

**Math Lessons That Teach Children to Use “For Example...,” “But Then
Again...,” “If...,” and “In That Case...”
Creating Lessons That Focus on “Starting Phrases”**

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1. Creating Lessons That Promote Cooperative Thinking Processes Using Students’ Mutterings

When considering approaches to cultivate the representation skills of students in math classes, there are several types of methods of expression to be considered:

1. Representation through expression (formula or calculation), tables and symbols
2. Representation through diagrams (Number line, Tape diagrams such as equally divided tape by the unit, and Proportional number lines such as proportionality between two intersection lines *Descartes, Geometry*)
3. Representation through pictures or operations
4. Representation through words

Types 1 and 2 are more highly abstract, formal ways of representation than types 3 and 4. However for children to be able to use representations 1 and 2 effectively which are specific to mathematics, they need to be able to express their understanding of a phenomenon concretely using methods 3 and 4.

However, children are too often asked to use formal textual representations in math classes conducted under the classic-style lesson format hence they have greater difficulty expressing their understanding of the material being studied.

Here, I will refer to a “hand-raising and appointing-style” lesson in which the teacher asks a question and the students raise their hands before answering. In this type of lesson, the children who do not raise their hands will not have a chance to speak. However, it is reasonable to assume that those who raise their hands and can express their own opinions are children who have considerable confidence in their own opinions. When this method is used, children who have a vague idea of the answer or who only know part of the answer do not get a chance to express what

they know. In math classes where the main goal is to cultivate the ability to think, it is important to create time when children can work with one another while they are thinking. To enable students to work together in the thinking process, teachers need to incorporate methods of conducting lessons in which they pay attention to the natural mutterings made by the children seated in the class.

Students have a variety of natural reactions as they listen to the ideas expressed by their peers during a lesson. For example, if they hear something they agree with, they nod, and when they have a question about something, they tilt their heads. If the students notice that there is a counterexample, they will mutter under their breaths, saying things like “But...” and “But then again...” In other words, they are constantly responding in some way to their peers’ opinions. To create lessons that enable all students to cooperate in their thinking process, it is important for teachers to notice these natural responses and make use of them as the keys to move the lesson forward.

While conducting lessons in which I pay attention to and use the quiet utterances of students, I noticed that the types of natural “starting phrases” uttered by the students can be divided into several types that lead to the development of their mathematical thinking.

This paper will focus on “starting phrases” among those natural expressions of children and will strive to help the reader to understand their value better.

2. Paying Attention to Students’ “Starting Phrases”

When a child really wants to convey his/her thinking to his/her peers the child will spontaneously use a variety of representations. It would be a mistake to think that all of the phrases that children use to diligently express their ideas in class are of the same type.

By focusing on “Starting phrases”, these phrases could be categorized into the following identified types:

◆ “For example...”

This phrase is used by children who want to replace their ideas with more concrete or specific examples. The ability to replace one’s thoughts with a

demonstrative example is not something a child can do unless he/she has a fairly solid understanding of the material. “For example, let’s make this 10. If we do that, then...” Making a point by replacing their ideas with ideas that are easier for their classmates to imagine is also helpful for the other students.

When a student says something abstract (or general) and moves on to give more concrete (or specific) ideas by saying, “For example...,” a listening peer who had been nodding might start to respond by saying, “No, that’s not it.” Thus, when a student starts to say something using the phrase “for example...,” it might be a signal that there are some misunderstandings developing among the students. It is useful to watch the students’ facial expressions when this happens.

If you do, you will notice that several of the students who are listening intently will have their heads tilted. When you notice one of these children, take the opportunity to call on him/her. In this way, minor inconsistencies among interpretations by the students can be discussed with the whole class.

There are two types of “for examples” used by children in this kind of situation:

- a) When the child wants to give a typical example (special case which has general idea) of a phenomenon he/she is trying to explain
- b) When the example being used is actually a special example which cannot be generalized rather than an ordinary case

A child who can give a type (a) example is likely to have understood the problem well. However, a student who explains using a type (b) example usually does not realize that the example he/she is giving is a special case. In this instance, it is hoped that the student will be corrected by the other children in the class. However, if the teacher is aware that these two types of thinking processes may occur in different students who use the phrase “for example”, it will help them to shape the development of the other children’s ideas.

Also, in the case of type (a), some children will try to explain the idea using a simplified version of the scenario while others will try using a different scenario of similar representation. In this case, if the teacher is aware that it is better to use the simplified examples to cultivate the children’s ability to think, it will be easier for the teacher to develop the lesson.

◆ “But then again...” and “But...”

These words are used by children when they try to proactively connect to what is being said by their teacher or peers. Their purpose may change depending on the situation. Sometimes they try to connect by offering a counterexample and sometimes by providing an explanation. However, if the child is earnestly listening to a peer, he/she will often utter these phrases spontaneously.

Since this often comes as an under-the-breath utterance during class, the teacher needs to be on high alert to be able to pick up these verbal cues. If the teacher hears a student use phrases, this indicates that he/she is trying to connect to what a peer is saying, the teacher should then call that student. This can help to enliven a sluggish lesson.

◆ “First.... Then...”

Children who start expressing themselves with this phrase are capable of examining their own ideas and later separating and organizing their thinking processes into segments. When a student starts off this way, it will be an interesting exercise to stop the student before he/she finishes his/her thought that is after he/she completes the “first...” phrase. Then ask the other students in the class to try to finish the thought.

Even the student who might not have understood the material initially, if given an opportunity, he or she can begin to work through it. This exercise, therefore, allows many of the students to get a taste of how fun it is to solve a problem.

Moreover, if the students understand that conveying their thoughts by dividing them up into segments is an easier comprehensible method of conveying something to their peers, they will then become aware of this when they speak up. Speaking by dividing one’s thoughts into segments helps teach children to listen to their peers by breaking up what they hear. As they become accustomed to doing this, speakers will start to speak in such a way that incorporates an awareness of the listeners, by asking the listeners, “Should I stop here?” Listeners will also learn to organize their thinking, for instance, by saying, “I understand what you said up to this point, but I don’t understand what you said after that. Please explain what you said after that

point more slowly.” Because understanding is a process of organizing and arranging knowledge within one’s own head, this kind of change within a child marks a considerable step forward.

◆ “In that case...”

This phrase can be viewed as a benchmark for evaluating a lesson since it is the phrase that students will use when they have become self-motivated. When classroom activities progress on a natural flow, the children can begin to work with an expectation of what will happen next. If the lesson consists of a series of fragmented questions, the children will not get the sense of the flow of the activities and therefore remain in a passive state.

If a child tries to move forward before the teacher, saying something like, “Teacher, then (in that case) what will happen if that number is larger?” the child should be highly praised. Developing such students is the precise reason these lessons are being taught. In other words, teachers must recognize that if their lessons do not produce students who can actively approach a problem using the starting word “in that case...,” then their lessons are likely lacking in certain continuity.

◆ “If...”

This is a very convenient phrase and may be the most needed phrase to be learnt by students in mathematics. Phrases that start with “if” are also used when trying to organize a phenomenon. It is like trying to explain a pentagon, which tends to be hard to understand by using triangles which are easier to understand. It is used when trying to understand, something by converting it into something simpler and easier to understand. This is similar to the use of the “for example” phrase.

It is also used after solving a particular problem and when thinking about how the solution could be applied to other situations. In this way, it is similar to the “in that case” phrase. “If” is a phrase that can be used in a variety of mathematical approaches. This is why I described it as being convenient at the very beginning. Teachers should intentionally praise children who unconsciously use this word in class and make efforts to encourage other students to use it as well.

During class, it is important for the teacher to explain how this phrase is being used and to make the students aware of its different roles. Emphasize the student's use of "if" as follows:

"That's a good question. See how Student A thinks. 'Would it work *if* it was a double-digit number?' He realized that this could probably be used in situations where the number is not a single-digit number. But I wonder whether it will work that easily?"

Students will learn to generalize phenomena by repeatedly experiencing these kinds of situations. It is an excellent sign when students begin to spontaneously use these kinds of phrases in their math lessons.

Next, we will look at specific examples of situations where these phrases are used in math classes.

3. Lessons Using "Children's Misunderstandings" Derived from "For Example"

In a lesson that I conducted on the division of fractions, I tried to get the students to think about how far they could go on their own before teaching them the formula used in the division of fractions, which is to switch the numerator and the denominator and then multiply.

At the beginning of the class, I wrote the following on the board and asked, "What number would have to go into the box in order for you to solve this problem?"

$$\square \div 1/4$$

This question was intended to make the students try to make connections with what they had already learned. The numbers that the children wanted to enter in the box were the "for example" representation of the children at this point. The specific numbers that the children used reflected the way they understood the problem when they said, "I can solve it if this number is entered." The children muttered different numbers.

"Try 1/4," said some students while the others suggested using "a fraction with a denominator of four." Others thought it would be easy if the box contained an integer. They drew on all the images of fractions they had developed so far.

First, I entered $1/4$. The problem $1/4 \div 1/4$ is easy to solve. A number divided by itself is always 1. Everyone agreed. However, when I tried to enter an integer, there were some misunderstandings. Some children wanted to enter 4 while others wanted to enter 1. Since these were integers, I thought their reasons for choosing integers might be fairly similar, but they were in fact quite different. Those who wanted to enter 4 thought the answer would be 1. They thought that $\div 1/4$ meant finding the value of one piece of an item that was divided into four pieces hence one portion of 4 that was divided into four pieces would be 1.

Other students disagreed.

“The term $1/4$ means that one is broken into four pieces. In that case, if 4 is entered, and four items were broken into four pieces, there would be 16 pieces. So the answer should be 16,” they argued.

I liked this explanation, as it was easy to follow. However, Student S, who had wanted to enter 4 in the box, still had her head tilted. Then another student started to give an explanation using another division formula.

“When you calculate $8 \div 2$, you ask how many times does 2 go into 8, right? In this case, we have to ask how many times does $1/4$ go into 4, see?” the student explained.

But Student S had a different idea.

“But doesn’t $8 \div 2$ mean that you divide 8 into two parts? So you have to figure out the value of one of the two parts. And that means it’s $1/2$.”

What she meant, in other words, was that $4 \div 1/4$ means finding the value of one of the portions produced by dividing 4 into four pieces. But if you look closely, $\div 2$ does not include a fraction. What Student S tried to do was $\div 2$.

Student S finally relented after hearing this explanation.

This minor misunderstanding would not have been brought to light had the students not been able to express themselves clearly. Because they could, the students learned that finding the value of one thing divided into 4 was the same as dividing by 4.

So what does it mean, then, to divide by $1/4$? One student answered that it meant figuring out how many $1/4$ s there were in a number, using the same method used to solve $8 \div 2$ by asking how many times does 2 go into 8. It is different from

dividing into two. The children were stuck. Unfortunately, this is where this lesson ended. However, it was significant that the children were able to think about the meaning of fractional division in two different situations. In Japan, students are taught that there are two types of division: Measurement division and Partitive division. These can be described as follows.

- $8 \div 2$ means asking how many times does 2 go into 8 (Measurement division). Thus, $4 \div 1/4$ means asking how many times does $1/4$ go into 4. Because it goes into 1 four times, it would go into 4 a total of 4×4 times, or 16 times. This approach is easy to understand.
- $8 \div 2$ means finding the value of one portion when 8 is divided into two portions (Partitive division).

Thus, $4 \div 1/4$ means finding the value of one portion when 4 has been divided into $1/4$ portions. But this leads to the question of what it means to “divide something into $1/4$ portions,” and this is where the students become even more confused. This remains a problem.

If an approach using the concept of Ratio or Proportion is not used from this point onward, the students will not be able to develop an image of the type of calculation they are being asked to perform. A word problem provides some help. Suppose an area of 4 m^2 can be painted with $1/4\ell$ of paint. Have the students calculate $4 \div 1/4$ by setting up a problem that asks how large an area can be painted with 1ℓ . In this case, it is easy to imagine the answer of 16 m^2 , but the problem is getting the students to recognize that this is a division problem.

In this kind of situation, you would have to change the approach of dividing 8 into 2 by asking, “There is a number that if doubled will produce 8. What is that number?”

In this case, $4 \div 1/4$ could be thought of as: “There is a number that will produce 4 if quartered.” Perhaps this path to achieve 16 as the answer will help the students see this as a division problem.

Some feel that it is better to teach that division is always a matter of finding the value of a single portion as shown in the previous paint example. However, in the

case of $\div 1/4$, this is not always easy to see.

Another student suggested that it would be easier to just write 4 as $16/4$, making things even more interesting. At the beginning of the lesson, some students said that the problem would have been easy to solve had it been written as $1/4 \div 1/4$. Other students actually suggested fractions with denominator of 4. These ideas, which initially seemed to be out of topic, now become important for solving the problem. If the problem is written like this, one student argued, then it is the same as $16 \div 1$. He strongly emphasized that he did not originally understand it because it was by a fraction (\div a fraction).

$$4 \div 1/4 = 16/4 \div 1/4$$

The other students liked this explanation and the student who presented it was praised by his peers. Student C, the presenter, seemed to be pleased with himself. However, some of the students realized something was wrong.

“But what if the denominator weren’t 4?” they questioned, their enthusiasm subsiding.

Other students explained that perhaps the fractions could be reduced to a common denominator, just as it is done when trying to add fractions with different denominators, reigniting their excitement again. They concluded that fractional division could also be performed by finding a common denominator.

Example: $3/5 \div 1/4 = 12/20 \div 5/20 = 12 \div 5$

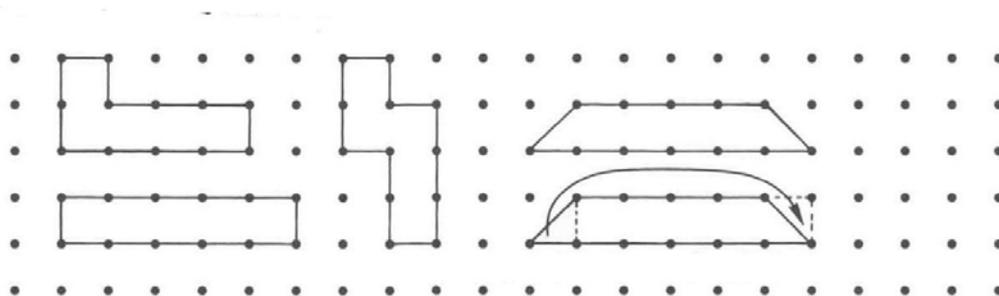
They were able to derive this new method of calculation using the knowledge and skills they already had. Children who have the experience of being able to discover a whole new way of performing a calculation all on their own are likely to have more confidence, knowing that they will somehow be able to solve new problems in the future even without being taught. This is an example of how to conduct a lesson by building on a student’s misconceived “example”.

By comparison, developing a lesson that leads to a new way of performing calculations by allowing students to try to solve the problem (saying, “For example,

if you enter this kind of number in the box, I can do it”) is an effective approach that can also be used to teach other methods of calculation. Such a lesson is no place for a child who was taught to just repeat what they have learned in the past. This is the kind of class where a teacher can directly impact his/her students’ view of math.

4. Creating Lessons That Proceed with the Phrase “In That Case”

I distribute graph paper with grid points to a class of fifth grade students. I instruct them to try to draw a shape with an area of 5 cm^2 by connecting points on the paper with straight lines. First, they simply connect points to make squares, but then they start to connect points in such a way that the square is divided into half.



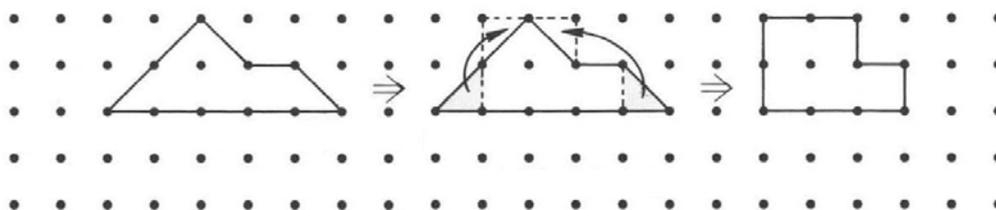
I check whether the areas of the shapes created by the students are actually 5 cm^2 . There is little resistance when I tell them that it is problematic for this exercise to try to figure out the area of triangles created by cutting rectangles or squares in half. While I am checking their work, someone realizes that all of the shapes created using five squares are made by connecting 12 points.

One student suggests to me, “Check the shapes by seeing if they connect 12 points.” When conducting this lesson, I try to wait until I hear this suggestion from one of the students. However, if no one seems to be heading in that direction, I will intentionally prompt them by focusing on the perimeter points of each shape.

For example, while copying Student A’s diagram onto the blackboard, I will intentionally make an unsure remark, such as, “Hmm, I wonder if this shape will work.” When I do, the other students start talking about the number of points, saying things like, “No, that’s not right. It’s connected to the point above that one.

Then it goes four points sideways.”

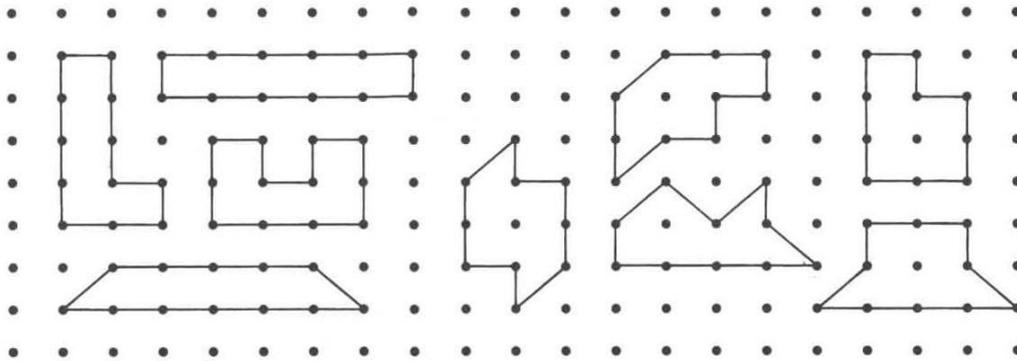
At this point, I ask, “So, how many points should be connected?” “Twelve points,” comes the reply. Once the students come to this realization, you will start to hear other students saying things like, “Hey, mine hits 12 points too. I created a different shape but used the same number of points.” This is because most of the students have hit 12 points. Then I ask, “Wait a second. Did everyone use 12 points?” At this point, someone speaks up, “No, I didn’t. I only used 10 points.” I purposely start giving that student a hard time, suggesting that if she only used 10 points, the area of her shape must be wrong. I check this student’s work by having her show her result on the blackboard.



However, the area of the shape turns out to be 5 cm^2 . It seems fine. The series of activities leading up to this point was a concatenation of sentences that started with “in that case...” Here is a summary of how it went:

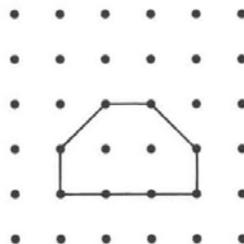
- Everyone’s shapes are made using 12 points.
- In that case, perhaps you can figure out if a shape has an area of 5 cm^2 only by counting the number of points hit.
- No, I only used 10 points.
- So then, doesn’t yours have a different area? Let’s check.
- No, it’s correct even with only 10 points.
- They realize that the number of points in the perimeter does not determine the area.

This does not necessarily mean that the words “in that case...” will literally be used, but you will inevitably start to hear it as the work of checking the students’ shapes proceeds. Here, I compare the 12-point shapes and the 10-point shapes.



The class starts to get more interesting as the students notice, “Hey, look at this!” and “Teacher, the 10-point figures have bellybuttons.” When a student says that the 10-point figures have bellybuttons, an adult might not readily understand what that student means, but his/her fellow students will immediately understand. That is, each of the 10-point figures has a single point in the middle of the shape. This is not the case with the 12-point figures.

Amid all this, one girl, looking rather perplexed says, “Teacher, the perimeter of my figure has 8 points. I double-checked it and it definitely has an area of 5 cm^2 . But it only connects eight points.” She speaks as though she is apologizing for ruining everyone’s discovery. I had her put her shape up on the blackboard so everyone can see it.



The class, which has suddenly grown quiet, erupts in the next instant.

“Teacher! Look! I’ve found something interesting.”

“Yeah, I see it too! I haven’t checked it, but...”

When I hear a student start to say, “Hmm, the bellybutton...” I stop them mid-sentence. Then, I ask the other students who are still staring at the diagram, “What do you think he was about to say?”

Many of the students, having realized through this exchange that the number of

perimeter points has changed from 12 to 10 to 8, now see that the center points are increasing accordingly from 0 to 1 to 2. However, while the more sensitive students will have gotten into the rhythm of this discovery, others may not have been able to keep up. I break the rhythm temporarily so that everyone can make the final discovery together. Then I give some time for those who had not noticed the pattern to consult with their peers.

This time is used by the students who did notice the pattern to make notes in their notebooks. This time can also be used by students to double-check whether the discoveries they made can be confirmed using other examples. Likewise, students who did not notice the pattern can walk around with their peers and talk about the diagrams. However, I instruct them not to talk to the students who are sitting and writing in their notebooks.

I do this because I want to give all the students a chance to taste what it is like to make a discovery on their own. To make discovery meaningful, one should face the challenge and solve the problem by oneself. This method can be used any time you want your students to not only just listen to students who have already discovered something but also to discover something themselves.

As in the common saying, “Out of the counsel of three comes wisdom”. Several of the students who were walking around and discussing the figures came up to the blackboard to explain their thinking processes while pointing to the center of the diagrams. I ask the other students to pay attention.

“What have these students discovered?” I ask.

As everyone watches the movements of the children’s fingers, in the next instant, even more students come to realize that “the number of points in the middle is increasing!” At this moment, the teacher really needs to be on high alert and watch the students’ reactions. I intentionally stop the rhythm of discovery here for a moment. When everyone is reviewing their work, I take the time to evaluate whether any of the students can see what is coming. I strive to notice the children who are moving toward an “in that case...” statement. If possible, wait to see if you hear this from one of the students who has been walking around discussing it with

his/her classmates.

“Teacher, if this is how it works, then a figure with six points will have three bellybuttons.”

This realization will ultimately develop out of this series of exercises. From here, the students can check whether creating a shape using the right number of perimeter points and interior points will, in fact, result in a shape with an area of 5 cm^2 .

While trying to solve one problem, the students come up with new problems, and they begin to work on new problem-solving techniques to check their solutions to those problems. This exercise is also helpful for students to fully understand about area since they are constantly calculating and double-checking the area of the figures they are drawing. Repetitive drills should be done in this kind of context, using activities of discovery that take advantage of the students' sense of anticipation.

5. “If” Is Often Used in Trial-and-Error Activities

When children are demonstrating their rich representation capabilities, teachers have a tendency to focus on diverse ideas or the discovery of more mathematical methods. However, if we want to improve the students' individual problem-solving skills, we need to recognize the importance of using more mundane, childlike, and straightforward means of representation. One area of importance is the representation children use when they make mistakes in their problem-solving efforts. We want the children to share their mistakes with each other and gain plenty of experience while working out new solutions by working together with their peers. Creating classes where students are allowed to repeat their trial-and-error attempts without fear of failure, to make self-adjustments to their efforts, and make interesting discoveries by discussing things with their peers, will give more students the chance to succeed.

I conduct the following lesson with the third grade students using an interesting problem based on multiplication tables.

The challenge is to use up all the number cards, from 0 to 9, by calling out five multiplication problems. This explanation is not really sufficient enough to understand the activity. It is the same as when you try to explain the problem to your students. The best approach is to have one student come up in front of the class so that you can demonstrate the exercise.

After choosing a student, I ask him to suggest a random multiplication question. He suggests $5 \times 6 = 30$. I then remove cards 3 and 0. I take the next turn by saying, “ $3 \times 8 = 24$,” and remove cards 2 and 4.

$$0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \quad \Rightarrow \quad \begin{array}{c} 1 \quad 5 \ 6 \ 7 \ 8 \ 9 \\ 30 \ 24 \end{array}$$

Only the 1, 5, 6, 7, 8, and 9 are left. By this time, many of the students will understand the exercise and will be thinking along with you. In this case, call on one of the students who seem to want to play. The student says, “ $7 \times 8 = 56$.” Then 5 and 6 are removed. Only four cards, 1, 7, 8, and 9 remain. This time, however, everyone will have a difficult time. You will notice several of the students performing calculations with their fingers in the air or tilting their heads back and forth as they think. “Finish! But it’s not going to work,” some will say. You will notice several of the children doing trial-and-error tests repeatedly in their heads. I call on one of those students.

“It’s no use. It’s done,” the student replies. Then another student chimes in, “No it’s not. You can keep going.” The student who said it does not work replies, “You can go one more round, but that’s it.” The third-grade student can think ahead quite clearly.

I ask him to explain his thought to the entire class.

“If you use $2 \times 9 = 18$, then only 7 and 9 remain. But there are no multiplication sets that produce 79 or 97. So, that’s it. It’s over,” he explains. While organizing these thoughts, the students actually used the following three “if” statements.

If you use 1 and 8, only 7 and 9 remain. That won't work.

If you use 1 and 7, only 8 and 9 remain. That won't work.

If you use 1 and 9, only 8 and 7 remain. That won't work.

This is an example of the organizational skills we want to cultivate in “number cases” which used to be taught in sixth grade. In a trial-and-error exercise, the children naturally attempt to organize a situation. In doing so, they often use the phrase “if”. While the students are thinking about the possibilities, it is important for the teacher to write the values up on the blackboard while repeating the possible combinations. This time, the numbers 9 and 7 are leftover, ending the game.

So I start a new round. This time, I have the students work in groups. Once again, things do not go on so well, and I hear the students say things like, “Ah, that's too bad. Only a few more. We have left 6 and 7,” or “We have left 9 and 2.” Eventually, you will hear some students get frustrated.

“9 again. I hate that number!” One student exclaims.

At this point, I make the students realize what is going wrong.

“So, I'm hearing several people say that the 9 was left again. What about the others?” I ask.

I hear lots of students report the same problem.

“In that case, we should use the 9 first,” someone suggests.

“But then again, there are no double-digit multiplication answers that include a 9” another student suggests.

Upon hearing this, I ask all the students whether that is true.

“Really, none of them have 9s?”

The students go through the multiplication tables quietly in their heads.

“I've got it!” one student shouts.

And it is true. There is one: $7 \times 7 = 49$. Understanding this, the students realize that $3 \times 8 = 24$ cannot to be used since it uses the 4.

“Oh, so that's why the teacher started out with $3 \times 8 = 24$. That was tricky,” the students realize.

The students laugh and try it again.

“This time, we should start with 49!” they exclaim.

The truth is, the same problem occurs with the number 7. It only appears in the multiplication table results of 27 and 72. That is, it has to be used in combination with a 2. Again, the students came to realize the teacher's true motives by starting out with 24, and they start to accuse the teacher of tricking them. The important point of this lesson is for students to learn from the failed outcome of their own trial-and-error efforts. I emphasize that carefully recording one's failures can be very helpful to help one proceed to the next step. The starting phrase "if" is particularly helpful in this organizational exercise.

Mathematician Yasuo Akizuki of the Tokyo University of Education once commented that, "One of the most important skills for children to learn in elementary school math is their organizational abilities." The use of the starting phrase "if" is a manifestation of those skills. It is a phrase used in many different scenarios, like that described above, when working through different trial-and-error options or when changing the conditions of a scenario while examining various possibilities.

6. Using "For Example," "If," and "In That Case" to Track Thinking Process

After this lesson, I have the students take notes of their activities in their notebooks or on drawing paper. That is, I have the children spend a designated period of time writing about how the lesson developed.

Every teacher knows that a lesson that involves a lot of discussion looks engaging but still usually gets only about 1/3 of the students actively involved. To ensure that each student acquires the skills intended by the lesson, it is therefore important to create a scenario that requires each of them to take his/her own position and express his/her own ideas. Teachers must recognize this even when conducting lessons that involve class discussion, and I believe it is essential to create a specific time for students to reflect back on their own thinking processes that is different from the time during which they work together to improve their thinking capabilities. This could be viewed as the time for them to have a conversation with themselves.

I have the students spend the entire 45-minute class period taking notes. However, as mentioned above, this has to come after you explain that there is great

value even in the failed results of their trial-and-error attempts, otherwise some of them will not know what to write. This exercise does not work well when the students think they are only supposed to write about their best ideas or the right answer. When their sense of values about the exercise are changed, children who experienced repeated failures during the trial-and-error exercise can write even more than those who might have had fewer failures.

It is important to continuously remind students that you want them to be able to write something like the following: “I kept experiencing the following kind of failure. After thinking about why this did not work, I thought of a different way to approach it.” I let them know that they did well even if they did not ultimately end up on a path that led to the right solution. I want them to learn that when they encounter a problem they cannot solve, they might be able to solve it if they just rearrange the way the problem is structured. This teaches them that the lesson does not end once a single problem is solved but rather helps them learn to approach new problems by asking themselves, “In this case, what would happen?”

I think it is best to start with lessons on topics on Function when teaching this kind of perspective. When working on function problems, students start by looking at the structure of the given problem. In the process of doing this, the children will spontaneously replace numbers or scenarios with one that is easier for them to understand the problem. Using the results they obtain from those, they are then able to discover the results that apply to the entire problem. Sometimes they realize that something is possible if certain numbers are used but impossible if other numbers are used. In this way, they are engaged in a trial-and-error process by repeatedly changing the numbers used.

In this series of processes, they are able to realize the ease of using formulaic expressions and of understanding explanations given using diagrams. Also, since it is natural to wonder if the rule could be applied even when the conditions are changed, they learn to have a positive attitude about working with and developing solutions to problems that are given to them. This is a time when they have a chance to try a lot of approaches, considering questions, such as “What will happen if the number is larger?” or “What will happen if I change the structure of the problem slightly?”

I do not want to treat the children mechanically, based on whether they understand or do not understand the problem. Instead I want to help them express where they are in the middle of a process, for instance “I know what to do if I take this action.” The students’ “starting phrases” that were described earlier are helpful in allowing students to work on a problem and create activities that develop into the next steps.

This is why the phrases “for example,” “if,” and “in that case” appear frequently in their notes. How often these phrases appear in their notes can be used as a benchmark for evaluating the teacher. The following two points should be reiterated and communicated to the students who are not sure what to write:

1. Try to describe the failing attempts you made and then think about what you should probably try next based on what you have learned.
“For example...” “In that case...”
2. You might not know the answer to a particular problem, but think about what changes can be made that *would* allow you to be able to solve the problem.
“If...” “In that case...”

7. Techniques for Improving the Students’ Representation Capabilities through Lessons That Promote Cooperative Thinking Processes

Finally, I would like to discuss two techniques that I use when conducting lessons aimed at promoting cooperative thinking processes.

(1) Interrupting students during their presentations

People who come to observe my classes often ask the following: “You often stop a child who is speaking and ask another child to finish what the first child was saying. What is the purpose of this?”

I do this because a class in which the students simply listen to presentations given by their peers is no different from one who learns everything by reading a book. Regarding my discussion on writing about their failures, we need to think deliberately about how to conduct lessons on subjects whose purpose is to cultivate

the students' ability to think.

For example, giving the faster thinkers a chance to express themselves allows the other students to taste the same sense of discovery. This is why, rather than having a single student give an entire presentation, I stop the student halfway through and then have everyone in the class think together about how to reach the conclusion.

One advantage of this approach is that it makes students realize that they can divide a single thinking process into several segments. For a child to be able to start saying, "I understand up to this point, but...", they need to have the ability to organize their own thinking processes into steps. Utterances by students, such as "I get it up to this point, but..." or "I understand up to there, but what comes next is harder," are proof that students are thinking about a subject theoretically. When this happens, more students start to interrupt and ask questions on their own while their peers are giving a presentation.

If the teacher arranges the class such that students are allowed to interrupt and express themselves during their peers' presentations, students will come to learn that this makes things easier to understand. "Wait a minute. I understood what you said up to that point, but I didn't understand what followed. Please explain it more slowly."

This technique helps to increase the number of students who get actively engaged in what his/her peer is saying by interrupting him/her. This could never happen in a formally conducted class, where the teacher instructs the students to sit quietly and listen to their peers' presentations all the way to the end. Teaching students to listen carefully to a presentation from beginning to end is certainly also important, but this needs to be premised on lessons in which the speaking students have learnt to say what they want to say clearly and concisely. Listening to a long, rambling presentation is unnecessary difficulty for children.

To enable students to interact naturally with one another, I think the classroom style should be changed from one of formal presentations to one where students are encouraged to respond interactively to their peers' ideas. The promise that one will listen to what a friend is saying all the way to the end is effective only after the friends have become actively engaged with each other.

When students know that they are being listened to carefully, they will try to express their ideas clearly and concisely. They will make special efforts to express themselves in a certain way because they have something they really want to communicate. Surely, conversing with attentive listeners will go a long way toward cultivating speakers with the ability to express themselves in diverse ways.

(2) Ask whether they understand the speaker's feelings

Rather than quickly correcting your peer's mistakes, try to think about the issue from his/her perspective. You may discover that even if your peer's conclusion is wrong, he/she followed the same thinking process that you did. Thus, I often ask students in my class, "Do you understand what this student is feeling?"

Teaching representation capabilities is fundamentally related to the way that children interact with one another. Students will learn to develop a straightforward way of expressing themselves only in supportive environments where they mutually support each other. Children who have been laughed at for having the courage to explain an idea they were unsure of will hesitate to express themselves again in the future and will instead tend to participate in the class with a kind of emotional armor in tact. However, if a student's friends continue to support him/her even after he/she expresses a vague opinion or idea in which he/she did not have a lot of confidence, that student will enjoy expressing himself/herself in class.

Common to both of these techniques is the concept of "putting oneself in someone else's shoes." Asking whether one understands the speaker's feelings is one way of cultivating children who can "put themselves in the shoes of the speaker." The approach of expressing oneself by dividing one's thoughts into segments is really the approach of a child who "puts himself/herself in the shoes of the listener. It is important that listeners try to understand the perspective of speakers and vice versa.

8. Conclusion

I would like to conclude this paper with a brief explanation of the value of focusing on self-representation capabilities in math classes. One benefit is that it

changes the way children perceive math.

If children repeatedly learn that the only way to express themselves is to raise their hand and give an answer, they will learn that expressing oneself means presenting oneself perfectly. However, there are a lot more situations in this world where problem-solving efforts result in conclusions that are less than certain. The experience of discussing one's hypothesis with others, identifying the weaknesses in one's predictions, and recognizing the faults in one's hypothesis resembles real-life experiences much more closely than the experience of going from hypothesis to conclusion all on one's own.

The ability to clarify and adjust one's own position by discussing it with peers is an important advantage of group learning. This teaches students to realize that the things they unconsciously see every day or use every day can be viewed in a certain way if they just change the way they look at them. This is very similar to the way we interact with each other. You can discover unexpected aspects of your friends by looking at them from a different perspective. Also, by experiencing what it is like to be able to solve a whole new problem by reconstructing something they have already learned, children come to realize that learning is not just studying but creating something on their own.

Learning that every problem leads to a new problem and that continuously solving those problems can be fun leads to the realization that math problems do not just occur when they are presented in a one-sided format by the teacher but also can be created by the students themselves.

Viewed from this perspective, it becomes clear that creating math lessons actually plays a significant role in the ways that children live their lives. In the future, we need to focus not only on just taking great pains to convey knowledge and skills but also on creating lessons that are deliberately intended to create well-rounded students equipped with important living skills.