What Is Mathematics?

A Pedagogical Answer with a Particular Reference to Proving

Guershon Harel
University of California at San Diego
harel@math.ucsd.edu
http://www.math.ucsd.edu/~harel

1

Two Fundamental Questions

1. **What mathematics should we teach in school?**
2. **How should we teach It?**

DNR's **stance: an overview**

2

Presentation Structure

Part I:

***DNR* as a conceptual framework: a synopsis**

Part II:

***DNR's* stance on the 2nd question:**
How should mathematics be taught?

Part III:

***DNR's* stance on the 1nd question:**
What mathematics should be taught?

3

Part I: What is *DNR*?

DNR-based instruction in mathematics is a theoretical framework consisting of a system of three categories of constructs:

- ***Premises***:
  explicit assumptions, most of which are taken from or based on existing theories.
- ***Concepts***:
  definitions oriented within the stated premises.
- ***Claims***:
  statements formulated in terms of the *DNR* concepts, entailed from the *DNR* premises, and supported by empirical studies.
  - ***Instructional principles***: claims about effects of *teaching practices* on *student learning.*

The term ***DNR*** refers to three foundational **instructional principles**:
–The ***D**uality* Principle
–The ***N**ecessity* Principle
–The ***R**epeated Reasoning* Principle

4

DNR Premises

Mathematics Premise

1. ***Mathematics:*** Knowledge of mathematics consists of all *ways of understanding* and *ways of thinking* that have evolved throughout history.

Learning Premise

2. ***Epistemophilia*****:** Humans—all humans—possess the capacity to develop a desire to be puzzled and to learn to carry out *mental acts* to fulfill their desire to be puzzled and to solve the puzzles they create. **Aristotle**
3. ***Adaptation:*** Learning is adaptation; namely, learning is a developmental process, which proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium—a balance between the structure of the mind and the environment. **Piaget**
4. ***Content:*** Learning is context dependent. **Cognitive Psychology**

Teaching Premise

6. ***Teaching:*** Construction of scientific knowledge is not spontaneous. There will always be a difference between what one can do under expert guidance or in collaboration with more capable peers and what he or she can do without guidance. **Vygotsky**

Ontology Premise

7. ***Subjectivity:*** Any observations humans claim to have made is due to what their mental structure attributes to their environment. **Piaget**
8. ***Interdependency:*** Humans' actions are induced and governed by their views of the world, and, conversely, their views of the world are formed by their actions. **Piaget**

5

Part II: The 2st Fundamental Question

How should mathematics be taught?

In *DNR*, teaching *effectively* entails:

- preserving the ***mathematical integrity*** of what we teach
- addressing the ***intellectual needs*** of the student
- assuring that students
  - ***internalize,***
  - ***Structure,***
  - ***retain***

  the mathematics they learn

6

The Necessity Principle

For students to learn the **mathematics** we intend to teach them, they must have a need for it, where 'need' refers to ***intellectual need***, not social or economic need.

7

Violation of the *Necessity Principle:* Some Examples

8

Premature introduction of algebraic symbolism

Tom and John are roommates. They decided to paint their room. Tom can paint the room in 4 hours and John, a perfectionist, in 8 hours. How long would it take them to paint their room if they work together?

Kate: … 4x + 8x …

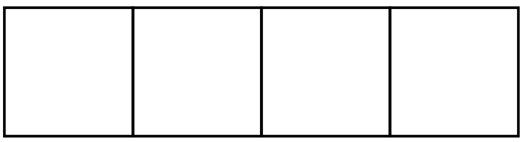
Teacher: What is x?

Kate: … x?... x is the house.

Teacher : You want to find x, … so you want to find the house?

Tom and John are roommates; they decided to paint their room. Tom can paint the room in 4 hours and John, a perfectionist, in 8 hours. How long would it take them to paint their room if they work together?

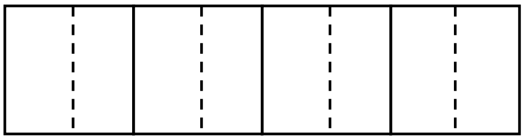
Kate:



Tom paints ¼ of the house in 1 hour.

Tom and John are roommates; they decided to paint their room. Tom can paint the room in 4 hours and John, a perfectionist, in 8 hours. How long would it take them to paint their room if they work together?

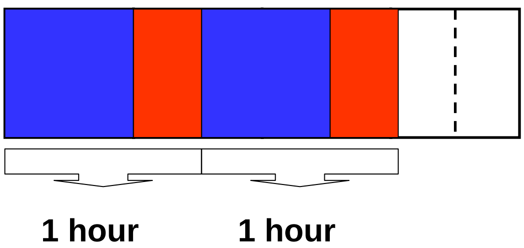
Kate:



John paints 1/8 of the house in 1 hour.

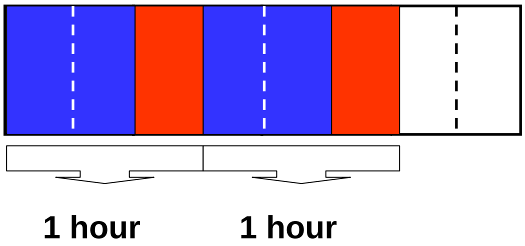
Tom and John are roommates; they decided to paint their room. Tom can paint the room in 4 hours and John, a perfectionist, in 8 hours. How long would it take them to paint their room if they work together?

Kate:



Tom and John are roommates; they decided to paint their room. Tom can paint the room in 4 hours and John, a perfectionist, in 8 hours. How long would it take them to paint their room if they work together?

Kate:



- ¾ of the house is painted in 2 hours, ¼ of the house will be painted in 2/3 of an hour.
- The whole house will be painted in 2 hours and 40 minutes.
- …This can't be right … there is no x …

13

intellectual need response

social need response

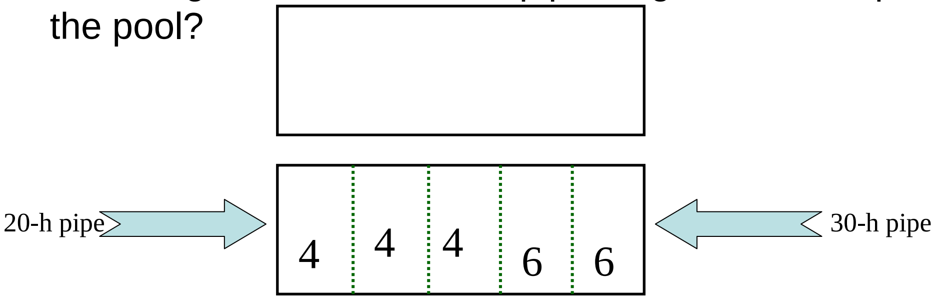
The house will be painted in 2 hours and 40 minutes

…This can't be right … there is no x …

14

Harriet’s solution to the *pool problem*:

A pool is connected to two pipes. One pipe can fill up the pool in 20 hours, and the other in 30 hours. How long will it take the two pipes together to fill up the pool?



15

Linear Algebra Textbooks

> “So far we have defined a mathematical system called a real vector space and noted some of its properties ....
>
> [In what follows], we show that each vector space V studied here has a set composed of a finite number of vectors that completely describe V. It should be noted that, in general, there is more than one such set describing V. We now turn to a formulation of these ideas.”

Following this, the text defines the concepts:

Linear independence

Span

Basis

and proves related theorems.

16

Is Students' Intellectual Need Considered?

- Can this "motivation" help students see a need for the pivotal concepts, *linear independence*, *span*, and *basis*?
- Can it constitute a need for the concept *finitely generated vector space* (alluded to in the "motivating paragraph)?
- Can it help students see how these three pivotal concepts contribute to the characterization of *finitely generated vector space*?

17

Five Categories of ***intellectual need:*** ***The 5 C's***

1. Need for ***c****ertainty*
2. Need for ***c****ausality*
3. Need for ***c****omputation*
4. Need for ***c****ommunication*
5. Need for ***c****onnection*

18

Categories of ***intellectual need***

(a) Need for *certainty*

Is it true that the sum of the angles in any triangle is 180˚?

(b) Need for *causality* (Need for *enlightenment*)

What makes the sum of the angles in any triangle 180˚?

(c) Need for *computation*

I am looking for a number whose square is less than its cube. Can I find all such numbers?

(d) Need for *communication*

How can I convince Jill that I am right?

(e) Need for *connection*

What does the negative solution of an equation mean in the physical reality represented by the equation?

19

Need for certainty
versus
Need for causality

20

Aristotle’s Definition of Science

“We do not think we understand something until we have grasped the why of it. … To grasp the why of a thing is to grasp its primary cause.” Aristotle, *Posterior Analytics.*

21

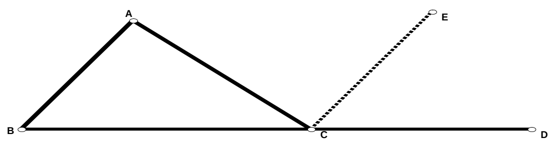
Mathematics is not a perfect science, argued 16-17th Century philosophers, because an "implication" is not just a logical consequence; it must also demonstrate the ***cause*** of the conclusion.

22

Euclid’s Proof to Proposition 1.32

The sum of the three interior angles of a triangle is equal to 180^0.

Proof:



23

“Mathematics is not scientific:” Examples of other arguments

- Proof by contradiction is not a causal proof since it does not provide sufficient insight of how the result was obtained.
- If mathematical proof are scientific (i.e., causal), then equivalent statements (i.e., “A iff B” statements) are an absurdity.

  “A implies B” means “A causes B”

  “B implies A” means “B causes A”

  Hence: “A causes A”—an absurdity.

24

Attempts to Conform to the Aristotelian Theory of Science

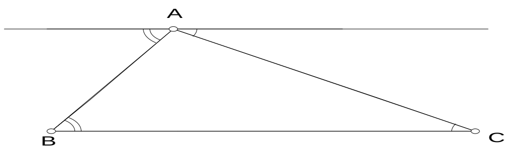
Rejection of proof by contradiction and use ostensive (causal) proofs

- Descartes’ appeal to a priori proofs against proofs by contradiction
- Cavalieri’s geometry of indivisibles
- Insistence on movement in geometrical proofs

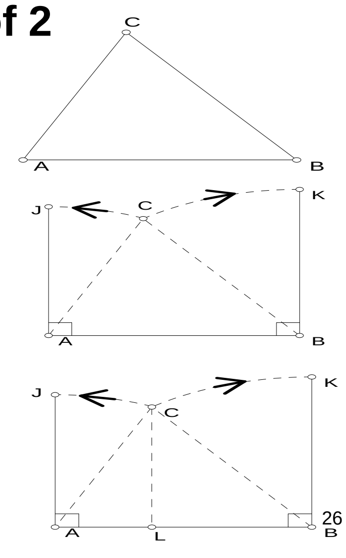
25

Theorem: For every triangle, the sum of the measures of its interior angles is 180^0.

Proof 1



Proof 2



26

Students' Conceptions of Proof: Selected Results

- Students justify mathematical assertions by examples
- Often students' inductive verifications consist of one or two example, rather than a multitude of examples.
- Students' conviction in the truth of an assertion is particularly strong when they observe a **pattern**.

27

- **Students view a counterexample as an exception—in their view it does not affect the validity of the statement**.
- **Confusion between empirical proofs and proofs by exhaustion.**
- **Confusion between the admissibility of proof by counterexample with the inadmissibility of proof by example.**

28

Teaching Actions with Limited Effect

- Raising skepticism as to whether the assertion is true beyond the cases evaluated.
- Showing the limitations inherent in the use of examples through situations such as:

The conjecture " $\sqrt{1141y^2+1}$ is an integer" is false for $1 \leq y \leq 10^{25}$. The first value for which the statement is true is: 30,693,385,322,765,657,197,397,208

29

Why showing the limitations inherent in the use of examples is not effective?

- Students do not seem to be impressed by situations such as:
  - The conjecture " $\sqrt{1141y^2+1}$ is an integer" is false for $1 \leq y \leq 10^{25}$. The first value for which the statement is true is: 30,693,385,322,765,657,197,397,208

- **Students view a counterexample as an exception—in their view it does not affect the validity of the statement**.

30

How can instruction facilitate the transition from empirical reasoning to deductive reasoning?

The Role of the *Need for Causality*

31

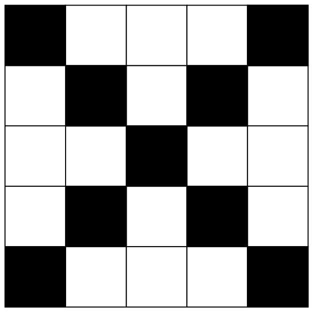
The Necessity Principle

For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to ***intellectual need***, not social or economic need.

To implement the necessity principle:

1. Recognize what constitutes an *intellectual need* for a particular population of students, relative to the concept to be learned.
2. Present the students with a **sequence** problems that correspond to their intellectual need, and from whose solution the concept **may** be elicited.
3. Help students elicit the concept from the problem solution.

32



33

Doris Solution

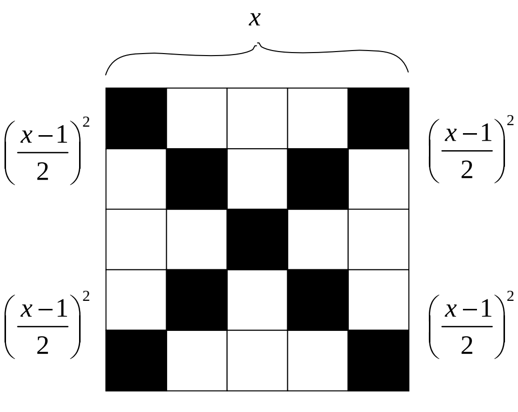
Dimension	Number of White Squares	
1	0	$(1-1)^2$
3	4	$(3-1)^2$
5	16	$(5-1)^2$
7	36	$(7-1)^2$

34

Doris Solution

Dimension	Number of White Squares	
1	0	$(1-1)^2$
3	4	$(3-1)^2$
5	16	$(5-1)^2$
7	36	$(7-1)^2$
x		$(x-1)^2$

John's Solution



$$4\left(\frac{x-1}{2}\right)^2 = (x-1)^2$$

Towns A and B are 300 miles apart. At 12:00 PM, a car leaves A toward B, and a truck leaves B toward A. The car drives at 80 m/h and the truck at 70 m/h. When and where will they meet?

Students' reasoning:

After 1 hour, the car drives 80 miles and truck 70 miles.

Together they drive 150 miles.

In 2 hours they will together drive 300 miles.

Therefore,

They will meet at 2:00 PM.

They will meet 160 miles from A.

37

Towns A and B are ~~300~~ miles apart. At 12:00 PM, a car leaves A toward B, and a truck leaves B toward A. The car drives at 80 m/h and the truck at 70 m/h. When and where will they meet?

38

Towns A and B are ~~300~~ 100 miles apart. At 12:00 PM, a car leaves A toward B, and a truck leaves B toward A. The car drives at 80 m/h and the truck at 70 m/h. When and where will they meet?

Students' reasoning:

It will take them less than one hour to meet.

It will take them more than 30 minutes to meet.

They will meet closer to B than to A.

Let's try some numbers:

$$\frac{50}{60}80+\frac{50}{60}70\overset{?}{=}100$$

$$\frac{40}{60}80+\frac{40}{60}70\overset{?}{=}100\ (Yes!)$$

39

Towns A and B are 118 miles apart. At 12:00 PM, a car leaves A toward B, and a truck leaves B toward A. The car drives at 80 m/h and the truck at 70 m/h. When and where will they meet?

Necessitating the concept of unknown

Students' reasoning:

It will take them less than one hour to meet;

It will take them more than 30 minutes to meet;

They will meet closer to B than to A.

Let's try some numbers:

$$\frac{50}{60}80+\frac{50}{60}70\overset{?}{=}118$$

$$\frac{40}{60}80+\frac{40}{60}70\overset{?}{=}118$$

$$\frac{30}{60}80+\frac{30}{60}70\overset{?}{=}118$$

Repeated Reasoning

40

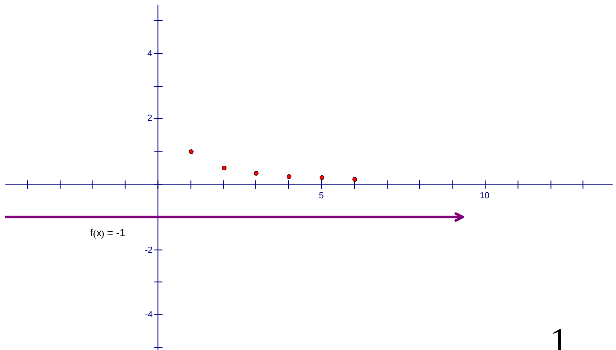
Goal: To necessitate the e-N definition of limit

Teacher: What is $\lim_{n\to\infty}\frac{1}{n}$ **and why?**

Students:

$\lim_{n\to\infty}\frac{1}{n}=0$ because the larger n gets the closer $\frac{1}{n}$ is to zero.

Teacher:



$\lim_{n\to\infty}\frac{1}{n}=-1$ because the larger n gets the closer $\frac{1}{n}$ is to -1.

41

Part III: The 1st Fundamental Question

What mathematics should we teach in school?

DNR's **stance:**

Mathematics consists of two categories of knowledge:

ways of understanding **and** ***ways of thinking***

Hence, it is necessary to focus on both categories of knowledge

42

ways of understanding* = *subject matter

E.g., definitions, theorems, proofs, problems, solutions, etc.

ways of thinking* = *conceptual tools

E.g., problems solving approaches, beliefs about mathematics

43

Ed's Background

1. Second grader
2. Reported by his teacher to be slightly above average in mathematics and average in other subjects
3. Ed's formal instruction had included addition and subtraction without regrouping
4. Ed was informally exposed to the basic notion of multiplication as repeated addition and the meaning of division as sharing equally
5. Ed was never taught any division strategy

44

Ed's solution to Division Problems

(Ed is a 71/2-year-old child)

1. What is Ed doing?
2. How would I respond to Ed?
3. What value, if any, do I see in Ed's solution?

45

Ed's Strategy for Solving Division Problems

Interviewer: How much is 42 divided by 7?

Ed: … That's easy …

40 divided by 10 is 4

3 plus 3 plus 3 plus 3 is 12

12 plus 2 is 14

14 divided by 2 is 7

2 plus 4 is 6

The answer is 6! (**triumphantly**)

Interviewer (to himself):

Okay, miracles do happen …

46

Interviewer : How about 56 divided by 8?

Ed: You do the same thing (**Impatiently**)

50 divided by 10 is 5

5 times 2 is 10

10 plus 6 is 16

16 divided by 2 is 8

5 plus 2 is 7

The answer is 7.

Interviewer : … And 72 divided by 9?

Ed: Are you going to ask me every single problem?

47

Ed: **… okay …** 72 divided by 9 …

70 divided by 10 is 7

7+2 is 9

1 and 7 is 8

The answer is 8.

48

A survey of mathematics teachers

- Why do we teach the *long division algorithm,* the *quadratic formula, techniques of integration,* and so on when one can perform arithmetic operations, solve many complicated equations, and integrate complex functions quickly and accurately using electronic technologies?

49

Teachers' Answers	Justifications
• "These materials appear on standardized tests." →	• Teachers must prepare students for tests mandated by their districts.
• "One should be able to solve problems independently in case a suitable calculator is not present." →	• Teachers must teach students to do calculations without a calculator, especially those needed in daily life.
• "Such topics are needed to solve real-world problems and to learn more advanced topics." →	• Teachers must prepare students for more advanced courses where certain computational skills might be assumed.

50

Teachers’ answers are external to mathematics as a discipline

The justifications for these answers are:

- **neither cognitive**
  - Role of computational skills in *one’s conceptual development of mathematics*
- **nor epistemological**
  - Role of computations in the development of mathematics
- **mainly social**
  - Role of computational skills in the context of social expectations

51

Why teach proofs?

Typical Answer:

- So that students can be *certain* that the theorems we teach them are true.

While this is an adequate answer—both cognitively and (by inference) epistemologically—it is incomplete.

52

The teachers had little to say when skeptically confronted about their answers by being asked:

- Do you or your students doubt the truth of theorems that appear in textbooks?
- Is certainty the only goal of proofs?
- The theorems in Euclidean geometry, for example, have been proven and re-proven for millennia. We are certain of their truth, so why do we continue to prove them again and again?

53

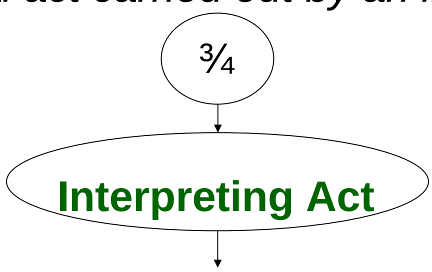
- Textbooks and teachers at all levels tend to view mathematics
  - in terms of *subject matter* such as definitions, theorems, proofs, algorithms, and problems and their solutions.
  - not in terms of *conceptual tools* necessary to construct such entities.
- Knowledge of *subject matter* is indispensable, but it is not sufficient.
- *Conceptual tools* constitute an important category of knowledge different from the *subject matter* category.

54

- What exactly are these two categories of knowledge, *subject matter* and *conceptual tools*?
- What is the basis for the argument that both categories are needed?

55

A *way of understanding is a particular product of a mental act carried out by an individual*.



Ways of Understanding (products)

- Three objects out of four objects
- The sum ¼ + ¼ + ¼
- The measure of the quantity resulting from dividing *3* units into *4* equal parts
- The measure of a *3* cm long segment with a ruler whose unit is *4* cm long
- The solution to the equation $4x=3$
- The equivalence class $\{3n/4n \mid n \neq 0\}$
- Two numbers with a bar between them.

56

A way of thinking is *a characteristic of a mental act*

Interpreting act

↓

Ways of Thinking (characteristics)

One's interpretation of symbols might be characteristically

1. inflexible: a symbol has a single interpretation
2. flexible: symbols can have multiple interpretations
3. non-referential: devoid of referents (quantitative, spatial, functional, etc.)
4. referential: a representation of an entity within a coherent reality

57

A *way of understanding is a particular product of a mental act carried out by an individual*.

Show that 2 is an upper bound for: $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$

↓

Proving act

↓

Proof 1

$\sqrt{2} = 1.41421 < 2$

$\sqrt{2+\sqrt{2}} = 1.8478 < 2$

$\sqrt{2+\sqrt{2+\sqrt{2}}} = 1.9616 < 2$

Therefore

Every term in the sequence is less than 2.

Proof 2

$\sqrt{2}$ is less than 2

Therefore

$2+\sqrt{2}$ is less than 4

Therefore

$\sqrt{2+\sqrt{2}}$ is less than 2

Therefore

$2+\sqrt{2+\sqrt{2}}$ is less than 4

Therefore

$\sqrt{2+\sqrt{2+\sqrt{2}}}$ is less than 2

58

A way of thinking is *a characteristic of a mental act*

proving act

↓

Ways of Thinking (characteristics)

One's proving act might be characteristically

1. inductive: based on examples or measurements
2. perceptual: based on visual perceptions
3. authoritarian: based on an authority (e.g., teacher)
4. deductive: based on rules of deduction

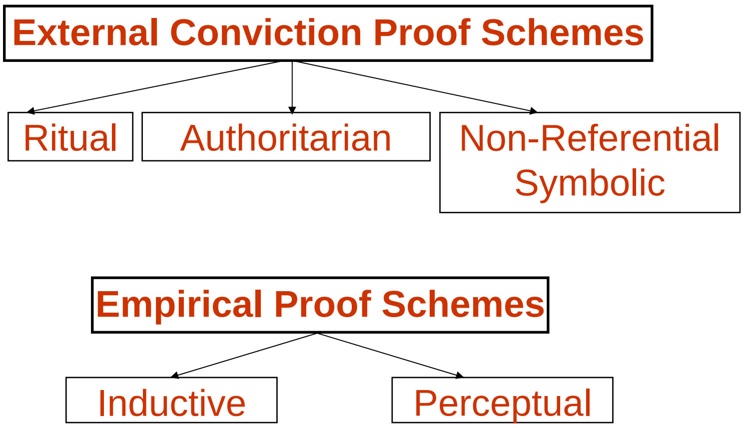
59

Proof Scheme

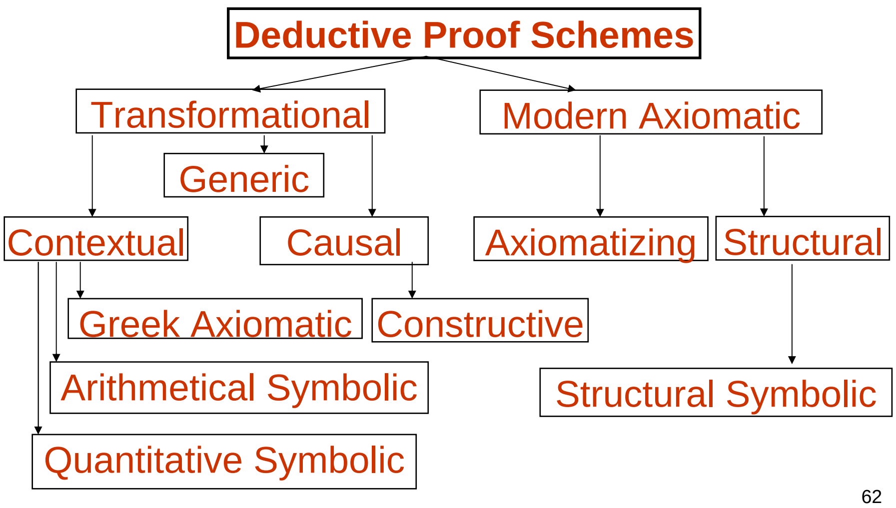
A person's ***proof scheme*** is a persistent characteristic of one's act of proving.

60

Taxonomy of Proof Schemes



61



62

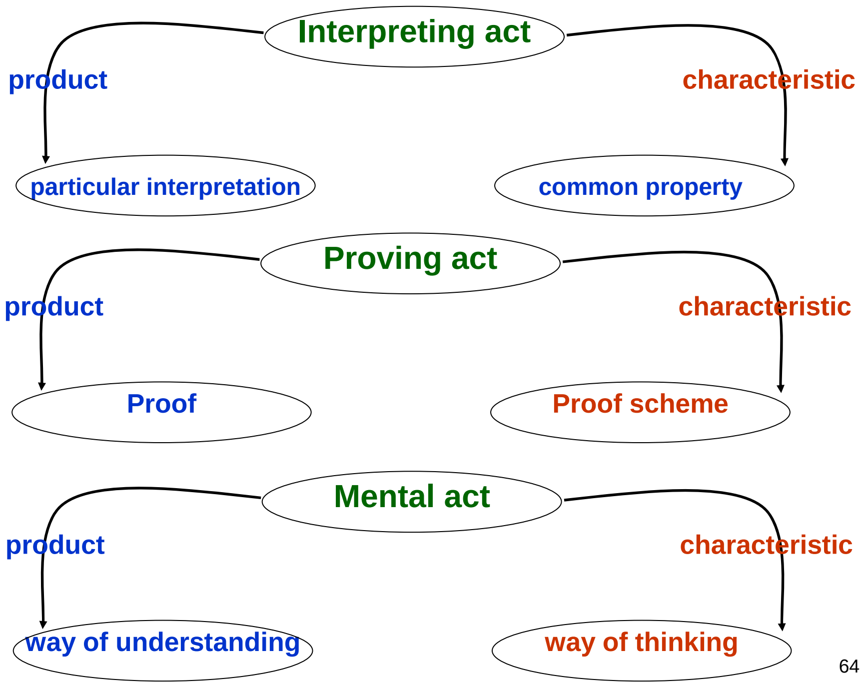
Mental Act

Humans' construction of knowledge involves numerous mental acts

such as:

representing, interpreting, defining, computing, conjecturing, inferring, proving, structuring, symbolizing, transforming, generalizing, applying, modeling, connecting, predicting, reifying, classifying, formulating, searching, anticipating, problem solving.

63



64

Mathematics **consists of two complementary subsets:**

The first subset is a collection, or structure, of structures consisting of particular axioms, definitions, theorems, proofs, problems, and solutions. This subset consists of all the institutionalized ways of understanding in mathematics throughout history. It is denoted by WoU.

The second subset consists of all the ways of thinking, which characterize the mental acts whose products comprise the first set. It is denoted by WoT.

$$M = WoU \bigcup WoT$$

65

Thinking in Terms Ways of Thinking

Pedagogical implications

Mathematics curricula at all grade level, including curricula for teachers, should be thought of in terms of the constituent elements of mathematics, *ways of understanding* and *ways of thinking.*

66

Thinking in Terms Ways of Thinking

Why do we teach the *long division algorithm*, the *quadratic formula*, *techniques of integration*, and so on when one can perform arithmetic operations, solve many complicated equations, and integrate complex functions quickly and accurately using electronic technologies?

67

Algebraic Invariance

Algebraic invariance is the **way of thinking** where one recognizes that algebraic expressions are manipulated not haphazardly but with the purpose of forming a desired structure while maintaining certain properties of the expression invariant.

computational skills

IS

algebraic invariance

68

Why Teach the Proof of the Quadratic Formula?

$ax^2 + bx + c = 0 \quad (a \neq 0)$ $\qquad$ $(x+T)^2 = L$

CHANGING THE FORM WITHOUT CHANGING THE VALUE

$$2x^2 + 5x + 3 = 0$$

$$2\left(x^2 + \frac{5}{2}x + \frac{3}{2}\right) = 0$$

$$2\left[\left(x + \frac{5}{4}\right)^2 - \frac{25}{16} + \frac{3}{2}\right] = 0$$

$$2\left[\left(x + \frac{4}{4}\right)^2 - \frac{1}{16}\right] = 0$$

$$\left(x + \frac{4}{4}\right)^2 - \frac{1}{16} = 0$$

$$\left(x + \frac{4}{4}\right)^2 = \frac{1}{16}$$

$$x = \pm\frac{1}{4} - \frac{5}{4}$$

69

Why Teach the Long Division Algorithm?

$0.14\overline{)12.91}$ $\qquad$ $14\overline{)1291}$

CHANGING THE FORM WITHOUT CHANGING THE VALUE

$$\frac{12.91}{0.14} = \frac{12.91 \times 100}{0.14 \times 100} = \frac{1291}{14}$$

Proportional reasoning as a way of thinking

A way of understanding the decimal system

Algorithmic way of thinking

70

Why Should We Teach Techniques of Integration?

$$\int \tan\theta d\theta \qquad \xrightarrow{u=\cos\theta} \qquad \int -\frac{1}{u}du$$

CHANGING THE FORM WITHOUT CHANGING THE VALUE

$$\int \tan\theta d\theta =$$

$$\int \frac{\sin\theta}{\cos\theta} d\theta =$$

$$\int -\frac{1}{u}du =$$

$$-\ln|u| + C =$$

$$\ln|\sec\theta| + C$$

71

Depriving Students the Opportunity to Develop Ways of Thinking

72

Multiplication and Division

(Junior-high school students)

A cheese weighs 5 pounds. 1 pound costs $12. Find out the price of the cheese. Which operation would you have to perform?

(a) **12÷5** (b) **5÷12**

(c) **5×12** (d) **12+12+12+12+12**

Correct responses (c and d): **83%**

A cheese weighs 0.823 pounds. 1 pound costs $10.50. Find out the price of the cheese. Which operation would you have to perform?

(a) **10.50+0.823** (b) **10.50×0.823**

(c) **10.50÷0.823** (d) **10.50-0.823**

Correct responses: **29%**

73

Teachers teach the following solution strategy:

A cheese weighs 0.823 pounds. 1 pound costs $10.50. Find out the price of the cheese.

1. **Replace the decimal numbers (0.823 & 10.50) by any whole numbers (say, 8 & 10):**

   A cheese weighs 8 pounds. 1 pound costs $10. Find out the price of the cheese.

2. **Find the expression that solves the *new* problem:**

   8 × 10

3. **Replace the numbers in this expression with the original decimal numbers:**

   0.823 × 10.50

74

A cheese weighs 0.823 pounds. 1 pound costs $10.50. Find out the price of the cheese. Which operation would you have to perform?

(a) **10.50 + 0.823** (b) **10.50 × 0.823**

(c) **10.50 ÷ 0.823** (d) **10.50 - 0.823**

Rita: None [none of the given choices is a solution to the problem].

T: How would you solve the problem?

Rita: (A long pause)

One thousandth of a pound costs 10.50 divided by 1000. Then I times that by 823.

75

Teacher: One pound of candy cost $7. How much would 3 pounds cost?

Tammy: Three times seven: 21.

Dan: I agree, 3 times 7.

Teacher: How much would I pay if I buy only 0.31 of a pound?

Tammy: It is the same. You only changed the number. 0.31 times 7.

Dan: No way! It isn't the same. … Can't be. It isn't times.

Teacher: How would you, Dan, solve the problem?

Dan: Divide 1 by 0.31. Take that number, whatever that number is, and divide 7 by it.

76

Developmental Interdependency

- ***Ways of Understanding***
  - Products of mental acts

- ***Ways of Thinking***
  - Characteristics of mental acts

77

Developmental Interdependency

- ***Ways of Understanding***

  definitions, problems and their solutions, algorithms, theorems, proofs, and so on

The *Duality Principle*

Students develop ways of thinking through the production of ways of understanding.

Conversely:

The ways of understanding students produce are impacted by the ways of thinking they possess.

- ***Ways of Thinking***

  algebraic invariance, algorithmic reasoning, proportional reasoning, and deductive reasoning

78

Duality Principle

Students develop ***ways of thinking*** through the production of ***ways of understanding***.

And:

The ***ways of understanding*** students produce are impacted by the ***ways of thinking*** they possess.

Necessity Principle

For students to learn what we intend to teach them, they must have a need for it, where ‘need’ refers to ***intellectual need***, not social or economic need.

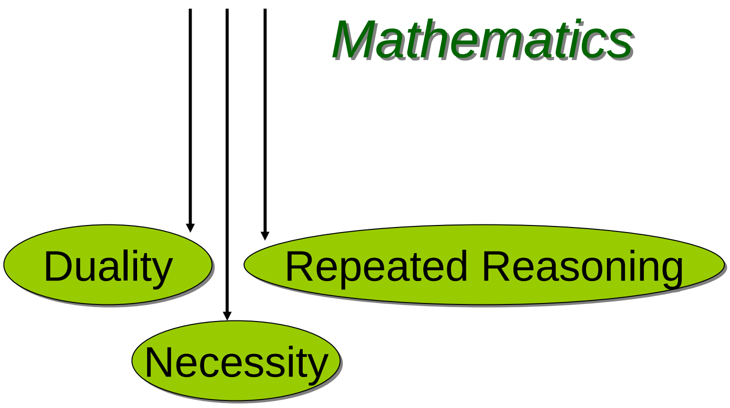
Repeated-Reasoning Principle

Students must practice reasoning in order to organize, internalize, and retain what they learn.

DNR Theoretical Framework

http://www.math.ucsd.edu/~harel

DNR-Based Instruction in Mathematics



80