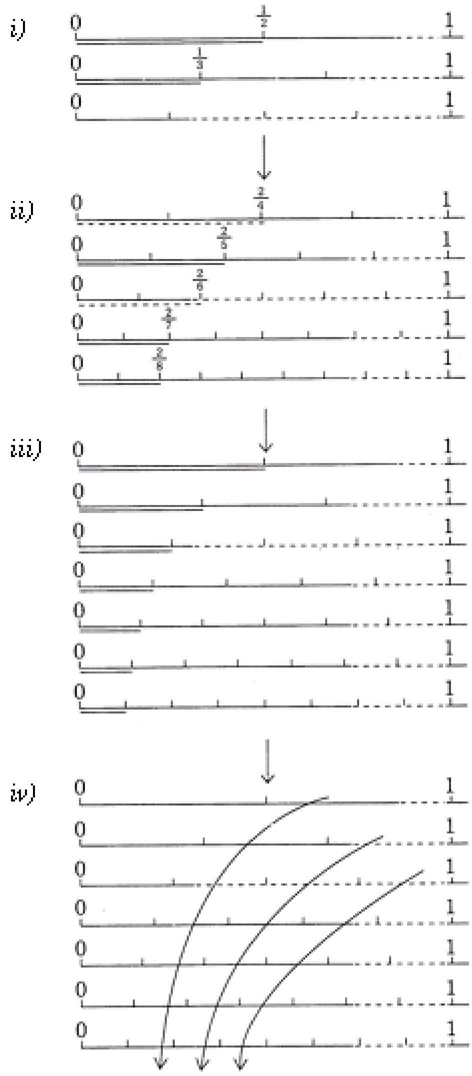


Figure 4

Observation and the Representation Process

Although it is difficult to pick out a portion of a complex thought process, Figure 4 offers a general view of what it might look like. Numbers *i)*, *ii)*, *iii)*, and *iv)*, which were added to avert misinterpretations, do not indicate simple visual realities. Rather, each one indicates a mental action for achieving a representation.

The action of seeking a new viewpoint or changing a viewpoint is nothing more than the representation processing seeking your own understanding of the object you are thinking about. It is an attempt to expand and organize partial understandings, seek relations, and then achieve total integration and structural consistency.

For example, when a representation such as number *iii)* is mentally formed, the student will likely recognize that number *iv)* is not all that difficult. In terms of mathematics education, the central issue is how to urge the student toward number *iii)*. The action in number *ii)* is crucial to that end. This is because number *iii)* is a mental action that reproduces the action of number *i)* and plays a role in forming number *iii)* based on a methodical contrasting of numbers *i)* and *ii)*. You could also say that the action of carefully comparing numbers and *i)* and *ii)* forms number *iii)*.

In other words, the process of gradually elevating from a quantitative visual domain to a qualitative concept is important. We get the following when we use a formula to simplify things.

The direction of the inequality sign changes ($1/a > 1/b$) based on the condition $a < b$ (when creating a unit fraction where a and b are denominators. In addition, even if the numerators are not 1 (but still the same), the student will gradually recognize that their relationship is maintained ($2/a > 2/b$, $3/a > 3/b$...) through a representation process similar to the one shown in Figure 4. Furthermore, cognitive organization resulting in the relation $m/a > m/b$ ($a > m > 0$) is carried out.

The same also applies to Problem 1. The viewpoint shifts from left to right as in a_1) (Figure 3). By recognizing that mental action, the student notices the generality that a_2) and a_3) are actually the same. Looking back upon what one did is nothing more than thinking about what significance the action you just did holds for yourself. It is also an operation in which you make sense of things in your own way.

The following can be derived, at minimum, through a discussion of the analysis of the issues up to this section.

(1) The success or failure of problem solving does not necessarily depend only on the memorizing of algorithm-like knowledge.

(2) The recognition of a general mathematical fact (prior knowledge) accompanies the non-verbal aspect of knowledge in that area.

(3) The organization of knowledge signifies an overall systematic understanding, and the representation process is crucial therein. It is mathematical behavior that underpins this. In particular, mental behavior is key.

2-1-2 Reflective Experience and Physical Experience

It is crucial, of course, to deepen the method of understanding. In this section, I will attempt to further examine the learning discussed so far with the “deepening of the method of understanding:” in mind.

Fraction line diagram no. 2

In contrast to the experience of carefully observing an object and perceiving the mathematic relation, universal nature, and generality that lie behind it (hereafter referred to as reflective experience), there is what I will call physical experience, which is the partial and sensory relation to an event.

In the case of reflective experience, it is simply a matter of where the verbal memory of *a)* and *b)* differs.

First of all, the following can be extracted as fairly active knowledge common to them both.

You should focus on either the numerator or denominator to judge fraction size. (Re_1)

This is also active in the point of being able to form a new point of view of a certain object that differs from the one up to now. However, it is more than conceivable that this ends up being the objective to achieve in the case of physical experience. What I am saying is, even if students focus on the numerator or denominator of the fractions they are trying to compare, they will predictably abandon their attempt to judge size if the fractions do not have either the same numerator or denominator.

In the case of reflective experience, however, the following knowledge is also added. This is a more active method.

You can judge size by transforming fractions into the same numerator or denominator. (Re_2)

This is an event that the students have not yet encountered, but situations where such a problem could occur are conceivable, and it makes them sense that the disposition of attempting to actively respond to it is developing. If the numerators or denominators of the presented problem were the same, Re1 would certainly be possible as well. But if they are not the same number, students might also end up believing it was not possible.

Use of Knowledge That Invites Creativity

On the other hand, we could say there is an elevation to knowledge(Re_2) that has the potential of creative use. In terms of mathematics education, reflective experience is an experience where, by looking back on one's actions, the stimulus source transcends the range of sensory reception and thereby we can expect an experience in which we can operate and process as if we can see with our eyes and an understanding in which partial visual facts can be perceived within a comprehensive dynamic understanding. This can also be called a leap. Mathematical activity can also be called a mathematical translation of this reflective experience.

For example, 2/13 and 5/13 are not written in the figures (for instance, Figure 2), but we can obviously estimate which one is greater, and at times, the following is also conceivable. If 6/13 and 3/7 are given, and this time we base the estimate on 0.5, we cannot judge size because they are both less than that. We could make their denominators the same, but the reduction method is cumbersome. However, when trying to judge size, we can also make their numerators the same rather than their denominators. So then why don't we make the numerators the same? It goes without saying that according to *b)* [6/13, 6/14 (3/7)], 6/13 is clearly greater than 6/14 (6/13 > 6/14), therefore 6/13 > 3/7.

Mathematical knowledge is enriched and understanding further deepened by the behavior (much of which depends on mental behavior) that underpins mathematical facts. Gradually revealing things you have noticed or things you have doubts about based on known concepts is the basis of the deepening of understanding through speculation. To make this happen, we need teaching that gives many opportunities for looking back upon one's own behavior.

Reflective experience promotes willing learning activity and is a source that stimulates the evolutionary expansion of thinking. In the next section, I would like to point out the theoretical basis according to the science of teaching that emphasizes its importance in mathematics education and explore the method for achieving the experience necessary for students through the teaching of arithmetic/mathematics.

2-2 The Importance of Reflective Experience: Connecting with the Future

2-2-1 Transcending "I've Done It Before"

In the previous section, I discussed the importance of mathematical experience while basing my argument on a line graph of fractions. Here I will consider the theoretical background that emphasizes mathematical behavior, based on J. Dewey's study on experience and nature.[5]

[Experience and Thinking]

1. The Nature of Experience. The nature of experience can be understood only by nothing that it includes an active and a passive element peculiarly combined. On the active hand, experience is trying̃a meaning which is made explicit in the connected term experiment. On the passive, it is undergoing. When we experience something we act upon it, we do something with it; then we suffer or undergo the consequences. We do something to the thing and then it does something to us in return: such is the peculiar combination. The connection of these two phases of experience measures the fruitfulness or value of the experience. Mere activity does not constitute experience. It is dispersive, centrifugal, dissipating. Experience as trying involves change, but change is meaningless transition unless it is consciously connected with the return wave of consequences which flow from it. When an activity is continued into the undergoing of consequences, when the change made by action is reflected back into a change made in us, the mere flux is loaded with significance. We learn something. It is not experience when a child merely sticks his finger into a

> flame; it is experience when a child merely sticks his finger into a flame; it is experience when the movement is connected with the pain which he undergoes in consequence. Henceforth the sticking of the finger into flame means a burn. Being burned is a mere physical change, like the burning of a stick of wood, if it is not perceived as a consequence of some other action.

This passage covers the following four questions:

(1) What constitutes experience?

(2) What decides the value of experience?

(3) What is the difference between experience and mere activity?

(4) What is meaningful experience?

I would like to explore each question using the learning of fractions discussed previously as an example.

(1) What constitutes experience?

To answer the question first, one of the constituents is the attempt to connect with an object. This is an active element. Another one is the attempt to accept what was returned as a result. This is a passive element. In the problem comparing the size of (2/9, 5/9) and (4/5, 3/5), we think of a variety of methods for the solution. This is because we understand the problem and want to do something about it.

When we do this, a sudden result arises and certain change takes place. In this case, the result would be the widely known rules that say, “Among fractions with same denominator, the fraction with the larger numerator is greater”···*a)* and “Among fractions with same numerator, the fraction with the smaller denominator is greater”···*b)*. This is, of course, one of the mathematical rules accepted by children.

The relationship between the two constituents can superficially perceived as mentioned above. Dewey’s argument, however, went beyond this obvious relationship. In short, he is emphasizing that there must be a special connection between what was done and what was accepted (the ability to agree with the meaning of what one has done). Borrowing the author’s words, an experience is only realized after the special connective relationship between trying and undergoing forms.

(2) What decides the value of experience?

Even if the problem is solved and *a)* and *b)* are learned linguistically, a look at the numbers in the problem makes it quite obvious. We know that elementary school students suffer quite a bit in trying to verbalize this. However, if we stop at the level of comparing the numbers in the fractions that were presented and simply replacing the visual facts with verbal terms such as *a)* and *b)*, it only amounts to an exceedingly weak connective relationship because if there is no figure or the numbers are changed even a bit, or if unfamiliar fractions appear, students will probably be unable to cope with the problem.

Consequently, a relationship with a strong connection is what underpins experience, and therefore, fruitful experiences are due to the degree of connection between the two, rather than simply having as many as possible.

(3) What is the difference between experience and mere activity?

As I already made clear, the linguistic replacement of partial visual facts certainly does produce superficial change, but it does not necessary bring with it a mental transformation. Dewey called this behavior “mere activity” to distinguish it from experience. As Dewey wrote, “Henceforth the sticking of the finger into flame means a burn. Being burned is mere physical change, like the burning of a stick of wood, if it is not perceived as a consequence of some other action.” This is what he referred to as a “meaningless transition.”

Even when we linguistically represent the problem of judging fraction sizes, as we did with *a)* and *b)*, it is really not much different than codifying $a + b = b + a$ because $3 + 2 = 2 + 3$ or stressing that $a(b\ c) = (a\ b)c$ is an associative law. This is due to the fact that based on that experience alone, it is not easy to come up with the idea of calculating 25 x 24 as 25 x (4 x 6) = (25 x 4) x 6. There is a wide gap between visually accepting a formula that replaces 25 x (4 x 6) = (25 x 4) x 6 with the symbols $a(b\ c) = (a\ b)c$ and knowing and becoming able to utilize the significance of $a(b\ c) = (a\ b)c$.

(4) What is meaningful experience?

So then what do you suppose meaningful experience is? And what do you suppose “peculiar combination” means? Experience is mere activity if the return wave from the result is not consciously accepted.

Dewey also wrote the following.[5]

> … When an activity is continued into the undergoing of consequences, when the change made by action is reflected back into a change made in us, the mere flux is loaded with significance.

Activity does not stop when we see a changed fact or when a result is achieved, but rather those things are continued. What do the changes caused by trying mean to oneself? What does the act of getting involved with an object mean to mean to oneself? That is what I am objectively considering. Dewey is saying it is passive element, but if you think about it, it is also an active conscious element.

There is meaningful experience with that “undergoing.” Dewey is saying experience is also thinking about what trying and understanding means to oneself, rather than saying it is merely trying and understanding.

A familiar example would be “a car stops when you apply the breaks,” and in understanding the fact that cars do not stop immediately, the matter of distance comes into play, or as Dewey would say “a meaningful experience.”

2-2-2 The Meaning of “a Meaningful Experience”

If students try to actively accept the return wave from the results of Problems 1 and 2 and if they can make either the denominators or numerators the same as was summed up as (Re1)(Re2), then they will be able to make sense that links to the future. Simply comparing 6/13 and 3/7 is actually what that means.

Alternatively, the mental behavior (Figure 5 *d)*) shown next also triggers. This time, it is slightly different than the situation of gradually understanding. Instead, it is an action where a conscious effort is made to discover a rule. It is thought that it may have been seen from the beginning, but because the focus was on making sense of the patterns in *a)* and *b)*, it was hidden temporarily in the background. That is what became apparent here.

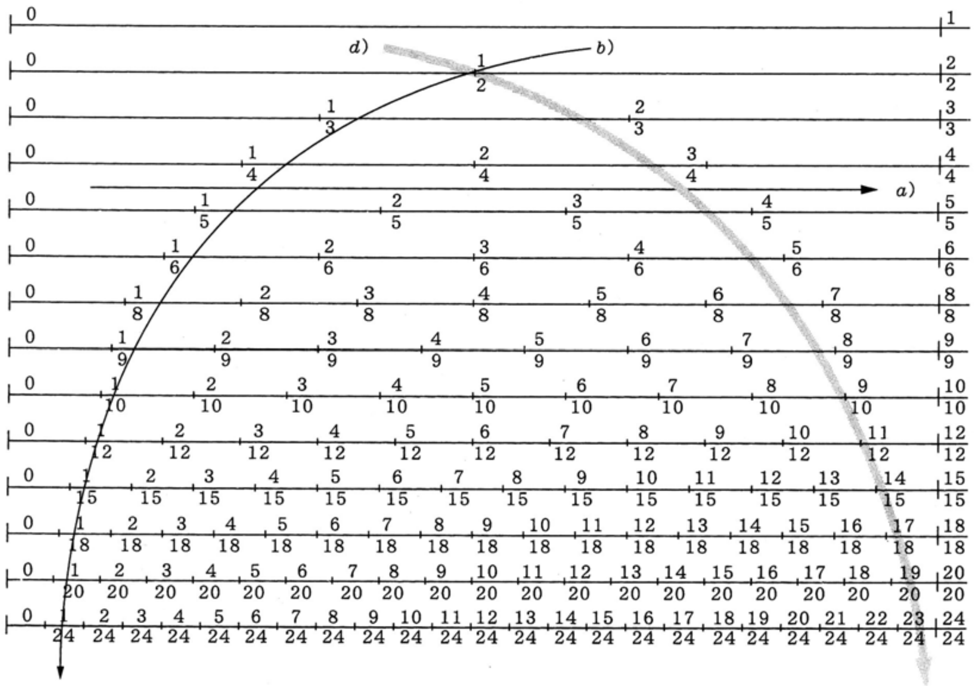


Figure 5

Now then, what do you suppose this means? Since this is a matter of predicting some sort of rule when considering the object, noticing the pattern is not difficult this time. Depending on how it is handled, even elementary school students would probably notice. Verbally, the rule would be as follows.

> Fractions given the same numerator or denominator are greater than then original fractions. d)

However, stressing the fact that “fractions given the same numerator or denominator are greater the then original fractions” is difficult even for junior high school student. In learning fractions, which are underpinned by multiplicative structure, it is necessary to further understanding by applying operations that fulfill additive relations, and this will no doubt easily give rise to confusion (for example, in the case of 4/3, 4+2/3+2 = 6/5 < 4/3).

In order to recognize this, we must add a certain condition. We can get this by considering the following problem. What I am saying is, *d)* is a proposition that can asserted only under the condition $0 < a/b < 1$ (i.e. a proper fraction). (This is related to the content of high school mathematics.)

Problem: Find the size of a / b and $a + m / b + m$ when $b > a > 0$ and $m > 0$.

However, since it fully possible that even a junior high school student would inductively recognize this based on a fraction line graph, we could, for example, conceivably pursue the judgment of size of 5/13 and 11/19 by using the following logic for advanced students.

$$5/13 < 6/14 < 7/15 < 8/16 < 9/17 < 10/18 < 11/19$$

Interestingly, the fraction 1/2 that I considered based on the 0.5 figure in the reasoning in Section 1-1 on page 4 just happens to pop up in the above fraction example.

$$(5/13 < 1/2\ (= 8/16) < 11/19 \quad \therefore \quad 5/13 < 11/19)$$

What Does Mathematical Activity Produce?

Such experience is not the only thing we produce. If we look back on the solution process for comparing the size of these fractions, we were also able to think based on the fraction 1/2 (0.5), but that also leads to the possible thought that you will not be able to make a comparison without using 1/2 as a standard. The return wave from that also stimulates the following behavior.

$$1 - 5/13 = 8/13$$
$$1 - 11/19 = 8/19 \qquad 8/13 > 8/19 \qquad \therefore \ 5/13 < 11/19$$

Here, we make 1 the basis by judging the size of the portion remaining after subtracting the fractions being compared. Alternatively, an adequate understanding of *a* and *b* is also possible by the following logical operation.

$$5/13 < 5/10 = 1/2 = 10/20 < 11/20 < 11/19$$

On close inspection, it is apparent that this is a codification of the response case (reasoning 2) that was previously given (refer to Section 2-1 on page 4). Such a concise representations will not always be possible, but it is possible to expand one's thinking by thoroughly understanding the meaning of *a* and *b* and then appropriately combining them.

What I am saying is, we know that mathematical activity inevitably leads to the formation of diverse views and ways of thinking.

Such behaviors require a little bit of numeric calculation, such as mental calculations, but on the other hand, as a tool for moving reasoning forward, they can likely be called mathematical behavior in which the subject that is considered employs symbols, or in other words, symbol initiative (refer to Section 1-2 on page 25 in the following chapter).

Chapter 3 Suggestions for Improving the Teaching of Arithmetic/Mathematics

3-1 The Importance of Meaningful Experience: Introspection

3-1-1 Meaningful Experience

If we look back on the structure of experience according to J. Dewey, which I discussed in the previous chapter, at the core, mental activity that meaningfully and actively accepts change and tries to make sense of it is suggested, as is the fact that we must hypothesize the existence of another self that objectively gazes at the self or one's own actions from a lofty position.

What I am saying is, it is the shape of the self-evaluation that naturally appears within the individual (student) that is experiencing this.

Leroy G. Callahan and J. Garofalo stated the following.[6]

> Mathematics instruction is focused too much on mathematical content and not enough on mathematical behavior. If we want our students to become active learners and doers of mathematics rather than mere knower of mathematical facts and procedures, we must design our instruction to help develop their metacognition.

To put it in terms of a fraction problem, a place must be provided for experience that promises high-level standards, including not merely replacing the rules *a)* and *b)* (refer to page 21) with language, or in other words, a universal fact that stands no matter how far the diagram is extended, and what should be done when numerators and denominators have different numbers, or in other words, the ability to guide, to a certain degree, behavior that can foresee leaps to the future.

Fraction line diagram no. 3

A careful observation of the fraction line diagram reveals many deeply interesting mathematical facts in addition to *d)*. I would like to limit the discussion here to an introduction of an article by A. B. Bennett, Jr.[7] Simply stated, by seeing (via the observation of the fraction line diagram) into the fact that "the difference and product of unit fractions where any two adjacent denominators are consecutive are equal," which is a problem relating to the sum total of $1/1\cdot 2 + 1/2\cdot 3 + 1/3\cdot 4 + \ldots + 1/n(n+1)$, it asks why can this problem be processed by the partial fraction ($1/2 - 1/3 = 1/3 \times 1/2$)? It makes you think, "To junior high school students, content that is this advanced..." However, upon closer examination, this, too, actually is already apparent in the earlier fraction line diagram.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{k(k+1)}\right]=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{k}-\frac{1}{k(k+1)}\right]=1$$

J. Dewey mentioned that predictive relation rather than prior existence is what is important in the knowledge gained through experience. In learning calculations, we find algorithms for seeking a result. This, too, is important, but we also learn about devising calculation procedures that minimize error as much as possible. Of course, that is not everything. There is also an attempt to also extend the behavior of exploring the principles or laws that are hidden there.

The Background of Experience

When we mention experience, we often use it in the sense of "pass through." The late Yoshinobu Wada (professor emeritus, Tokyo University of Education) stated that what is important in experience is "What to do from this point on." He is pointing out the need to distinguish between 'past experience' and 'background of experience' in mathematics education. This suggests the question of what the subject did to acquire that experience. He also stated that experience is "making an effort not to repeat the same mistake twice." He is saying that teaching mathematics to students is crucial in school instruction, but we must not rely too much on words and symbols and thereby forget that humans have natural gifts (including cognition, judgment, and creativity) and do not function like machines.

The explanations instructors give in class sometimes do not sink into students in the form of knowledge as readily as they expect. Instructors must constantly consider how students are reacting to their explanations and how students are deepening their understanding by talking with their friends. I think the explanations of instructors should be thought of as one type of information for students. Students can then use that information to build their own new knowledge.

Experience is something instantaneous and it also certainly becomes a part of the past at the point it is incorporated into a time series. However, experience must be something for the future. For that matter, if what was experienced is not pointed toward the future, it is not really meaningful. In that sense, the fact that Dewey was clearly positioning experience as a concept oriented to the future is extremely important.

I would like to give the following three minimally required considerations for realizing reflective experience in students in mathematics education, based on an examination of the concrete case studies thus far, which were based on Dewey's thinking.

i) First-hand

ii) Activity of the mind

iii) Practical wisdom

We must highly value mathematics instruction that takes into consideration the direct connection with the target, the expectation of the manifestation of active mental behavior, and its transformation into wisdom that will prove useful as a powerful tool for students' activities in their social lives in the future. In short, the chain of behavior consisting of trying first hand, carefully thinking about what you did, and then actually putting it to use is what further enriches the experience, stimulates desirable activity, and fosters viable knowledge. (See the note on the following page.)

3-1-2 Restoration to Symbolic Initiative

If we look at the circumstances of students that are being taught mathematics in schools today from such a viewpoint, it seems as if the instruction is uni-directional. Activity ends up stopping once a result is achieved, and it would seem the value as mathematical knowledge is being overlooked. I would once again like to cite a passage from R. L. Wilder.[3]

> A good case can be made for the thesis that man can be distinguished from other animals by the way in which he uses symbols
>
> Man possesses what we might call symbolic initiative; that is, he can assign symbols to stand for objects or ideas, set up relationships between them and operate with them on a conceptual level. So far as has been ascertained, no other animal has this faculty, although many animals do exhibit what we might call symbolic reflex behavior.
>
> As an aspect of our culture that depends so exclusively on symbols, as well as on the investigation of relationships between them, mathematics is perhaps the furthest from comprehension by the nonhuman animal. However, much of our mathematical behavior that was originally of the symbolic initiative type drops to the symbolic reflex level. We memorize multiplication tables and then learn special devices (call algorithms) for multiplying and dividing numbers. We memorize simple rules for operating with fractions and formulas for solving equations. These are justifiable laborsaving devices, and the professional mathematician often puts much effort into devising them. However, the professional mathematician understands the purpose of what he is doing, while the pupil who learns only the devices usually does not even comprehend why they work. Processes whose understanding demands symbolic initiative have been placed on the symbolic reflex level.
>
> As a result, a considerable amount of what passes for “good” teaching in mathematics has become of the symbolic reflex type, involving no use of symbolic initiative.

One method for finding out the size relation of fractions is to reduce them to their (lowest) common denominator. Even if students do not think that is the only means for solving the problem, they will end up thinking that making the denominators the same is troublesome in the size comparison of 5/13 and 11/19. They reflexively assume a reduction calculation the moment they see the problem.

Practical Wisdom

Examples of such reactions by students can also been seen in word problems. They will attempt to answer a problem by combining the indicated values using one of the four rules of arithmetic. This is an educational problem. The problem lies in the fact that the knowledge of students has become a rigid structure in which they only execute calculations using this sort of reflexive behavior. There is no sense of humanity.

The students focus on computing and at some point forget the original meaning of “reducing.” If asked, “What is reduction?” they would no doubt respond with nothing more than “It’s making the denominator the same.” This is, of course, not wrong, but without the ability to verbally express the meaning of reduction in a simple manner, there is no connection between what is being learned and real life.

This is a simple matter if you think about it a bit. All you have to do is find the commonality (unit) that will serve as the standard common to the two or more fractions you are trying to compare and then figure out how many of those units each fraction has without changing the size of the original fractions. Simply stated, it is a matter of making the dividing method uniform.

The act of making things uniform is something that is actually done as a basic action making comparisons even in real life. It is related to how to set what is to serve as the standard (viewpoint for making a judgment) in situations where you have to make a selection between one thing or another. This happens to be simply a matter of providing 0.5 or 1 as a judgment standard in comparing fractions.

In addition, the observation of a fraction line diagram and the resulting discoveries when the numerators or the denominators are in a sense providing a standard. In actual problems, students must decide on a standard within a more complex situation. In an abstracted world (arithmetic/mathematics), there is little noise and logic is clear and easy to understand. We must recognize that this is not necessarily far removed from real life.

Note: In relation to this point, an important matter was indicated in the “Discussion Report”8 (Central Council for Education, Ministry of Education, Culture, Sports, Science and Technology, p. 16, 2/13/2006). Though not limited to arithmetic/mathematics, in the practice of teaching, learning by acquisition and learning by enquiry should not be exclusive of each other. The author states, “By stressing the positioning of the process of utilizing knowledge/skill between acquisition and inquiry, we clearly define the relation between the acquirement and use of knowledge/skill and between applied thought or activity and inquiry-based thought and activity, and then move forward with an examination that enables the synergistic nurturing of this according to the child’s development among other factors.” This is an aspect I would like to stress in the improvement of curriculum standards from this point forward.

Problems for Students

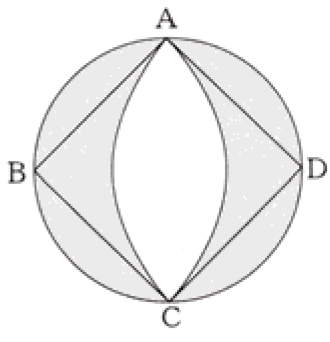
This sort of an examination provides something extremely important in arithmetic/mathematics education thus far. For example, after wrapping up a lesson on judging the size of fractions with different denominators by saying “reduce fractions with different denominators to a common denominator,” the instructor would wrap up the results by saying that method was “finding the common divisor to reduce a number of fractions and then simply changing them into fractions with that as a denominator.”

“Compare the size of the fractions” is the problem. However, the question is whether it a problem for young students. Realizing what sort of means are to be used to compare and what sort of idea can be used in the comparison is what was the problem for young students. Realizing that “if they made the division method uniform, they could make a comparison, and that there were a number of techniques for fractions using this idea, but reduction was only one method for judging size” was the problem for students. If this point is not forcefully taught, this ends up being an evaluation of only whether the problem was solved or not as well as learning that is disconnected from real life.

Hippocrates’ (470 B.C. to 410 B.C.) Lunes

Typical processing brings about a major reduction in the time and effort spent thinking. We could even say processing concludes without any excess thought. However, there are also cases where a roundabout way is required, depending on the situation. That is how one is likely to perceive the joy of thinking or the beauty of the world of mathematics. Consider the following problem.

Problem: Show that the area of square ABCD in the following diagram is equal to the shaded area.



A figure can easily be drawn by using a compass on the perfectly symmetrical shape. The problem is showing that the sum of the area of the left and right lunes is equal to the area of the inscribed square. Although it is slightly complicated, as long as you firmly decide on a solution policy and are able to represent it using a formula, a result will be obtained after carrying out the proper calculation.

For example, if we let *r* be the radius of the circle circumscribing the square ABCD, we can process the problem as follows.

The area of the shaded are will be:

$$2 \cdot \{ \pi r^2/2 - (\pi (\sqrt{2}r)^2/4 - (\sqrt{2}r)^2/2) \}$$
$$= 2 \cdot \{ \pi r^2/2 - (2 \pi r^2/4 - 2r^2/2) \} = 2 r^2$$

In addition, the right side matches the area of the inscribed square. (∵The length of one side of the square is $\sqrt{2}r$.)

The fact that experience is supported by peculiar combination with the object is confirmed. You could probably say that an understanding of this problem was demonstrably achieved by properly executing such a calculation and getting the result. You could also no doubt call this an experience. What I am saying is, within such algebraic processing there is concern that intuition, or estimation, analogy, discovery, or even sometimes agreement is unavailable and is passed through.

The Message from the Figure

What sort of secrets do you suppose are hidden in this problem? Anyone could see how the figure that was drawn is highly refined. However, on reflection it is very interesting to think how the area of a rectilinear figure and the area of a figure enclosed by a curve are not very often the same. Actually, it is already widely known that there was someone who had doubts and wrestled with the problem. Why do you suppose such things happen? It is possible that responding to such doubts oneself is meaningful experience.

Though you may think a similar problem has never been encountered thus far, the content at the right has already been learned. “The area of the figure that is enclosed by the half circle that forms the diameter of the hypotenuse and the half circle that forms the diameter of the two sides that are between the right angle of the right triangle is equal to the area of that right triangle.”

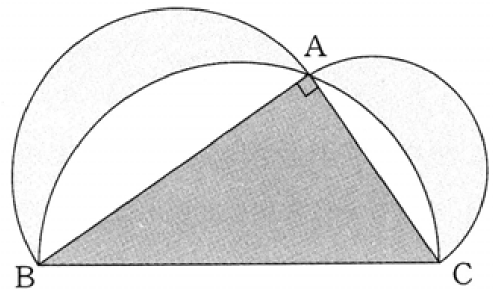


Figure 6

This fact should also hold true even for right isosceles triangles. Based on this, the problem comes into view as follows. Although it is not necessarily easy, it is one conclusion that is derived by the desire to understand or try to understand better.

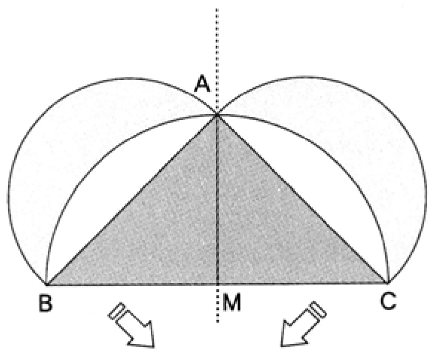


Figure 7

The figure in the problem will be achieved anytime, anywhere, and by anyone by rotating around point A as the center. This sort of understanding is formed clearly within the mind the more it transcends perceptual understanding by the manipulation of the figures before you and the more you try something else, but find that it is troublesome.

J. Piaget called this operative understanding. For the record, this is not merely gaining an understanding by performing an operation. In all other cases (includes those where the radii differ, it is the understanding that we were able to foresee that the same result would be obtained with the same operation.

What to Look at and Where to Look

We human beings do not act in direct response to signal stimuli. We must forget that we try to perceive the true nature of an event before we act. We express what we tried to think with figures and symbols and then use those figures and symbols as tools to explore the meaning of the event. You could also say we are testing our own thoughts. Perhaps that is what R. L. Wilder's symbolic initiative is talking about.

Based on my consideration thus far, we can see that mathematical behavior is that which can be deepened from a perceptual level of the existing domain of knowledge to a level that can perceive its generalities, from a sensible domain to a perceptible domain, or from an observable domain to a conceptual domain.

Most of the content learned in mathematics are abstract concepts. As a result, such concepts cannot be understood simply by memorizing facts that are presented using words. Neither is it possible to memorize each and every fact that appears. The content of school mathematics takes that into consideration, but if we are not careful to stimulate the behavior that underpins that content, i.e. the aforementioned mathematical behavior, we will likely be left with temporarily knowledge of memorized facts that are just waiting to be forgotten.

Stated another way, to acquire truly useful mathematical knowledge, we must ensure rich

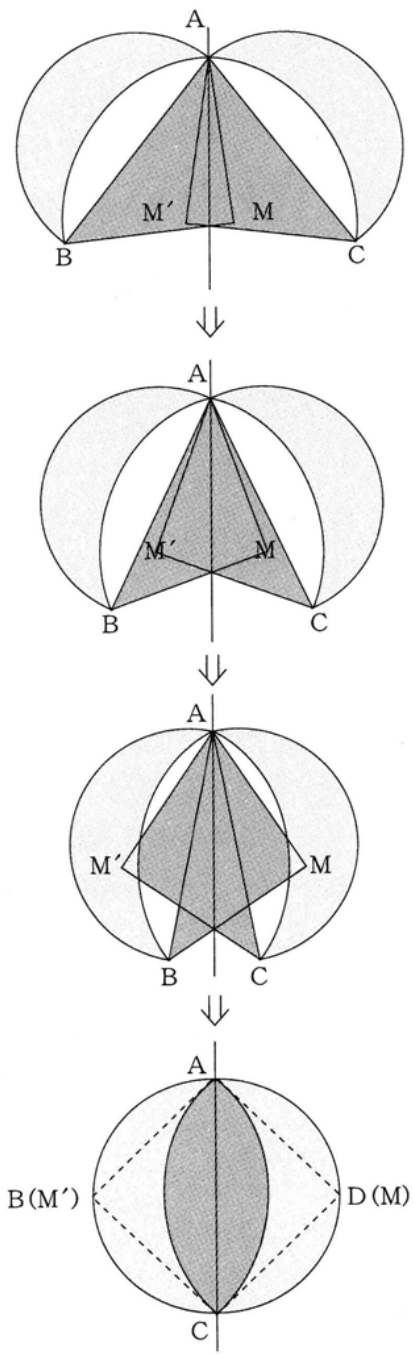


Figure 8

representation and experience that can make sense of that, or to once again borrow the words of Dewey, reflective experience is required in learning.

What I am saying is that this activity reconfigures what one has done within the self and makes sense of it as certain knowledge. A result is thereby achieved algebraically. However, you then look back and explore the meaning in your own way. Relearning and unlearning are important. I believe that continually abstracting abstracts is a vital activity in learning mathematics that gets to the root of problems.

The aforementioned "through the activity of..." is an enduring basic principle of education that has been stated throughout ages past. Now then, in mathematics education, what do you suppose is being referred to when "mathematical activity" is mentioned?" And what do you suppose is a key aspect in mathematical activity? Furthermore, how is this basic principle positioned in mathematical education?

To answer these questions, I introduced examples of Hippocrates' lunes and the initial teaching of fractions that demand relativistic thinking and discussed in detail the activity required in learning mathematics. I think we see that action for perceiving generality and setting a stage for stimulating cognitive organization, even though a result was obtained, are necessary, as is appropriate exemplification that is able to be conscious of that behavior.

3-2 Improving Arithmetic/Mathematics Teaching: Encouragement of Unlearning

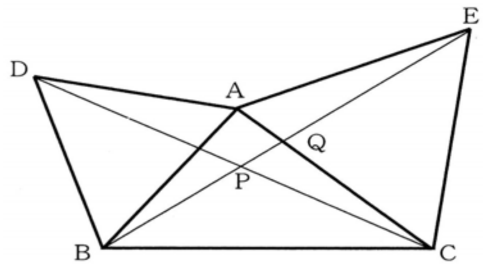
3-2-1 Actively Monitoring One's Own Progress

Trying to Understand What You Are Doing

What sort of things do you suppose come to most people's mind when they hear the term math class? If you ask, they will tell you that they were often bedeviled but came across surprising and unexpected discoveries, including the calculation of literal expressions, factorization, quadratic equations, and quadratic functions, and for geometric figures, the proving of the congruence theorem or the Pythagorean theorem. The sparkle in their eye also conveys their emotion.

It would seem that that the experimental and operational activity that employed the hard-won fruits (teaching tools) of instructors in the learning of geometrical figures is difficult for students to forget. However, you might ask whether mathematics is ultimately the memorization of laws and formulae or assumption, conclusion, and proof. The reason the learning of mathematics is thought of in this manner is our problem here. Let's consider this problem through the solving of the problem that follows (second year).

Problem: There is the following triangle ABC. In the diagram, triangles ADB and ACE are two equilateral triangles drawn with respect to the sides AD and AC of triangle ABC, respectively. Let a point P be the intersection of line BE and CE. Find the vale of Draw the regular triangles ADB and ACE. Then, find the value of ∠EPC.



For proofs, a two-stage build up of logic is required. Insight is also required. What I am saying here is that it is common to derive the fact that angle EPC is 60º from the nature of exterior angles of triangles AQE and CQP by first showing that triangles ADC and ABE are congruent and then using the fact that angles ACD and AEB are equal. Students will probably find even showing the congruency of triangles ADC and ABE troublesome. Although it is troublesome, they sense they might get a solution if they at least try some input when there is such a problem.

If we suppose something else other than proof is expected, what do you think it might be? And how would we get students to understand that? What is needed here is the activity of looking back at and thinking about the processing up to the solution and the resulting conclusion. Rather than telling students, “You did it. Now for the next problem.” it is important to say, “Nice work, but let’s look carefully at (observe) the figure once more.”

Not seeing things you thought you did is surprisingly common. Trying to observe the figure in this case entails rethinking what you did and trying to find something you did not see until now. It is relearning. Such questioning must be done on a daily basis for it to connect with the behavior of students. That is the labor of making sense in one’s own way for self-action.

Questioning Yourself About What You Are Doing

When students look back, we know that they checked conditions and gathered and selected a variety of information for a solution, including what they need to know to come to a conclusion, why an equilateral triangle is drawn, and the fact that the corresponding two sides can form equal triangles, but this is not a key to a solution, or whether there have been any other similar problems.

But that is not all. They also look back and think what if triangle ABC were an equilateral triangle? And they wonder what if the triangle ABC were an obtuse triangle? (∠BAC ≠ 120º)

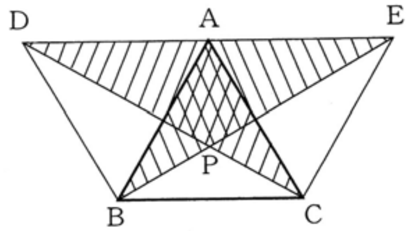


Figure 9

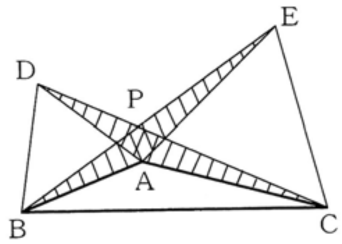


Figure 10

They would no doubt notice things by unconsciously questioning themselves. For example, they might wonder if the geometric figure were drawn based on the same conditions, would angle EPC always be 60º? Or, for angle EPC to be 60º, what relationship must it have with the equilateral triangle drawn based on the conditions?

Then, when they look carefully, they will see the geometric figure as in the figure to the right. It comes into view. In short, they, for example, rotate one of the two planes (the top side is transparent paper) on which the congruent triangles are drawn. In this case, the angle to adjust the corresponding two lines is the same as the angle of rotation. With that in mind, it is quite obvious that the angle is 60º. They come to understand that the problem that was cited was merely studied based on certain conditions (in this case, drawing equilateral triangles wherein any two adjacent sides are one side of the other).

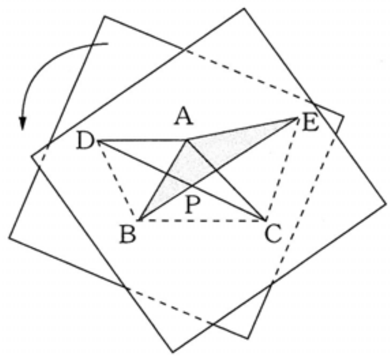


Figure 11

If that is the case, then what if the geometric figure given is not triangle ABC, but rather rectangle ABCD? What would be the size of angle FPC when drawing an equilateral triangle with AD and CD each as a side? (Figure 12)

Even if the geometric figure is changed to a rectangle, the answer is clear, even without using a diagram. The learning of mathematics is not only about producing an answer. It is also about questioning yourself about what you are doing while enabling yourself to think. This is something that is important.

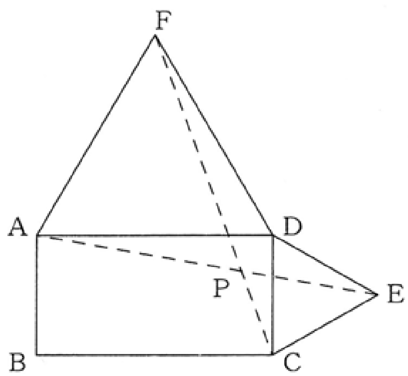


Figure 12

Trying to Put What You Know to Work

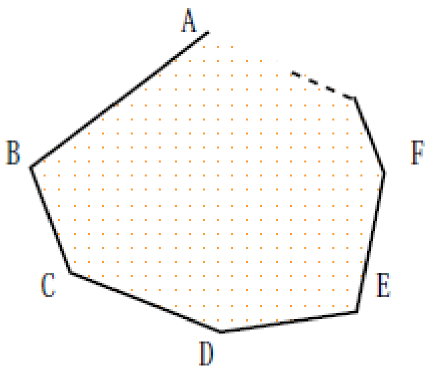
It is said that acquiring knowledge no matter what is important. What lies behind this is the enabling of correct decisions and judgment afterwards in order to live as a human being.

We expect that this will make it possible to expand one's world more than ever before, avert mistakes or at times danger, or boldly overcome difficulties. In short, the meaning of the phrase "It is better to know than not to know" is slighting different. What is important is how that knowledge was come to be known. No matter what content students learn through mathematics, whether it is prime factorization, the nature of parallel lines, the sum of the interior angle of a polygon, the inscribed angle theorem, or the solution of a quadratic equation, it is all the same.

For example, parallelism is a fundamental concept underpinning Euclidian geometry. It is a powerful tool for studying and elucidating the nature of a wide variety of geometric figures, such as the length and angle size of line segments. An inscribed angle is a revision of the perception that a circle is drawn radii from a fixed point to the view of an expected angle from two fixed points, while the general solution of a quadratic equation is a revamping of a known linear equation as well as an argument about the potential of drawing a solution.

Learning is not merely a matter of finding the angle and length or memorizing formulae and finding solutions. It is also a matter of acquiring viewpoints and concepts as a person, including what to do for a basis when expanding thinking logically or what knowledge to employ when trying to overcome a problem being faced. Perhaps we could say that the learning of mathematics is people learning how to think.

If you inductively consider the sum of the interior angles of an n-sided figure (convex polygon) as follows, you can determine it, including the case when it is a triangle, by using 180º x (n - 2)($n \geqq 3$).



Triangle	180º	(= 180º x ?)
Quadrilateral	360º	(= 180º x 2)
Pentagon	540º	(= 180º x 3)
… … … …		
n-sided figure	180º x (n - 2)	

Figure 13

Since it is not that difficult to memorize, the sum of the interior angles can be found whether the figure has 10 sides or 17 sides. I doubt students would forget this because it is not difficult. However, is it really possible to definitely state that they will never forget? When they try to remember, they may very well start worrying and saying things like "I remember the sum of the interior angles of a triangle is 180º and I have the feeling that the answer is n - ?, but what is the question mark in this case?" This is the so-called shedding of knowledge. It is, therefore, not surprising that there is a great need to reform education that is inclined toward the memorization of facts.

In problem solving, there must be some sort of key that lets students believe they can find a solution by using this or that or by thinking in a certain way. Measurement is possible with a protractor, but it will not do if results differ with each measurement, and there might not always be a protractor at hand when it is needed.

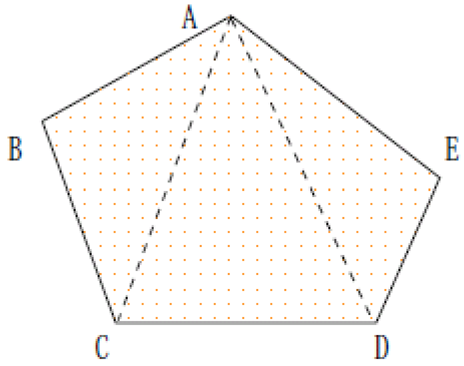


Figure 14

Since the sum of the interior angles of a triangle is known, it is seemingly possible to solve the problem based on that fact. As shown in the figure to the right, a diagonal line is required in order to divide the polygon into a number of triangles. The diagonal line cannot be drawn between vertex A and the adjoining points B and E. In other words, it is possible to divide any sort of polygon into (n - 2) triangles by drawing a diagonal line to the vertex, excluding two adjoining points. Rather than memorizing n - 2, I would like to encourage the attitude of trying to consider the meaning of subtracting two in this case.

Rather than rote memorization in the form of inorganic symbols, I would like to come to understand that numbers and characters appropriately represent the situation, fulfill a certain role, and are trying to say something to us. That is to say, we must value thinking that questions what to use as a basis, how to think, and what method to use when solving a problem, and then effectively translating all this into experience for students.

It goes without saying that bestowing effective solution methods is not school education. Conversely, I would like to make it so that thoroughness does not end up prioritizing the memorization of formulae as a result. This is deeply connected with what to do when we forget what we previously did to understand something. Fostering an attitude of humble inquiry, rather than the ignoring of suspicions, is of vital importance.

The significance of operational activity and testing lies in how you look back on your own behavior and how you make sense of it rather than in the behavior itself. Hence, learning activity that does not end up being calculation for calculation's sake is desired.

While it is efficient to offer an explanation and then leave it at that, you are left wondering if you said enough to make students realize what you were talking about. If you are trying to value self-evaluation, then try to create many opportunities to have students say what they realized and what they are thinking.

In addition, it is vital that instructors themselves look back and think about such things as what problems to present, how students should think about them, and what they want students to realize.

3-2-2 The Importance of the Word "Oneself"

The point of evaluation is self-evaluation. If there is an error, having the quality and ability to realize it and revise your own thinking, or to strive to come up with a better problem solution is important for human beings. Taking responsibility for what you have done is a requirement for living in society, but appropriate self-evaluation is required to ensure this. The learning of mathematics can be thought of as something that gives students the ability to do that and further refine it themselves.

Tatsuro Miwa (professor emeritus, University of Tsukuba) said the following.[9]

> Knowledge is something people organize themselves amid their circumstances, not some sort of stored information that is collected and kept in a library. The fact that comprehension is only achieved when one understands and accepts something means that knowledge must be organized by oneself in order for it to become one's own. No matter how wonderful stored knowledge might be, it must be understood and organized by oneself in order for it to become one's own. Since the learning of mathematics is built upon such understanding, the concept of "oneself" is decisive. In short, "learn by oneself and think by oneself" is essential in learning mathematics. People are enlightened by what is taught by teachers, and they can achieve that which would not be possible on their own. This has been experienced by countless people, and therein lies the raison d'etre of schools. However, a teacher's words will go to waste without the activeness of "learning by oneself" and the attitude of "thinking by oneself" about what was taught.

The existence of subjects in formal education came about due to their usefulness, but I cannot help but reexamine how useful the learning of mathematics is in the sense discussed earlier, i.e. in fostering the nature that supports the essential part of being human.[10]

> Mathematics is not something connected with society by merely materialistic factors that are transmitted to production. Rather, mathematics is something that suggests, in the most concise form, what to think and what was achieved, as well as what to think and what should be achieved. In light of this, we must not forget to focus on the human being, and we must also not forget to contemplate the universe through reflection on figures and the space that fills them, or to touch on the solemnity of mathematical harmony to contribute to the formation of one's own understanding of the world.

Reflect on oneself (Introspection)

J. Perry (1850-1920) once said the following.[11]

> Now in my experience there is hardly any man who may not become a discoverer, an advancer of knowledge, and the earlier the age at which you give him chances of exercising his individuality

> the better. [...]
>
> Let him know that he is expected to be making discoveries all the time; not merely that the best established law is not complete, but that is the very simplest things it is not so much what he is told by a teacher, but what he discovers for himself, that is of real value to him, that becomes permanently part of his mental machinery. Educate through the experience already possessed by a boy; look at things from his point of view --- that is, lead him to educate himself.

Self-negotiation or creating situations in which one has to think for oneself is not as easy to do as it is to talk about. However, it is necessary in the understanding of highly abstract mathematical concepts. In that sense, arithmetic/mathematics education that leads to reflective experience and stimulates mathematical activity in students is in strong demand. “Lead him to educate himself” is something J. Perry said in one of his speeches some 100 years ago. I would like the reader to take this to heart.

There is something I alluded to in the paper’s body, but which I would now like to expand upon here. The prefix un- is generally used to represent negation or a lack of something by adding it to an adjective or noun, as in untouchable, unlimited, or unmeaning. Consequently, there is a tendency to see the prefix un- as signifying passive or negative behavior and action. Do you suppose that is really true?

It is said that the philosopher Shunsuke Tsurumi was told the following by Helen Keller, who knew he was student at the time, when he met her once in New York.[12] The situation is not clear, but we can surmise that they would have been encouraging words for a young student who was devoting himself to his studies.

> I learned many things at the university, but I had to unlearn them later. (Helen Keller)

It is said this was the first time Tsurumi had heard the word unlearn. Does the word unlearn mean “not learn?” From the above-mentioned situation, we can surmise that is not the case. Rather, Helen Keller was using the word in a positive sense. If that is the case, then what was she trying to convey?

Tsurumi said he came to understand her words as follows.

> I imagined I had knit a sweater in a conventional manner and then unraveled it into the yarn that comprised it and knitted it again while holding it up to my own body. The knowledge learned at universities is, of course, necessary. However, it is of no use if it is simply memorized. That which results from unlearning such knowledge becomes flesh and blood.

To unlearn is to make something your own through efforts that surpass learning. At the same time, I am convinced that the spirit of this unlearning and the reflective experience for making what you have learned a part of you are one in the same.

Finally, I would like to add the following proposal about the formation of arithmetic/mathematics curriculum. With the cooperation of specialists, intellectuals, and others at the Ministry of Education, Culture, Sports, Science and Technology, earnest discussion is underway regarding the next “Teaching Guide.” The discussion is centered on revising those guidelines.

From the standpoint that the spirit of unlearning is something that leads to the reflective experience I covered in great length in this paper, we expect the establishment of a new “Teaching Guide” that can realize classes in which young students will be capable of unlearning in order to further deepen their understanding of arithmetic/mathematics.

Summary

The term “mathematical activity” was incorporated for the first time into the objectives of the present Course of Study (for mathematics) (notification dated December 2000). Soon afterward, the debate deepened under the “Discussion Report” from the Central Council for Education (February 13, 2006), and the “Discussion Summary” (provisional title) is to be announced. In light of the circumstances leading up to this, math and science education, which forms the foundation of science and technology, is naturally the focus, and mathematical activity is also emphasized. In an attempt to make that possible, I believe improvements will be made to enable specific deployments, including activity that gives rise to mathematics, activity that uses mathematics, activity that communicates mathematically, and activity that you can experience firsthand mathematically. That is what I truly wish.

The idea of reflective experience was born from research based on J. Dewey’s theory, but it would not be an overstatement to say that mathematics education translation is actually a mathematical activity. To begin with, the present Course of Study (for mathematics) was the result of just such a spirit, and the author has positioned that guide as a plan for ensuring the deployment and implementation of mathematical activity. Furthermore, it is activity that attempts to get closer to the true nature of things. Repeating and relearning what you have studied and making an effort to understand it in your own way is vital. In other words, unlearning is important.

Arithmetic/mathematics make it possible to check one’s own behavior (solution process) by oneself, and if there is an error, fix it oneself. In short, we must not overlook the fact that learning arithmetic/mathematics fosters an independent spirit. If you think about it, a spirit of autonomy and independence is essential in living in a democratic society. A place for unlearning by arithmetic/mathematics is also required for fostering an independent spirit.

The Fundamental Law of Education was revised in December 2006 after sixty years. Because of this, education in Japan is clearly in a transitional period. And it just so happens that improvements to the standards for the upcoming curriculum are currently being discussed. If we are to champion our nation built on science and technology, now is the time for us mathematics instructors to consider pooling our resources to make the most of such arithmetic/mathematics in formal education with an eye on a future 10, 20, or more years from now.

Acknowledgements

I gave the lecture at the important APEC International Conference several weeks before submitting this paper. While I strived to make the paper as clear as possible, I hope you will pardon me for any

disorganized writing you may have come across. I would like to end by saying how fortunate I feel to have been given this precious opportunity at the conference. I would also like to sincerely thank Masami Isoda, Professor, the University of Tsukuba, and the many others who made this opportunity possible.

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