Emergent Modeling and Iterative Processes of Design and Improvement in Mathematics Education

Koeno Gravemeijer

Eindhoven School of Education
Eindhoven University of Technology
The Netherlands

Lesson studies

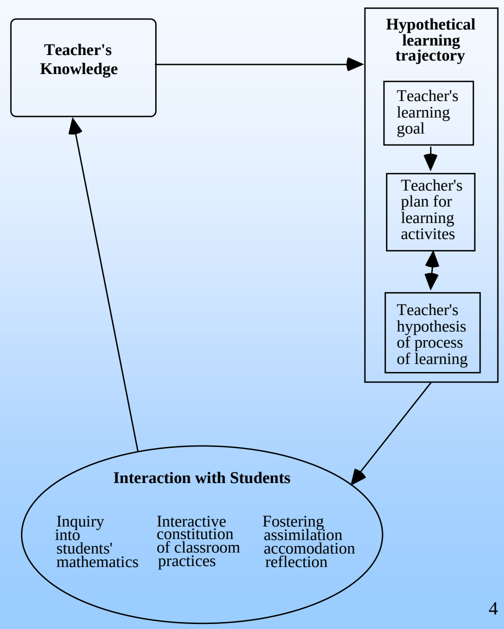
- Teachers design, try out, observe, analyze, and improve innovative lessons collectively
- Alternative for top-down innovations

Hypothetical Learning Trajectory

- “constructivist instruction”
- Teachers try to anticipate what mental activities the students will engage in when they participate in the envisioned instructional activities, and consider how those mental activities relate to the end goals one is aiming for.

3

Simon’s mathematical teaching cycle



4

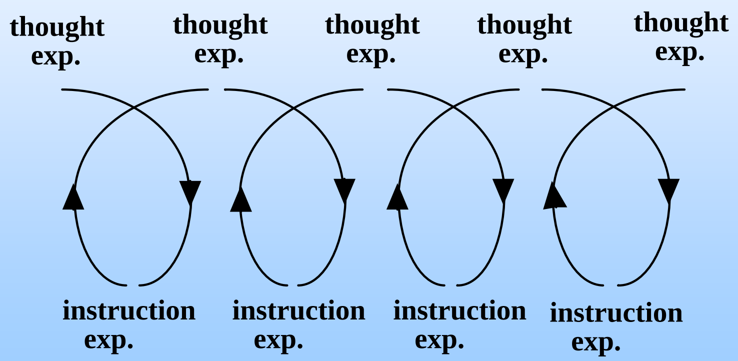
Local Instruction Theories

- If you want to build on the ideas and input you have to plan ahead
- You have to create experiences for the students on the basis of which they may come up with productive ideas
- In this context it is helpful to disgn instructional tasks that may generate a variety of solutions

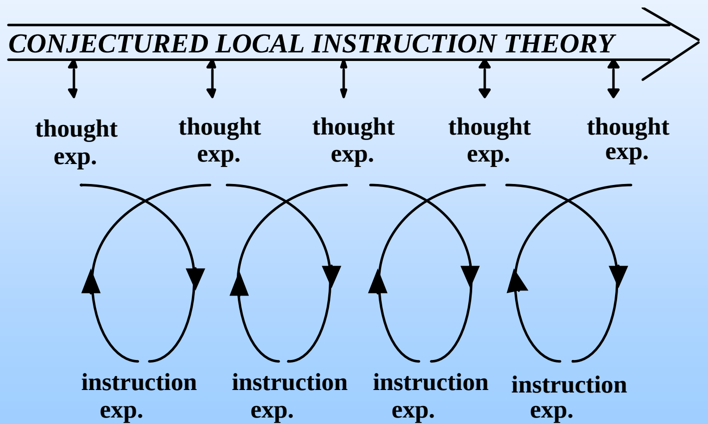
Local Instruction Theories

- A theory about a possible learning process, and the means of supporting that process
- Local = tailored to a given topic, such as addition of fractions, multiplication of decimals, or data analysis

Design Research



Design Research



This talk

- Point of departure: necessity of *local instruction theories* for helping teachers in helping students in constructing, or reinventing, mathematics
- Backbone of *local instruction theories*: RME instructional design heuristics, especially ‘emergent modeling’
- First: Need for ‘constructing’ versus ‘instruction’
  ⇔ What makes mathematics so difficult?

What makes mathematics so difficult?

A common view on learning

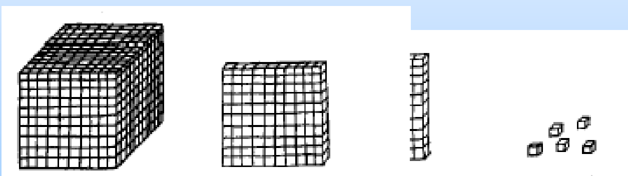
- Common view: Learning by making connections between what is known and what has to be learned

→Learning Mathematics: making connections with an abstract, formal body of knowledge

Designing visual and tactile models to bridge the gap

Didactical Models

- Didactical models: trying to show the mathematics



- But how are the students to see the mathematics they do not know yet?

Didactical Models

1128 Supporters want to visit the away soccer game of Feijenoord.
One bus can carry 38 passengers.
> How many busses will be needed?

13

Didactical Models

38 / 1 2 9 6 ¥

14

Didactical Models

38 / 1 2 9 6 ¥ *3*

<u>1 1 4</u>   ← How many tens? *3* x 38 = 114

Didactical Models

38 / 1 2 9 6 ¥ *3* *4*

<u>1 1 4</u>

1 5 6

<u>1 3 2   ← How many ones? *4*x38=132</u>

4

people busses

38 / 1 2 9 6 ¥ 3 4

1 1 4

1 5 6

1 3 2

4

Auburn

Auburn 'Grade 1'

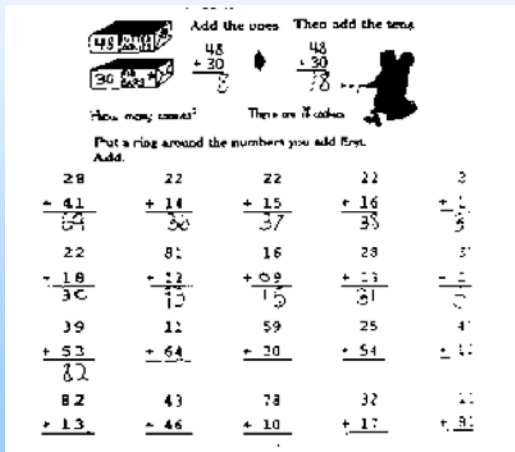
- 16 + 9 =
- 28 + 13 =
- 37 + 24 =
- 39 + 53 =

Auburn

Auburn ‘Grade 1’

- 16 + 9 = 25
- 28 + 13 =
- 37 + 24 =
- 39 + 53 =

Auburn



Worksheet

Auburn’s solution:

$$\begin{array}{r} 16 \\ \underline{9 +} \\ 15 \end{array}$$

interviewer (I), Auburn (A):

I : Is that correct that there are two answers?
A : ?
I : Which do you think is the best?
A : 25
I : Why?
A : I don't know.
I : If we had 16 cookies and another 9 added, would we have 15 altogether?
A : No.
I : Why not?
A : If you count them altogether you would get 25.
I : But this (15) is sometimes correct?
Or is it always wrong?

A : It is always correct.

A : It is always correct.

Two answers two worlds: school mathematics & reality

Problems with the common view on learning

- 1: The new mathematical knowledge the students have to connect with does not yet exist for them.
- 2: The learning paradox
  - – The symbols that one needs to get into the new mathematical domain, derive their meaning from that very domain.

The new mathematical knowledge does not exist yet: Early number as an example

- Young children don´t understand the question: "How much is 4+4?

  Even though they know that "4 apples and 4 apples makes 8 apples"
- Ground level: Number tied to countable objects: "four apples"
- Higher level: 4 is associated with number relations:

  4 = 2+2 = 3+1 = 5-1 = 8:2

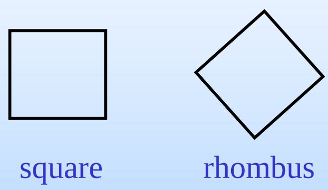
Miscommunication between teacher and students

- Student are thinking at the level of countable objects
- Instruction on the level of number relations;
  - Note: Telling students that 2+2=4, etcetera, will not help if the students do not know what '2+2' means.

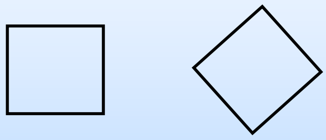
Gap between teacher and student knowledge: Different frameworks of reference

- Problem identified by the Van Hieles
- Van Hiele (1975): Teachers and students have different frameworks of reference
- It is as if they speak different languages;
- Or worse: They use the same words but with a different meaning

Van Hiele example: the concept ‘rhombus’ in geometry



Van Hiele example : the concept ‘rhombus’ in geometry



square rhombus

- Sides are two by two parallel
- All sides have equal lengths
- Diagonals intersect orthogonal
- Facing angles are equal

29

Consequences of the common view

- The body of knowledge only exist in the minds of teachers and textbook authors; how can students connect to a body of knowledge that does not exist for them?
- The learning paradox: Mathematical symbols derive their meaning from a certain mathematical domain. However, you need to understand those symbols to enter that domain.

30

Consequences of the common view

- Some people manage to reinvent mathematics even if it is not taught that way (but as "Learn first, understand later")
- Most don't, they learn definitions and algorithms by heart →
  - Problems with applications
  - Problems with understanding
  - Math anxiety

Alternative: Learning mathematics as a process of personal growth

- Helping students to expand and build upon their own (informal) mathematical knowledge:

- Structuring quantities;
  4 apples = 2 apples + 2 apples
  4 marbles = 2 marbles + 2 marbles
  - Curtail counting; explain & justify

- Investigating geometrical relations (rhombus)

Freudenthal: Mathematics as an activity

- It is an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter. This can be a matter from reality which has to be organized according to mathematical patterns if problems from reality have to be solved. It can also be a mathematical matter, new or old results, of your own or others, which have to be organized according to new ideas, to be better understood, in a broader context, or by an axiomatic approach.

Freudenthal: Mathematics as an activity

- Freudenthal (1973): mathematics as an activity of doing mathematics; most importantly, an activity of organizing or mathematizing subject matter,
  - Subject matter from reality
  - Mathematical matter
- Mathematizing: generalizing, formalizing, proving, curtailing, defining, axiomatizing
- And this we teach: Anti-didactical inversion

Realistic Mathematics Education

- Mathematics as an activity
- Students should be given the opportunity to reinvent mathematics
- Instructional-design heuristics
  - Guided Reinvention/mathematizing
  - Didactical Phenomenology
  - Emergent modeling

Guided Reinvention Through Progressive Mathematizing

- A route has to be mapped out that allows the students to (re)invent the intended mathematics by themselves
  - history of mathematics
  - informal solution procedures

Long Division

1128 Supporters want to visit the away soccer game of Feijenoord.
One bus can carry 38 passengers.
A reduction will be given for every ten buses.

37

1296 : 38

1296
38 -   1 x
1258
38 -   1 x
1220
38 -   1x
1182
38 -   1 x
1244
38 -   1x
….

Repeated subtraction

38

1296 : 38

38/1296 ¥ *34*		38/ 1296	¥ *34*	38/ 1296 ¥	*34*
380 -	10x	380 -	10x	1140 -	30x
916		916		156	
380 -	10x	760 -	20x	152 -	4x
536		156		4	
380 -	10x	76 -	2x		
156		80			
38 -	1x	76 -	2x		
118		4			
38 -	1x				
80					
38 -	1x				
42					
38 -	1x				
4					

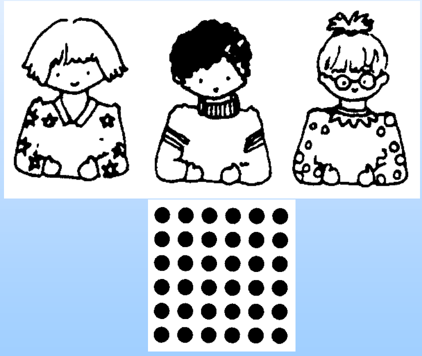
Various levels of curtailment

Didactical Phenomenology

- Phenomenology:

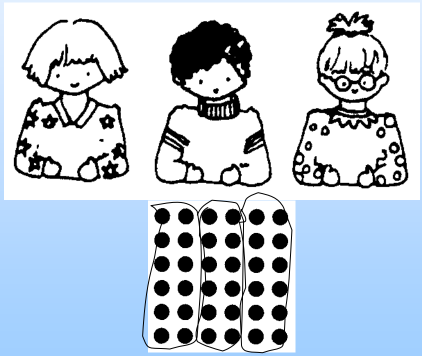
  how mathematical “thought things”, like tools or concepts help organize certain phenomena
- Look for applications & points of impact
- Goal: To find the phenomena and situations that may create the need for the students to develop the mathematical concept or tool we are aiming for.

Didactical phenomenology in **division**



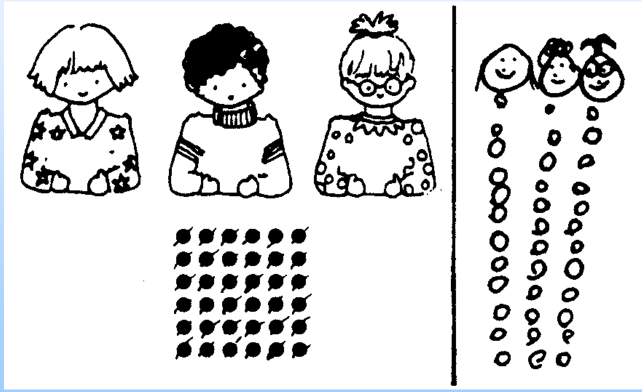
Three students dividing 36 sweets

Didactical phenomenology in **division**



Geometric division

Didactical phenomenology in **division**



Piece wise distribution

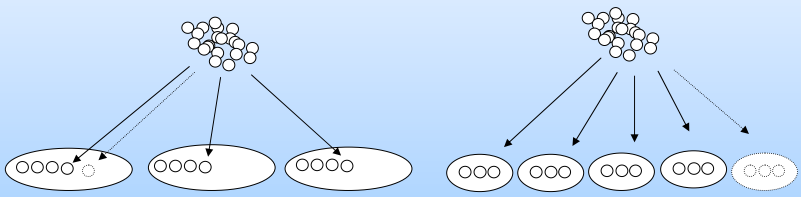
43

Didactical phenomenology in **division**

triads

44

Two phenomenologically different forms of division



distribution division ratio division

Emergent modeling

- Mark that we may discern three types of modeling in mathematics education
  - Use of didactical models
  - Mathematical modeling
  - Emergent modeling

Mathematical modeling

- Mathematical model and the situation modeled are treated as separate entities: *“goodness of fit”*
- Problem solving activity
- Learning process

  (Where does the mathematics come from?)

Emergent Modeling: a long-term learning process

From a *model of* the students' situated informal strategies

Towards a *model for* more formal mathematical reasoning

Key in this process:

a shift in attention: from context to mathematical relations => building a framework of mathematical relations

Emergent modeling: Long division

38/1296 ¥ *34*

380 -	10x
916	
380 -	10x
536	
380 -	10x
156	
38 -	1x
118	
38 -	1x
80	
38 -	1x
42	
38 -	1x
4	

repeated subtraction as a *model of* transporting supporters

Emergent modeling: Long division

38 / 1 2 9 6 ¥ *3 4*

1 1 4	⇔ minus 30 x 38 = 1140
1 5 6	
1 3 2	
4	

repeated subtraction as a *model for* mathematical reasoning

Emergent Modeling

Situational level:
Activity in the task setting, in which interpretations and solutions depend on understanding of how to act in the setting (often out of school settings)

51

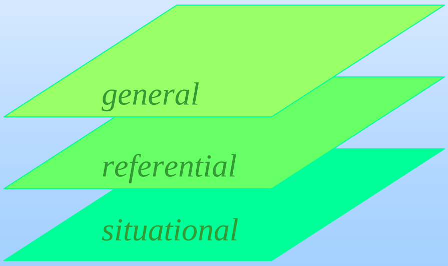
Emergent Modeling

Referential level:
Referential activity, in which the model derives its meaning from the reference to activity in the task setting, and functions as a <u>model of</u> that activity.

52

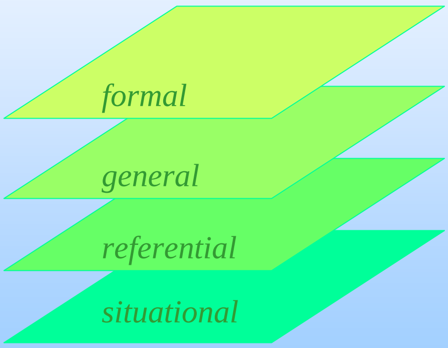
Emergent Modeling

General level:
General activity, attention shifts towards mathematicals relations, the model starts to derive its meaning from those mathematical relations, and becomes a <u>model-for</u> more formal mathematical reasoning



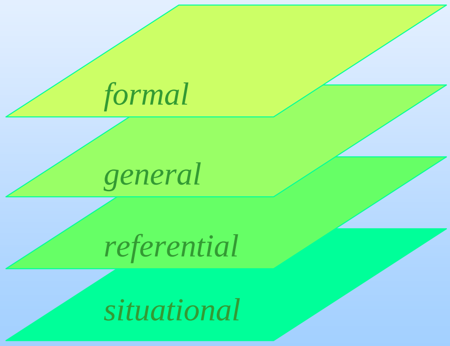
Emergent Modeling

Formal level:
Formal mathematical reasoning, which is no longer dependent on the support of models



Emergent Modeling

Formal level:
More formal, in that it relates a framework of mathematical relations that is new to the students.
(New mathematical reality).



Emergent Modeling

- *"The model"* as an overarching concept
  = a series of consecutive *sub models* that can be seen as various manifestations of the same model
- Shift in the role of "the model" on a more global level = various models that take on different roles

model of => model for

- initially, models refer to concrete situations, which are experientially real for the students
- the model gets a more object-like character
- becomes a base for mathematical reasoning

A *model of* informal mathematical activity becomes a *model for* more formal mathematical reasoning

The Empty Number Line

- Example local instruction theory (developed at Vanderbilt University)
- Local instruction theory on flexible strategies for addition and subtraction up to 100

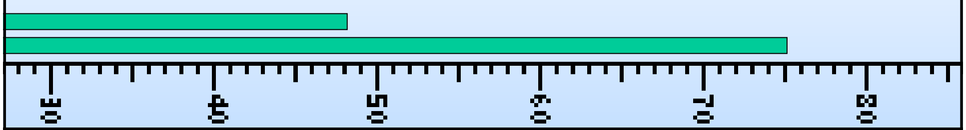
The Empty Number Line

- instructional sequence on flexible strategies for addition and subtraction up to 100
- informal solution procedures of students
  - splitting tens and ones
    - 44 + 37 = …
      40 + 30 = 70; **4 + 7** = 11; 70 + 11 = 81
  - counting in jumps
    - 44 + 37 = …; **44 + 30** = 74; 74 + **7** = 81,

    or:
    - 44 + 37 = …; **44 + 6** = 50; 50 **+ 10** = 60;
      60 **+ 10** = 70; 70 **+ 10** = 80; 80 **+ 1** = 81

Linear-type problems: Comparing lengths

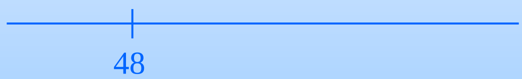
difference between 48 and 75 cm



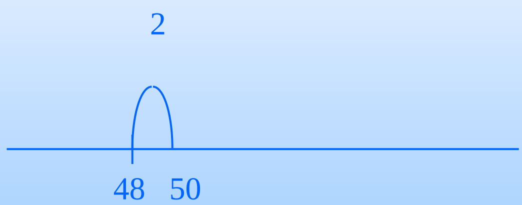
*Reasoning

48+2=50
50+10=60
60+10=70
70+5=75 difference: 2+10+10+5=27

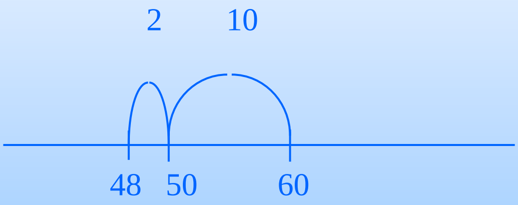
Empty number line



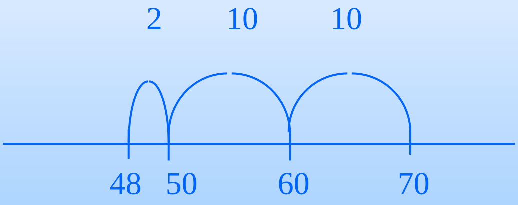
Empty number line



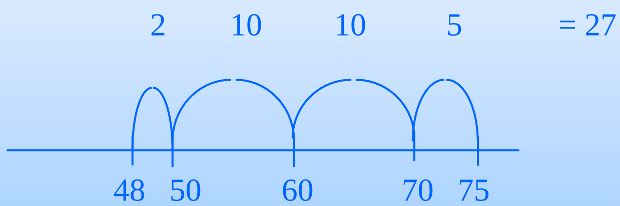
Empty number line



Empty number line

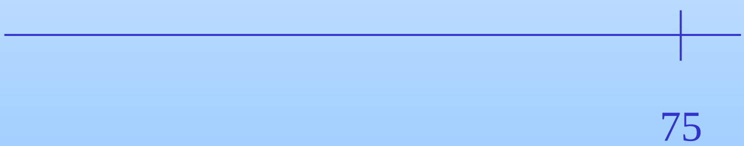


Empty number line

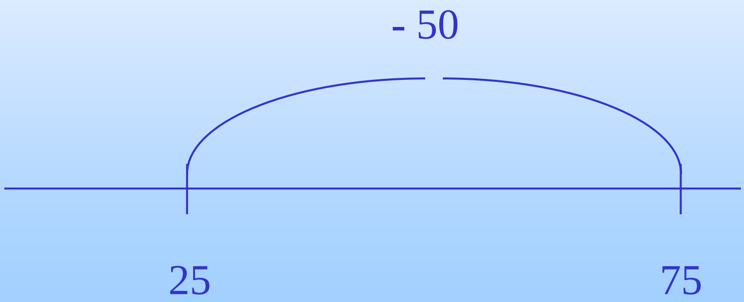


48+…=75

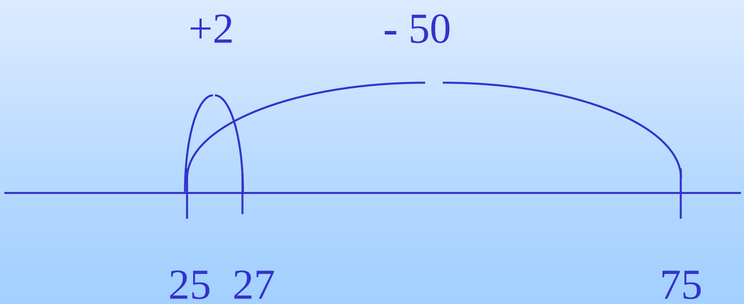
75 - 48



75 - 48

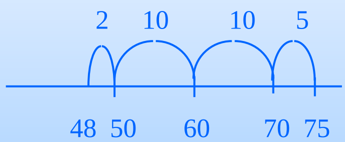


75 - 48

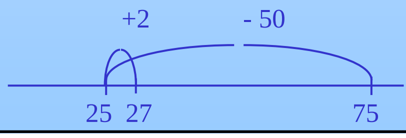


Empty number line

- Initially, the focus is on the relation between the context problem and the number line.



- Later the numerical/mathematical relations become more important

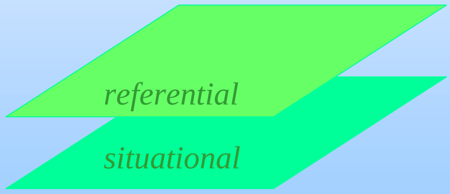


Emergent Modeling

Situational level:
Measuring context

Emergent Modeling

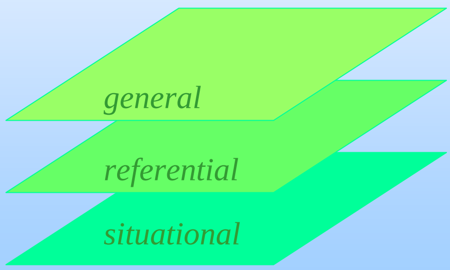
Referential level:
Describing strategies for reasoning in the measuring context with jumps on the number line



71

Emergent Modeling

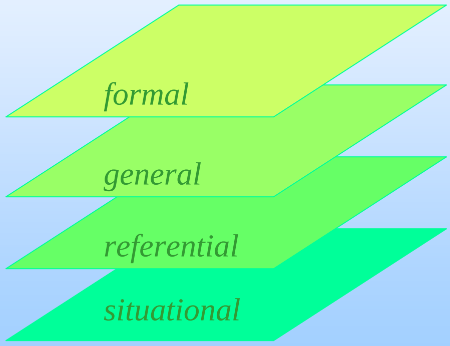
General level:
Describing strategies for reasoning with number relations with jumps on the number line



72

Emergent Modeling

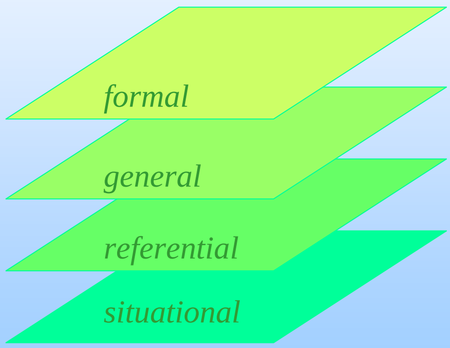
Formal level:
Reasoning within a framework of number relations without the support of the number line



73

Emergent Modeling

Formal level:
Students have created a framework of mathematical relations that is new to the students.



74

The shift from model-of to model-for is reflexively related with the creation of mathematical reality

The student's view of numbers transitions from

- numbers as referents of distances to
  - "37 feet"
- numbers as mathematical objects
  - "37"

⇔network of number relations:
37=30+7
37=3x10+7
37=20+17
37=40-3
etc.

RME as an Alternative

- Guided reinvention as a means for designing a learning route along which students can construct mathematics
- Didactical phenomenology as a means for finding potential starting points
- Emergent modelling as a means of circumventing the learning paradox by a dialectic process of symbolizing and development of meaning

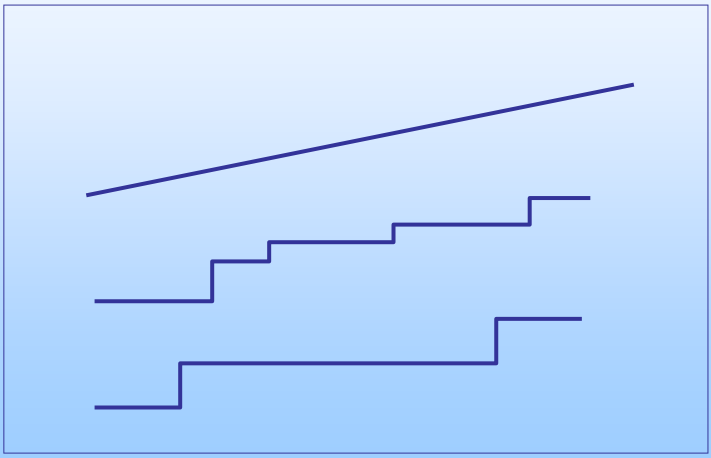
Emergent Modeling →
Starting points for instructional design

- *What constitutes the new mathematical reality we want the students to construe?*

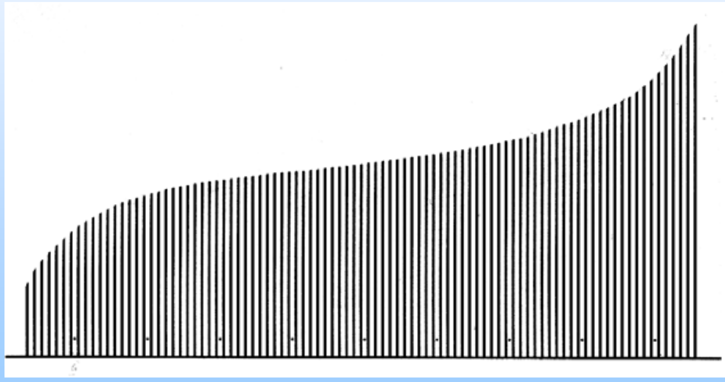
  → *What are the mathematical relations involved?*
- *What is the overarching model, and what do the underlying sub-models consist of?*

Interlude

- We pace 100 men in a row from small to tall. What will this row look like?

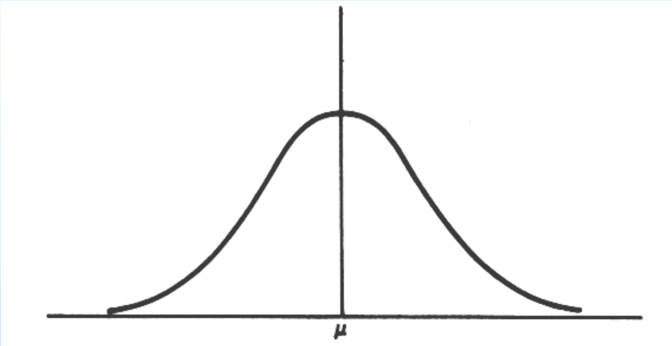


79



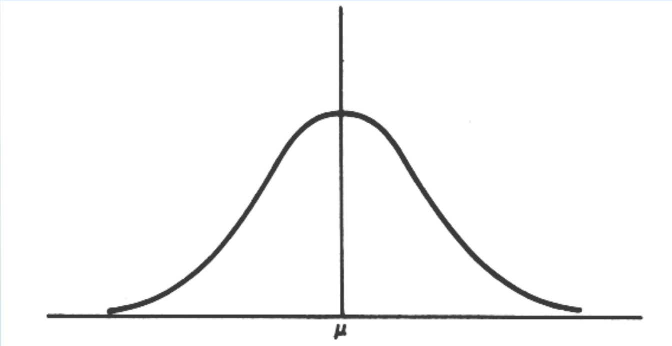
80

Bell curve



➢What does a point on the curve represent?

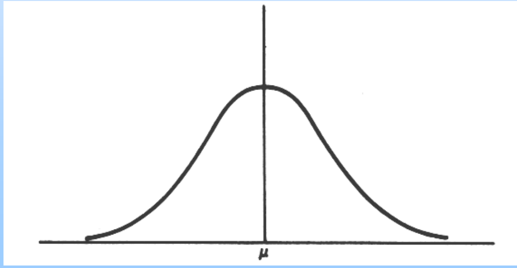
Bell curve



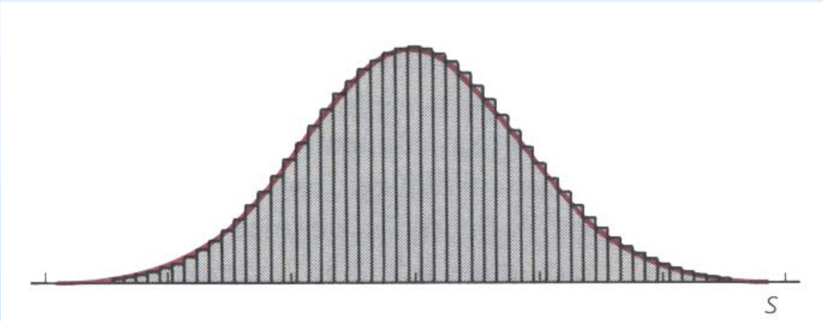
➢What does a point on the curve represent?
Height 178.23682 meter?

Learning Paradox

The picture of a bell curve does not tell you what a normal distribution is.



limit histogram
$\Delta \rightarrow 0$

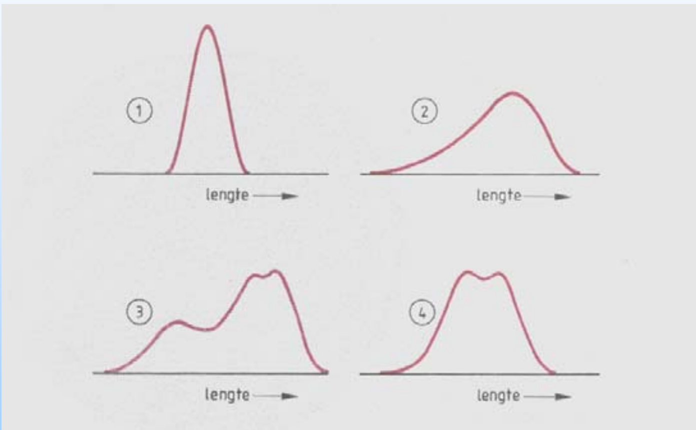


Local Instruction Theory on Data Analysis (Vanderbilt University)

- Traditional goals of a beginners course in statistics (grade 7 & 8): Mean, mode, median, spread, quartiles, histogram, ….
- *What constitutes the new mathematical reality we want the students to construe? What are the mathematical relations involved?*

  → Distribution as an object; density, shape, skewness, spread, …

Distribution as an object

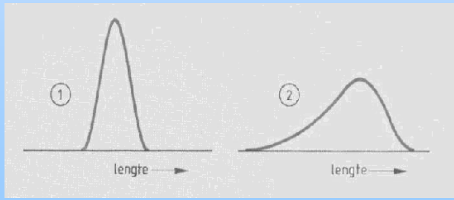


a) All Dutch,
b) All married Dutch
c) Families with parents <30
d) Dutch adult men

Distribution as an object

- Area ⇔ probability/density distribution
- Graph of a density function
  - height = density of data points around that value

- Distribution can be thought of in terms of shape and density
  - Spread
  - Skewness
  - Position



Design Heuristics in the context of data analysis

- Guided reinvention
- Didactical phenomenology
- Emergent models

Guided reinvention

- Reinventing distribution as an object
- Reinventing tools & measures (median, quartiles etc.)
  - means for getting a handle on a distribution => characteristics of a distribution
- Starting points experientially real
  - Data creation (⇔ reason)
  - Informal graphical representations

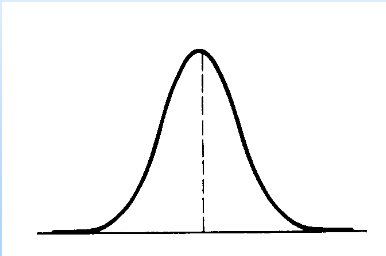
Didactical phenomenology

- Starting points: problem situations that may give rise to situation-specific solution procedures

- *Phenomenological analysis:* how the mathematical "thought thing" (nooumenon) organizes the phenomena

Phenomenological analysis

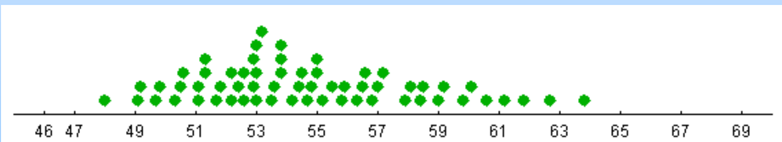
- Thought thing distribution
- “shape”



- density function = tool to organize density

Phenomenological analysis

- Thought thing density = tool to get a handle on how data points are distributed in a space of possible outcomes



Phenomenological analysis

- Thought things data points in a space of possible outcomes = tools to organize a set of measurement values

    - 48
    - 52
    - 61
    - 54
    - 59
    - 53

Didactical phenomenology

Designing an instructional sequence

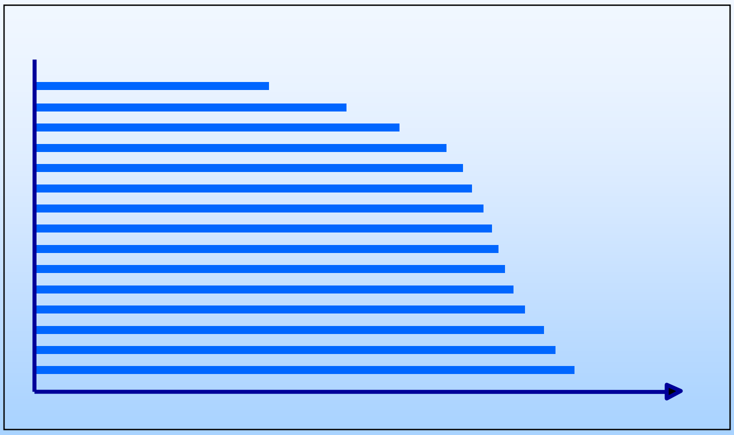
Solving applied problems, which gives rise to mathematizing or organizing:

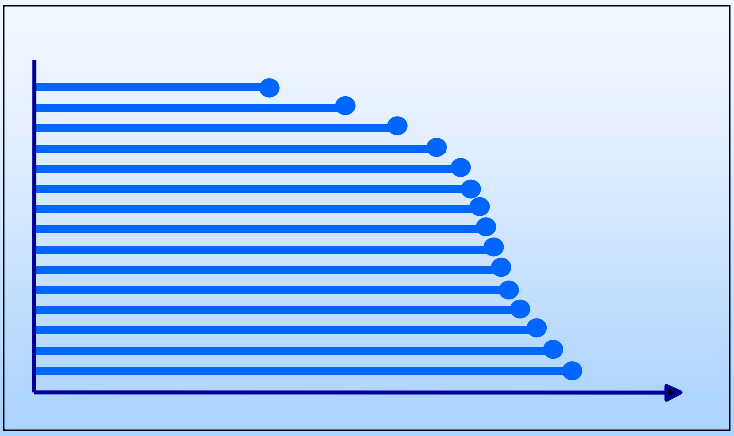
1. Organizing measurement values → data points on an axis
2. Organizing the distribution in of data points → density
3. Organizing density → density function

Emergent Modeling

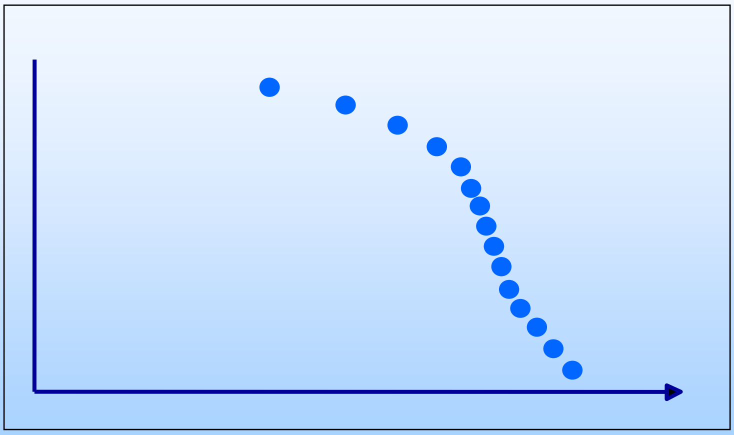
- Didactical Phenomenology informs Emergent Modeling:

  → Series of submodels

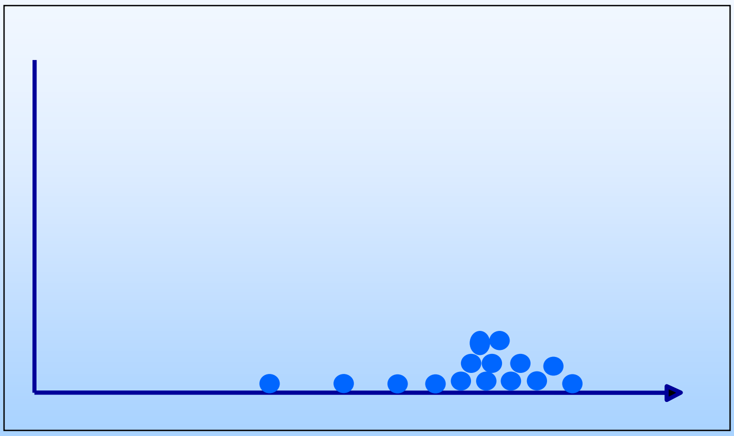




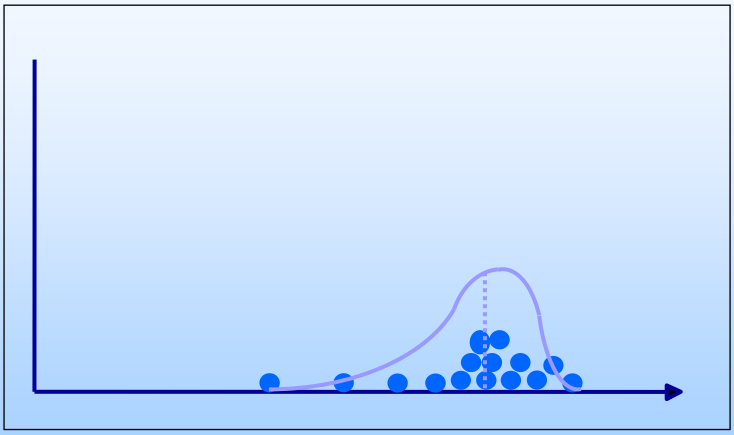
97



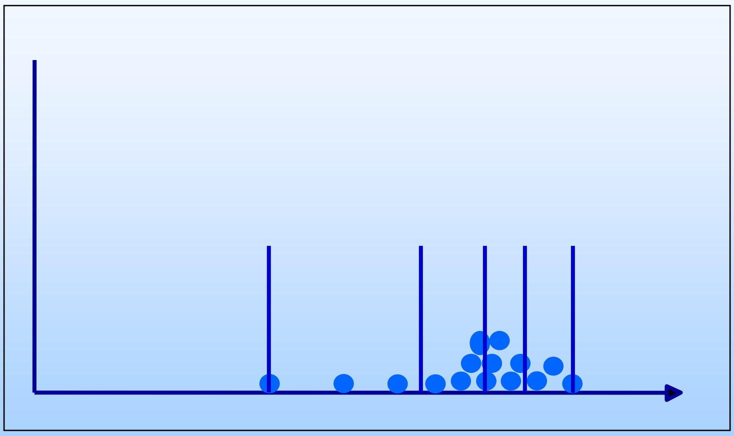
98

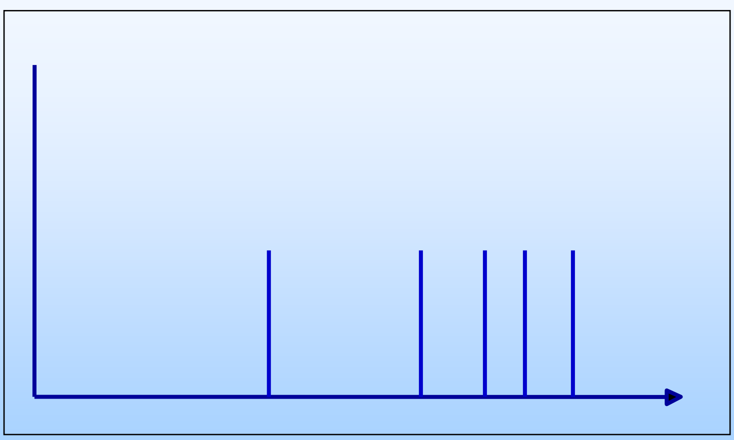


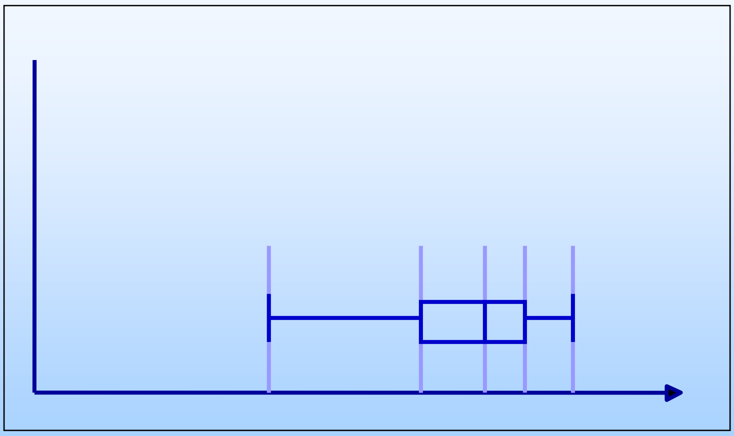
99

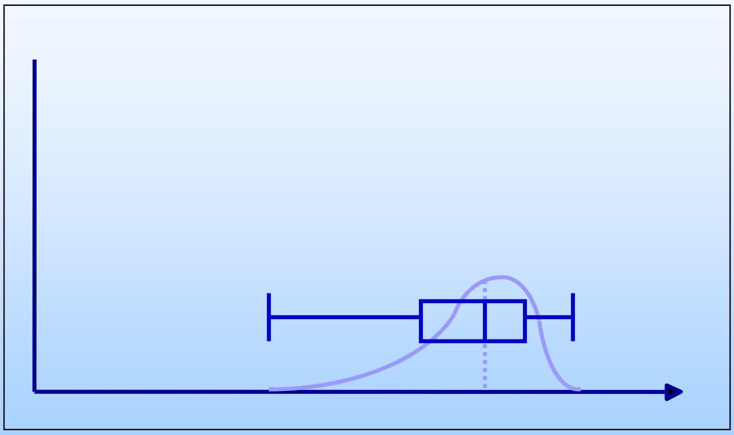


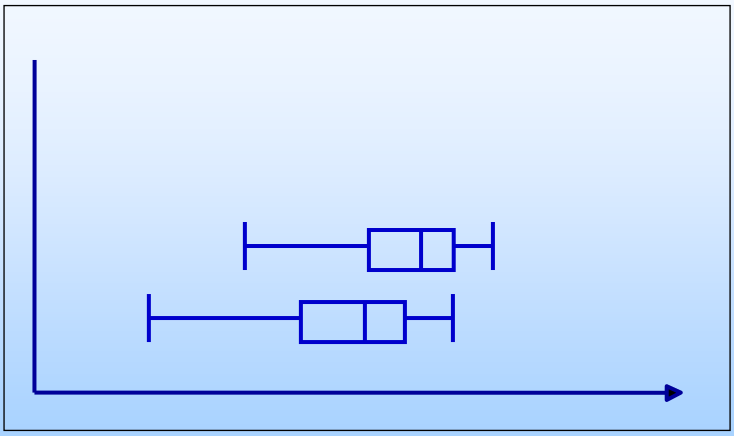
100



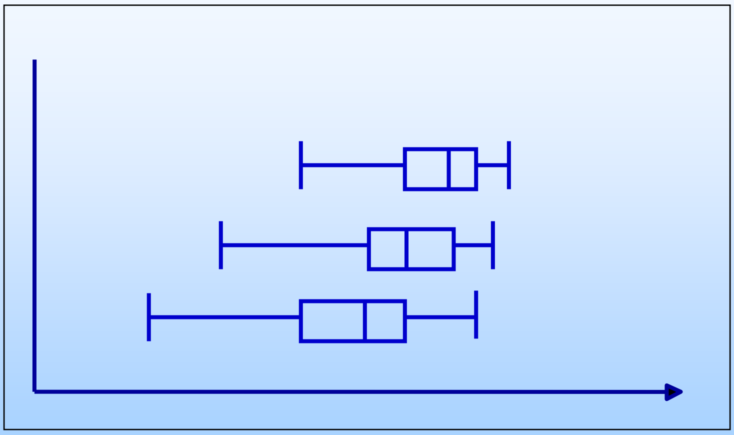




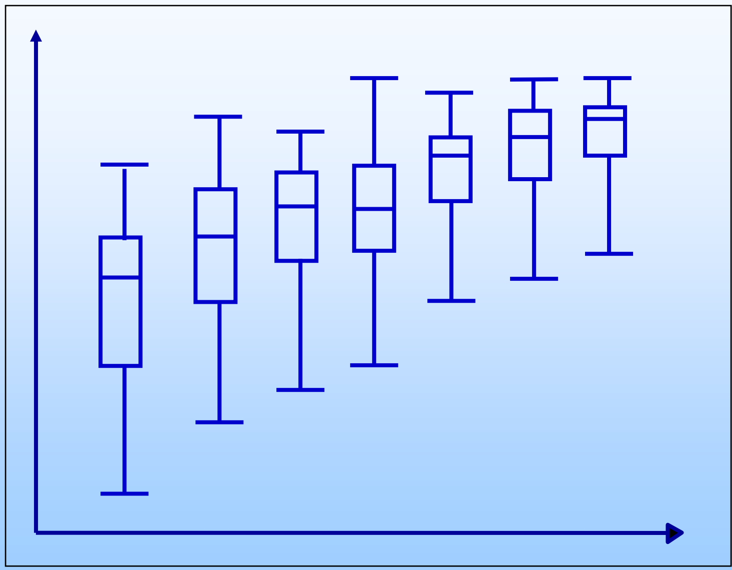


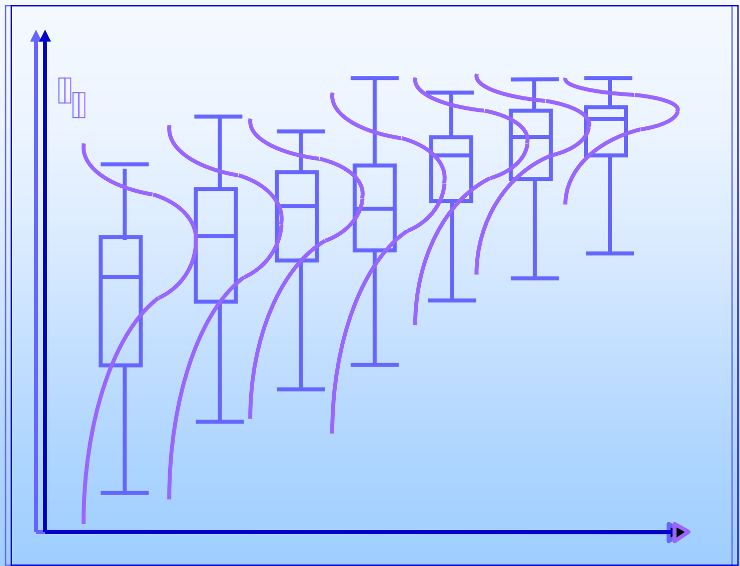


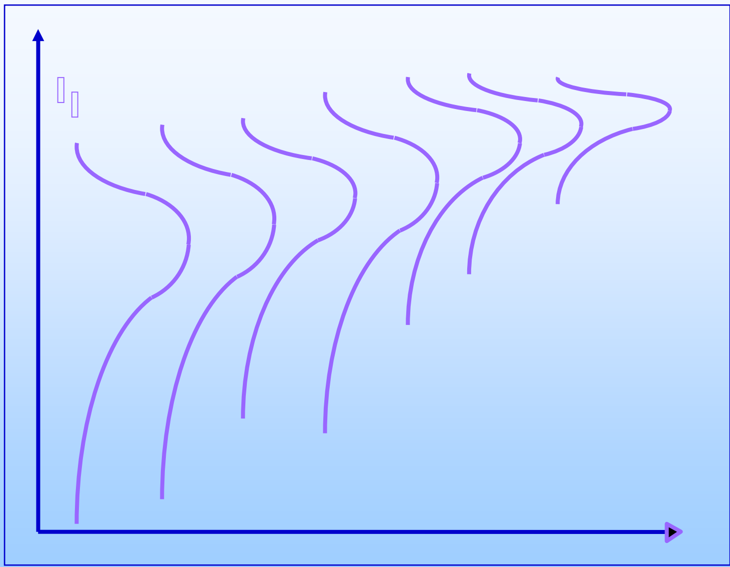
105



106

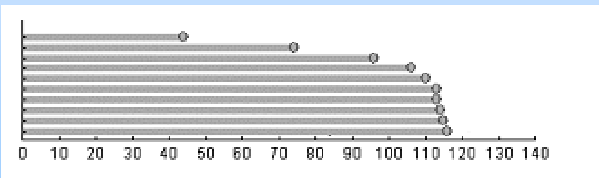






Emergent modeling

- The model: A graphical representation of the shape of a distribution
  - pre-stage of the model, where the distribution is still very much tied to the situation
  - Model of a set of measures



  - lifespan of batteries

Emergent modeling

- Potential endpoint: Box plot as a model for reasoning about a distribution

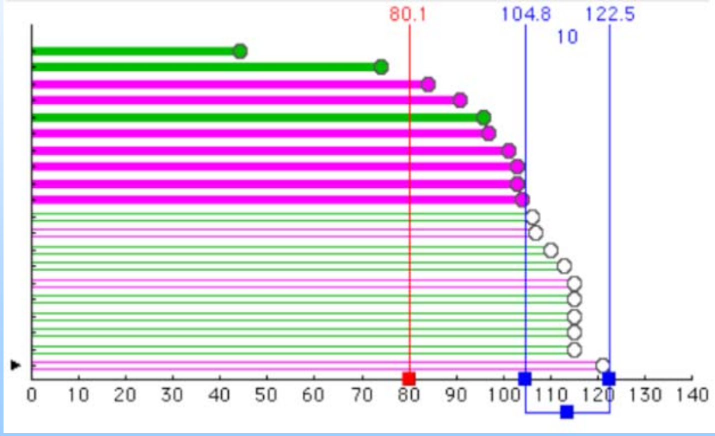
- skewed to the left
- density

Instructional Activities

- Comparing data sets (samples/badges of data) for a reason
- Question or a problem => Data creation
- Talking through the process of data creation
- “realistic” data sets & questions tailored at significant statistical issues

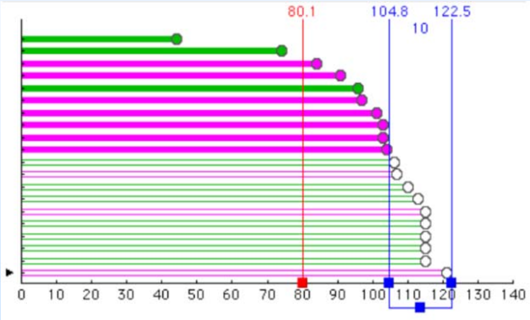
Classroom episodes

Battery life span



Batteries; life span

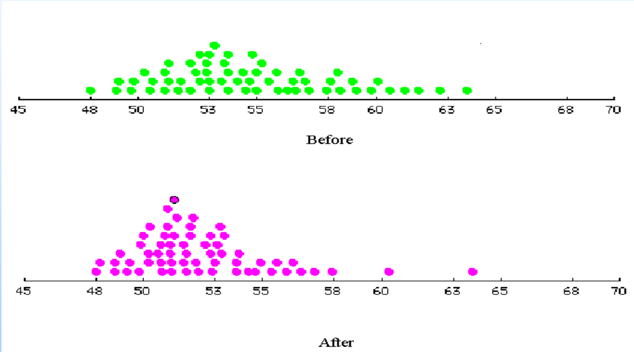
"consistency" versus "total"



- The life span of two brands of batteries

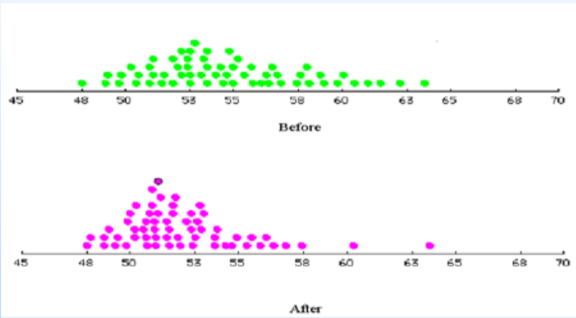
  "I would rather have a consistent battery (…) than one that you just have to try to guess"

Speed trap



Data of the speeds of cars before and after.

Speed trap: “the hill shifted”



“If you look at the graphs and look at them like hills, then for the before group the speeds are spread out and more than 55, and if you look at the after graph, then more people are bunched up close to the speed limit which means that the majority of the people slowed down close to the speed limit.”

Directions for Instructional Design

- Think through the endpoints of a given instructional sequence in terms of what mathematical objects, and the corresponding framework of mathematical relations
- Think through the model-of/model-for transition, consider what informal situated activity is being modeled, and what a potential chain-of-signification might look like.

Emergent Modeling Informs Teachers

- Emergent modeling explicates what mathematical relations to aim for.
- Emergent modeling clarifies what the mathematical issues are that are to become topics of discussion

Emergent Modeling Informs Teachers

- Emergent modeling informs teachers about the series of sub-models and about the process in which symbols/models and meaning co-evolve.

Central role of the teacher

- The teachers will have to respond to the students' thinking, they have to decide, for instance, which mathematical relations students start to grasp, and which are still to be worked on.

Central role of the teacher

- Teachers will also have to judge when a new sub-model might be introduced, and check whether that new (sub-)model is experienced as ‘bottom up’, which means that it signifies earlier activities with earlier (sub-)models for the students.

Iterative processes of design and improvement

Design Research, Local Instruction Theories, &Lesson Studies

- A combination of design research on local instruction theories, and lesson studies that build on, and feedback into, those theories might offer a powerful combination for improving mathematics education.

Thank You