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Learning to listen: from historical sources to classroom practice

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Abstract Listening to students in productive ways seems to be at the core of teaching practices aligned with the basic tenets of the constructivist world view. We present a definition for productive ways of listening, discuss the challenges involved in implementing it, and propose a way to support the "decentering" needed to learning to listen for teacher education programs. The proposal is based on reading and understanding historical texts as a way to exercising the adoption of the 'other's perspective.' We describe the materials developed for teacher workshops, their implementation and what participants, and we, learned from the experience.

Key words learning to listen · history of mathematics · constructivist teaching · teacher education · teacher–student interactions

1 Introduction

The purpose of this paper is to describe and analyze an approach to develop teachers' productive listening capabilities. Firstly, we define 'listening' (Section 1) and discuss its challenges and difficulties (Section 2). We then pose the question "How can we support the development of listening"? and propose an approach to answer it through the history of mathematics (Sections 4 and 5). Finally, we describe the way in which we implemented the approach (Section 6) and we report what we learned from the experience (Section 7).

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2 'Listening' to students

Two inter-related assumptions underlie the application of the constructivist philosophy to the practice of mathematics education:

"1. When students genuinely engage in solving mathematical problems, they proceed in personally reasonable and productive ways.
2. Researchers and teachers must learn to listen and to hear the sense and alternative meanings in these approaches." (Confrey, 1991, p. 111)

Inspired by this quote, we propose to define 'listening' to students as follows: giving careful attention to hearing what students say (and to see what they do), trying to understand it and its possible sources and entailments. 'Listening,' as we envision it, is not a passive undertaking, since it should include the following components:

- Detecting, taking up, and creating opportunities in which students are likely to engage in freely expressing their mathematical ideas;
- Questioning students in order to uncover the essence and sources of their ideas;
- Analyzing what one hears (sometimes in consultation with peers) and making the enormous intellectual effort to take the 'other's perspective' in order to understand it on its own merits; and
- Deciding in which ways the teaching can productively integrate students' ideas.

Thus, "Listening...promotes reflection which can lead to a conception of teaching grounded in adaptation, a condition necessary for professional development" (Cooney & Krainer, 1996, p. 1181).

Several researchers, have described different ways of listening to students as an integral component of the instructional process. For example, Moschkovich (2004) studied in detail how a tutor based his interaction with a student on the student's ideas in order to advance her. She presents and analyzes the construct of "appropriation," whose existence and function is based precisely on mutual teacher–student listening. "Appropriation involves joint productive activity, a shared focus of attention, and shared meanings ... also involves taking what someone else produces during joint activity for one's own use in subsequent productive activity ..." (p. 51).

Beyond being at the service of implementing a constructivist approach in the practice of mathematics education, listening can have other important intellectual and affective functions. For example, the development of teacher listening may be a strong component of "a caring, receptive and empathic form" (Smith, 2003, p. 498) of teacher-student conversations. If often modeled by teachers, students would feel respected and valued. Moreover, the listening modeled by the teacher may become internalized by students as a habit incorporated into their repertoire of learning techniques and interpersonal skills.

Listening may also benefit the listeners themselves. "Thinking ourselves into other persons leads us to reflect about our own relationship to mathematics" (Jahnke, 1994, p. 155). In other words, effective listening may influence 'listeners,' by making them re-inspect their own knowledge, against the background of what was heard from others. Such re-inspection of the listener's own understandings (which may have been taken for granted) may promote the re-learning of some mathematics or meta-mathematics. There are several candid self-reports on this phenomenon. For example, Aharoni, a research mathematician involved in experimental teaching in elementary school, reported on his own experience that "...what surprised me most was that I learnt mathematics. Actually, a lot of it."

(Aharoni, 2003, p. 1) Similarly, Henderson (1996) reported on his undergraduate teaching that: “At first I was surprised – How could I, an expert in geometry, learn from students? But this learning has continued for 20 years and I now expect its occurrence. In fact, as I expect it more and more and learn to listen more effectively to them, I find that a larger portion of the students in the class are showing me something about geometry that I have never seen before.” (p. 46)

Thus, as a competence which may serve a number of professional development purposes, listening should become a central teaching capability to be learned, developed and continuously nurtured. However, listening is not an easy task.

3 The challenges of ‘listening’

The following are some of the challenges and difficulties of listening and their possible sources.

3.1 “Packaged” knowledge

Once we know and understand a mathematical idea, or a corpus of knowledge, we tend to forget, and possibly even (unconsciously) dismiss, the process we (or others) experienced during the learning of that knowledge. The expert’s knowledge seems to undergo an irreversible progression towards ‘compactization’ or ‘packaging’ which erases some of the memories of learning. Freudenthal (1983, p. 469), describes this as follows: “I have observed, not only with other people but also with myself...that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question.” Similarly, also “...historical disputes, however fierce, tend to be completely forgotten once mathematicians manage to reconcile themselves with the problematic notions” (Sfard, 1994, p. 130). Thus, to a modern mathematician “nothing could appear simpler...” (ibid.). In other words, “Highly practiced cognitive and perceptual processes become automatized so there is nothing in memory for experts to “replay,” verbalize, and reflect upon” (Nathan & Petrosino, 2003, p. 907). This phenomenon may impede empathic listening to those in the midst of the process of learning. Therefore, it seems that the type of listening we propose has to be developed by running counter to the expert’s natural tendency, and it should include “unclogging automatisms” (Freudenthal, 1983, p. 469), unpacking issues taken for granted by artificially setting aside much of what we know so well. If this is the case, an important part of learning to listen to students would include, paradoxically, some kind of ‘mathematics unlearning’ on the part of the teacher. (The expression “unlearning” for teaching is borrowed from Ball (1988) although here we may be applying it differently than intended in the original.)

3.2 “Decentering” capabilities

As teachers, sometimes we may not only need to set aside our own knowledge and ways of knowing, but we also have to develop a capacity to ‘read through’ idiosyncratic ways of expression and learn to appreciate their underlying logic and potential. In other words, “Making sense of children’s ideas is not so easy. Children use their own words and their own frames in ways that do not necessarily map into the teacher’s ways of thinking.” ... “The ability to *hear* what children are saying transcends disposition, aural acuity, and knowledge, although it also depends on all of these" (Ball, 1993, p. 378). Thus listening not only implies an exercise in unclogging our own mathematics knowledge relevant to the situation at stake, but also requires the development of a "decentering" capacity. We borrow and adapt the idea of decentering from Piaget (see, for example, Ginsburg & Opper, 1979, p. 65) and, in our context we take it to mean, the capacity to adopt the other's perspective, to 'wear her conceptual spectacles' (by keeping away as much as possible our own perspectives), to test in iterative cycles our understanding of what we hear, and possibly to pursue it and apply it for a while. Such a decentering involves a deep intellectual effort to be learned and exercised.

3.3 Different ways of listening

A further difficulty would consist in deluding ourselves that we listen carefully, whereas we do not really do so. Confrey (1991) contrasts the approaches of discovery learning with the constructivist view of mathematics, a distinction which when stretched into teaching may result in very different ways of listening: "evaluative" (Davis, 1997) as opposed to "attentive listening" (as defined by Smith, 2003, following Davis's "hermeneutic listening," ibid.), respectively. The former, which is more common, consists of listening against the background of an expected correct answer. It implies a virtual 'measurement' of the 'distance' between the student's present state of knowledge and the desirable goal, providing straightforward feedback based on 'correctness,' and applying subsequent 'fixing' strategies. For some teachers, this may be a common practice envisioned as 'listening.' However, such listening may tend to disregard the students' thinking, the sources of their idiosyncratic ideas and their potential as a source for learning.

3.4 Unavoidable biases

Suppose one has had experience with and has developed some capacity to 'unclog' and decenter towards attentive, rather than evaluative, listening. Still, the difficulties may be far from over. Researchers have described the phenomena of teachers' "under-hearing" or "over-hearing" (e.g., Wallach & Even, 2002). These phenomena can be attributed to constraints under which teachers may work, or more fundamentally, to a human built-in impossibility: teachers (and human beings in general) will "always *hear through* various personal factors or 'screens'; ... it is unrealistic to expect an 'accurate' teacher understanding of what students are saying and doing. Hearing through is not something that could be overcome. Rather, it is a fundamental characteristic of one person's understanding of what another person is saying, doing, and feeling." (Even & Wallach, 2004, p. 491)

3.5 Timing

Listening to students implies a skillful application of different ways to sustain conversations, questioning and probing, aimed at unpacking, and double checking, as faithfully as possible, the other's perspective. This is time consuming and its results may not be immediate. The understanding of the other's perspective may occur when we review (mentally or from a recording) conversations with students long after these conversations are over. Probing and at the same time understanding (even partially) a student perspective can be very demanding, and at times even impossible to do in real time.

4 Research questions

We claim that, even when it may be impossible to fully capture students' thoughts and understandings, there is much to be learned in between complete 'deafness' (or indifference) and fully opening our ears and minds to students' ideas. Thus, given the importance of 'listening' towards understanding the students' point of view and in spite of the challenges described above, we assume that (at least to some extent) teachers can and should learn to listen. The main question is how? What kind of experiences should be orchestrated in order to develop desirable listening capabilities? What kind of "curriculum for professional learning" (Ball & Cohen, 1999, p. 20) and "pedagogy for professional development" (ibid. p. 25) can be designed to nurture and develop an ability to unfold and understand student ideas, their sources and potential entailments? How such a curriculum may work in teacher courses and to what extent the goal of learning to listen can be attained?

There are several approaches to tackle these questions. For example, the Japanese model of "lesson study" includes, among other components, the possibility for teachers to observe classes of their colleagues. Thus, teachers who, in their own classes, are too busy to pay enough attention to the ways students respond, are free to observe and listen. This practice of observing and following what students say and do enables teachers to develop "eyes to see the children" (*kodomo wo miru me*). The observing teachers 'comb' the classroom for evidence of student learning and motivation, focusing on how children, including the quietest, speak up or engage in non-verbal communication. This provides the observing teachers with opportunities to think more deeply about their own students than their daily practices may allow for. What students say and produce are always a crucial issue in lesson study discussions as a source and a basis for (a) comparing and contrasting interpretations with fellow teachers, and (b) developing successful classroom activities attuned to students (see, for example, Isoda, Stephens, Ohara, & Miyakawa, 2007, and Lewis, 2002).

For other type of experiences see, for example, Crespo (2000), who reports on a study in which some teachers taking a special methods course found that listening enabled them to change how they interpreted students' understanding, and thus to change their teaching.

5 The rationale for our approach

We hereby propose an approach to develop and nurture the ability to decenter oneself in order to listen to the other's perspectives and ideas. Our approach consists of a special way of work designed to support the understanding of certain type of primary sources from the history of mathematics. Our basic assumptions are that:

(a) In order to fully understand the ideas behind a historical (mathematical) source, we need a similar kind of decentering to that needed for listening to students;
(b) Such a decentering can be learned; and
(c) Learning environments to support this learning can and should be designed.

History of mathematics can provide many solution approaches (to problems) which are very different from what is common nowadays. Such solution processes may conceal the thinking behind them. Thus, one has to engage in a 'deciphering' exercise in order to understand what was done, what could have been the reasoning behind it and what is the mathematical substrate that makes an unusual method/approach valid and possibly general. Engaging in such an exercise bears some similarities to the process of grasping what lies behind our students' thinking and actions. We do not claim that there may be parallels between the mathematics underlying primary sources and that of our students. What we do claim is that experiencing the process of understanding the mathematical approach of a primary historical source can be a sound preparation towards listening to students. We base this claim in the following.

- Many students' responses, which differ from the expected, are often easily dismissed. When facing a historical source with a solution approach (foreign to us), we know that the best minds of their time and culture were behind it. Therefore, an historical text cannot be that easily dismissed on the basis of the right-wrong dichotomy, as is commonly the case with students' answers. A historical source has to be attended to in all its idiosyncrasy, and many times our own understandings cannot be immediately projected onto it; thus one has to delve deep into the text's own nature. If so, a first hurdle in the hard task of decentering is removed. Repeating these exercises may support both the development of the habit of not dismissing any solution and the search for its idiosyncratic mathematical approach.
- When facing an initially cryptic historical source, one has to develop tools to make sense of it. The main tools may consist of: parsing the source, posing questions to oneself (or to a peer) around it, paraphrasing parts of the text in our words and notations, summarizing partial understandings, locating and verbalizing what is still to be clarified, and contrasting different pieces for coherence. In a sense, this implies some kind of "hermeneutic" (interpretive) practice: "History of mathematics is essentially a hermeneutic effort: theories and their creators are interpreted. Interpretation comprises a circular process of forming hypotheses and checking them against the material given..." (Jahnke, 1996, p. 173). Inspired by the traditional hermeneutic school, Isoda (2002) and Isoda and Kishimoto (2005) identified the components of interpreting mathematics in general, and in particular, the reading of primary sources, as follows. Firstly, when we deliberately or unconsciously use our own mathematics to make sense of a mathematical text, we may not advance and even feel contradictions. At this point, we may start to seriously engage with the questions: What is written? Why did the author write in such a way? What are the hidden assumptions? If this text says A, and A entails B – where is B in the text? This questioning may lead us to adopt the 'writers' perspective'. This practice may help develop an understanding which then needs to be re-confirmed with a somehow recursive process (e.g., applying our understandings to similar texts, examples or problems). We propose that such hermeneutic tools and processes suitable for historical texts can serve teachers well in attempting to understand students' ideas.

In sum, learning to read and understand certain primary sources may result in learning the skills and the tools necessary for learning to listen to students.

6 Description and analysis of the materials

In this section, we present a sample activity around 'Egyptian Mathematics,' adapted from previous work described in Arcavi (1987) and Arcavi and Bruckheimer (2000), followed by the description of its implementation and the results thereof.

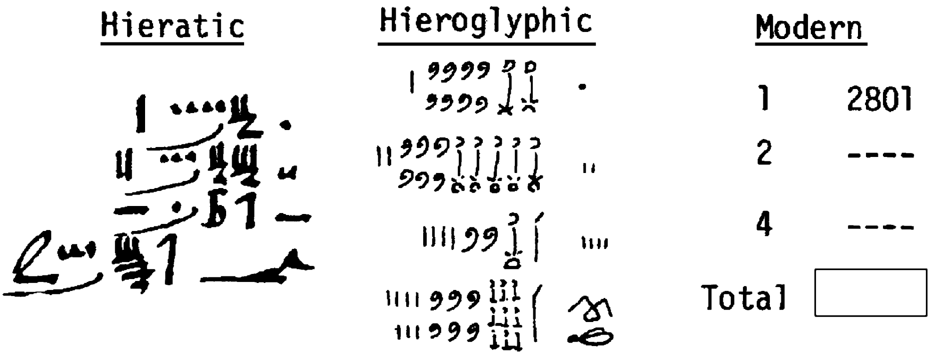


Fig. 1 Calculation performed during the solution to problem #74

Readers, especially those unfamiliar with ancient Egyptian mathematics, are invited to pause after reading each stage (see boxed tasks below), and to try to answer the questions (before reading the commentaries), to reflect on them and on the experience.

First stage:

The following text (Fig. 1) from the Rhind Papyrus presents a calculation, in Hieratic and Hieroglyphics, performed during the solution to problem #74.

1 On the basis of the initial translation to modern notation, indicate what each of the Hieroglyphic signs may mean.
2 Complete the blank spaces.
3 Explain what calculation was performed in this extract.

6.1 Commentary

The questions provide guidance towards the understanding of the text. The calculations in this and the following stage will be needed in the third stage.

The first question is introductory and directs attention to the building of a 'dictionary.' One can observe that there are two distinct columns of numbers, that a stick stands for the number 1, and the numbers between 1 and 9 are written as aggregations of sticks. The sign  stands for 100, the lotus flower  stands for 1,000, and also these signs are aggregated to compose numbers. Whereas the meaning of these signs can be inferred directly from the partial translation to modern notation, the meaning of the "bent finger," which stands for 10,000, should be conjectured and checked from the calculations and from its two appearances. This leads to fill in the blanks as follows.

1	2801
2	5602
4	11204
Total	19607

This translation provides an implicit feeling for the characteristics of the numeration system: the writing is from right to left, the system is, in a sense, decimal (one symbol represents ten identical 'lesser' symbols), numbers are produced by juxtaposition of symbols, but there is no place value (if the symbols designating a number are rearranged, they still represent the same number), thus no symbol for zero is needed in such a system.

At this stage, one can have a first idea of a procedure. The word *total* suggests an addition of the numbers in that column, which can be easily confirmed. However, this is not yet a full explanation of what was done. What is the meaning of the numbers in the first column? A further check helps us to realize the doubling of the previous number in both columns. Thus, what was calculated is, actually, 1×2801+2×2801+4×2801, or in other words, 7×2801. Now, we have a clearer idea of "what" was done. Still, we may want to speculate why the multiplication was performed in such a way. It is possible that this has to do with the characteristics of the writing system, in which doubling and adding is simple by aggregating the symbols (and substituting 10 of them by a 'larger' one). In this case, the deciphering did not bring to bear 'our' way of multiplying at the beginning, because we did not know which calculation was performed. Thus, in this case, we did not have the a priori option to judge against what we do, and the focus was on understanding what was done (comparisons between methods can certainly be done a posteriori). Note also, that the understanding of this text had two clear levels: the translation which leads to the realization that an addition was done, and then the realization that, in fact, the calculation is a multiplication performed by an idiosyncratic method based on the distributive law. Finally, one can speculate, on the basis of the number system, why the calculation was done in this way. Still, a more comprehensive mathematical reflection is due about the generality of the method, and that is the focus of the next part.

Second stage:

The following (Fig. 2) presents another calculation, performed during the solution to problem #52

1. Given your experience of the previous step, what questions would you pose to the new text?

   Pause for collection of questions
2. On the basis of the initial translation from the Hieroglyphics to modern notation, indicate what each of the signs may mean.
3. Complete the blank spaces.
4. Explain what calculations were performed in this case, and what is the meaning of the slash marks?
5. Calculate 13×27 by the Egyptian method.
6. Can one multiply any pair of numbers by this method? Explain

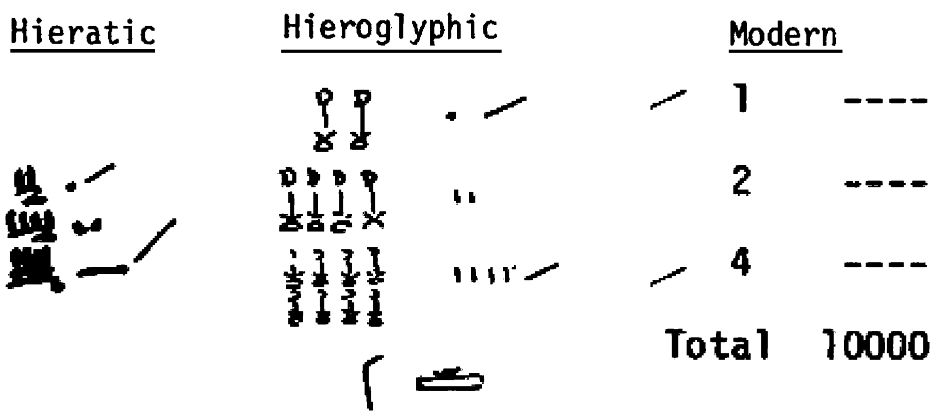


Fig. 2 Calculation performed during the solution to problem #52

6.2 Commentary

In this example not all the doubled amounts are required to perform the calculation 5×2000, but they are all presented as intermediate steps to obtain the addends needed ($1\times 2000+4\times 2000$). In this example, a more general feature of the method becomes clear: building *and* choosing the right addends. Note that before the guiding questions are provided, there is a request to produce one's own questions as a way to start exercising the understanding of a text without external support. Then our questions are provided again. This may support reflection on productive questioning for understanding texts.

The calculation of 13×27 is aimed at consolidating understanding, by providing an iterative cycle, this time actively *applying* the 'other's perspective' to a new example. The two factors for the multiplication were deliberately chosen such that the calculation is not immediate. Lastly, the question of the generality of the method is discussed. Were the Egyptians aware of the generality of their method? We leave this issue open, as an historical research question.

Note that when coping with understanding students' "products" we need similar tools: good questions, translation into a more intelligible language preserving the core ideas, uncovering implicit underlying knowledge, checking for generality (vs. locality) of a method or an argument, relating it to existing knowledge available to the student and considering how to integrate it in further mathematical activities.

Third stage:

The following (Fig. 3) presents the solution to Problem #24. The English translation (Peet, 1970) presents omissions for the purpose of this task.

1 Generate questions whose answers would help you understand the solution process for the linear equation.

Pause for collection of questions and discussion

2 Write, in modern notation, the equation corresponding to the first sentence, and solve it.

Questions 3–9 are aimed at helping to understand the solution method

3 Look at the first step:

/ 1	7
/$^1/_7$	– – –

Apparently, the Egyptians approached the problem by substituting a trial number, and seeing what happens.

(a) Which number did they try?
(b) Complete the blank.
(c) What result was obtained?
(d) Why do you think they chose the number they did?

4 Look at the second step:

1	8
/2	– – –
$^1/_2$	– – –
/$^1/_4$	– – –
/$^1/_8$	– – –

(a) Complete the blanks.
(b) What calculation has been performed, and what is the result?

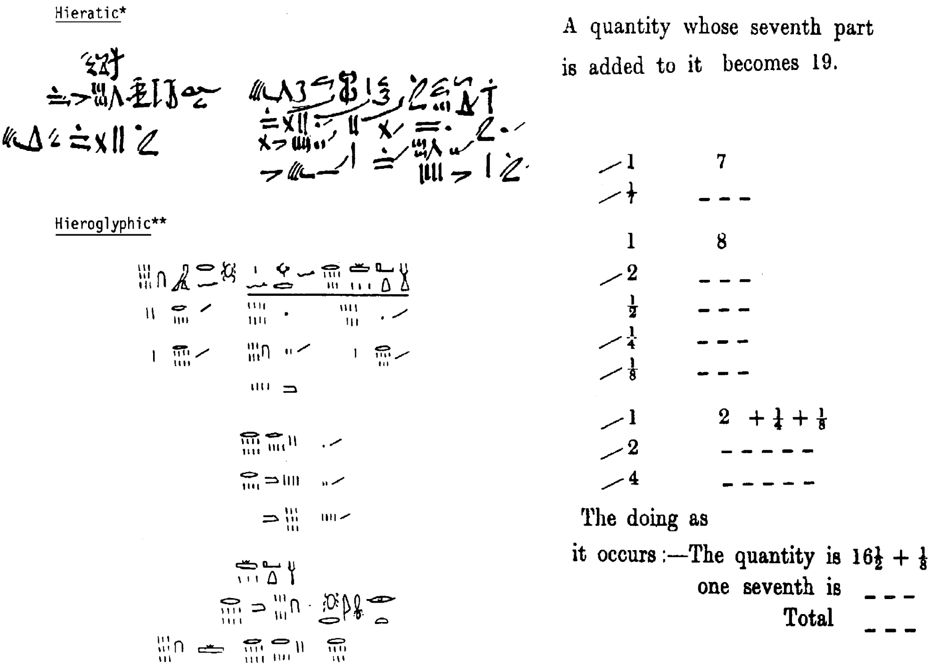


Fig. 3 Solution to problem #24

5 Look at the third step:

$$
\begin{array}{ll}
/1 & 2+1/4+1/8 \\
/2 & --- \\
/4 & ---
\end{array}
$$

(a) Complete the blanks.
(b) What calculation has been performed, and what is the result?

7 Look at the final step (Fig. 4):
Complete the blanks and explain what has been achieved in this step.

8 Reproduce and summarize the method of solution and explain it.

9 Write down the solution of the problem, as it would have appeared in the Papyrus (but in modern notation), if the first trial number had been 14 instead of 7.

10 Problem #25 in the Papyrus is "A quantity whose half is added to it becomes 16." Solve the problem as you would today and using the Egyptian method (as you think it would appear in the Papyrus, but using modern notation).

Fig. 4 Final step

The doing as
it occurs:—The quantity is $16\frac{1}{2} + \frac{1}{8}$
one seventh is ---
Total ---

6.3 Commentary

For those not acquainted with this solution method for solving linear equations, it is cryptic, certainly at first sight and possibly also after some efforts to make sense of it. It does not resemble the symbolic and straightforward solution method we teach and use today.

In the previous stages, our questions model the following tools: 'parsing' the text, identification of mathematical operations (based on what was done in previous stages) and the synthesis (see the questioning analysis below). At the beginning of this stage, participants are invited to reflect on which type of questioning may be productive and why.

The first question we propose requests our modern solution. The simplicity of our symbolic method stands, at least at first sight, is in sharp contrast to the Egyptian method, which seems lengthy, involved and purely numerical. This contrast may nudge the interpreters to set aside the modern method as the basis for making sense of the Egyptian method. As described above, a first step of the hermeneutic (interpretative) effort would be such feeling of contradiction or contraposition to drive the search of the 'other's point of view.'

Working through the next questions may not lead to the full understanding of the method. Thus, one may produce good answers to questions 3, 4, 5 and 6 above, and yet have only a descriptive account of what was done at each stage (addition, multiplication, division, etc) without having a sense of what is the method and why it works. Question 7 requests the reader to put together all the pieces and produce a full explanation. It may take a while to fully articulate it, for example, as follows: start with a guessed number which is convenient for the calculations of the problem, as if it were the sought solution. After applying the conditions of the problem to it, one adjusts the result using proportional reasoning. In our case, applying the calculations to 7 yields 8. The ratio between that result (8) and the desired result (19), namely how many times "8 goes into 19" should be the number of times that our trial number, 7, "goes into" the unknown we look for. The key mathematical idea underlying the Egyptian method is proportionality, in its several manifestations (including the hidden assumption that "1 is to 8, as 2 is to 16, as 1/4 is to 2, as 1/8 is to 1, and also as (2+1/4+1/8) is to (16+2+1)."

We also request the reader to apply the method starting with another trial number, and then to attempt the solution of a different problem by this method. The purpose is to consolidate the understanding of the method by being able to reproduce it, in order to gain a further feeling for it and its rationale – this corresponds to the iterative cycles, an important component of the hermeneutic process.

The next stage, not described here, consists of understanding an excerpt from *L'Arithmetique* by the French mathematician J. Peletier, published in 1549, discussing and applying the Rule of False Position (the pre-symbolic method of solving linear equations which has similarities to the Egyptian method – see, for example, Smith (1958) vol. II, pp. 437–441).

The types of questions we designed in order to facilitate the understanding of the historical texts consist of:

- The creation of 'dictionary' type translations of ancient notations to ours.
- The request for the description of a calculation. Note that such description can be provided at different levels of depth. For example, in the first stage, one may first notice a surface feature (the addition performed) and only later the deep structure of the method (the multiplication).
- The solution of a problem with our method and the realization that one can hardly map it into the Egyptian method – thus, indirectly nudging towards taking the 'text's perspective'.

- Parsing of the text into its components and concentrating on the 'local' understanding of small parts.
- The request to 'paste pieces together' towards a global understanding of the whole.
- Applying the Egyptian method to a new problem.
- Clarifying the kinds of previous knowledge relevant for understanding the text.
- Investigating the mathematical substrates of a solution method (e.g., assumed mathematical properties, generality, etc).

7 The workshop

A workshop consisting of two sessions was conducted with the above materials in a course for pre-service mathematics teachers at the University of Tsukuba, Japan. The course is part of a Master's Degree Program in Mathematics Education which prepares/upgrades the participants as teachers for the upper secondary school level. Fifteen out of the seventeen participants were recent graduates from a Bachelor's Degree in mathematics, architecture or engineering, or graduates from a teachers college majoring in mathematics. The remaining two participants were experienced teachers who returned to graduate school. The first workshop session lasted three hours without a break and the second lasted four hours and 10 min without a break. The conductor of the workshop (the regular instructor for the course and the second author of this paper), had minimal interventions except for the opening, leading the discussions and providing a summary. The whole workshop was videotaped, and two brief questionnaires were administered.

The workshop had two distinct components: the historical and the pedagogical, as described below.

Historical Component:

- *Introduction*. Starting with a general question: "In your opinion, what can be learned from the history of mathematics for the mathematics teaching practice?" After the discussion, the workshop leader explained that the aim of the workshop is to learn from the participants' experiences and opinions.
- *First stage*. (See above)
- *Second stage*. (See above)
- *Third stage*. (See above)

[End of first session]

- [Beginning of second session] Reminder of the main steps of the Egyptian solution method for solving a linear equation with one unknown.
- *Fourth stage*. Working on Peletier's text.
- *Questionnaire 1*. Requesting participants to write in which ways they think this workshop experience may be useful for them as teachers.

Pedagogical Component:

- *Fifth stage*. Presenting the following task borrowed from Even and Wallach (2004).

  Ahuva, a fifth grade teacher, wanted to assess whether her students know how to find the whole when a part is given. She administered her students a quiz that included the following problem: "3/5 of a number is 12. What is the number? Explain your solution." This is what Ron wrote: "12 * 2=24, 24:6=4, 24−4=20."

- Is Ron's solution correct?
- If you were Ron's teacher, what would be your assessment of Ron's knowledge?"

The purpose of this assignment was to embark on an interpretative exercise, taking a student written solution as "text."

- *Sixth stage*. Presenting a video with fragments of a mathematics lesson in a 5th grade in Japan, in which students produced many division exercises derived from and equivalent to 5.4:3=1.8. Some of their productions were expected, some were unexpected and sophisticated (e.g., 13.5:3), some correct and some reflecting interesting mistakes (e.g., 15.12:9). The task consisted in figuring out what may have been the thinking behind each answer.
- *Discussion*. Sharing impressions about and reflecting upon the workshop and the connections between its historical and pedagogical components.
- *Questionnaire 2*. Asking participants, for a second time, in which ways they thought this workshop experience may be useful for them as teachers.
- *Summary and closing*: The workshop leader explicitly shared the intended goal of the workshop related to learning to listen.

8 Learning from the workshop

In this section we address the double entendre implicit in this title: the workshop was a learning experience for both the participants and for us (the authors of this paper). The findings we report below are based on data from the two written questionnaires, from the recorded dialogues, discussions and solutions, and from our subjective observations.

8.1 Is the approach feasible?

The workshop participants were on task at all times, and their partaking was lively and enthusiastic. To the impression of the first author of this paper, who acted as an observer without knowledge of Japanese, there were moments in which the classroom atmosphere seemed to exude suspense towards a sought outcome. This was especially noticeable when participants were collectively engaged in producing an explanation of the Egyptian method (see question 7, third stage above), or when they worked on understanding Ron's solution. The emergence of the full explanation for the Egyptian solution took approximately 40 min, during which participants engaged collectively in producing it. Several students took turns to come to the board, and many times the whole class felt that an explanation was partial, either because it concentrated on a specific step, or because it was a mere description of the calculations performed. After several presentations, one participant came to the board and pasted all the pieces together producing the global explanation which satisfied everybody and the whole class applauded him. The process of searching and finding out the explanation (whose detailed description merits a full length paper of its own) of the intriguing Egyptian solution was as motivating as a detectivesque endeavor. We emphasize the collaborative nature of the work of these participants, their sense of dissatisfaction with partial explanations, their persistence in pursuing the search for a global one, the recognition of it and the subsequent satisfaction when reached. Thus, the proposed activities were shown to be both meaningful and engaging.

8.2 What is 'teaching practice'?

As mentioned, at the beginning of the workshop we posed to the participants the following question: What can be learned from the history of mathematics for the practice of mathematics teaching? One may argue that in order to answer this question, previous knowledge and experience in the two areas (history and teaching) may be required. However, the question is legitimate in any case, because, in the absence of experience, the answers would reflect beliefs and expectations, which can serve as an anchor against which to study possible attitude changes. In general, the oral answers collected centered on the mathematics (and the history) a teacher should know in order to have good resources for classroom use, even if there is no time or place in the curriculum to directly address the history of mathematics. The implicit view of teaching practice that emerged from these answers is mostly related to the subject matter knowledge a teacher must have in order to answer student questions or in order to have a repertoire of tools to variegate teaching.

As mentioned, besides the initial question, participants also replied twice to a written questionnaire, after completing the historical and the pedagogical components, respectively. As expected, the answers in the first questionnaire were related to the contents of the preceding assignments. The answers to the second questionnaire (administered after the pedagogical component) reflected the connections participants made between interpreting historical texts and understanding students' ideas. A frequent comment referred to the realization of how much mathematics can be done with a 'restricted' background, and how important it is for teachers to be aware of that. Thus, there is some indication that this workshop may have contributed to enlarging the conception of history of mathematics for "teaching practice" from content knowledge per se to include also content knowledge at the service of understanding others' thinking.

8.3 Learning to ask

Posing exploratory questions is essential for understanding the other's perspective. During the whole workshop, the participants engaged in questioning, and in the implicit comparison of different questions. At first, most of the questions were related to deciphering, e.g., "What is the '/'?" (Second stage), "What calculation was done here?" (Third stage). As the workshop progressed, other types of questions surfaced in order to address the need for uncovering hidden, missing or pre-supposed knowledge. We claim that by exercising such questioning (which was supported by the questions we provided), participants may have started to "appropriate" (in the sense of Moschkovich, 2004) the types of guiding questions we proposed in the task. The "micro-evolution" of the types of questioning – from mere guides for deciphering towards tools to direct the understanding of underlying ideas – can be regarded as an indication of the development of "attentive listening."

8.4 The evaluative stance

As mentioned, evaluative listening is pervasive and deep rooted. As a result of this workshop, we became more aware of how evaluative listening can stand in the way of attentive listening. For example, a task designed with the purpose of adopting the other's perspective may unintentionally foster value judgment, rather than discourage it. The intention of requesting the solution of the Egyptian problem with our modern notation was to highlight the apparent big differences between our symbolic method and the Egyptian method. We assumed that such differences would nudge participants to abandon attempts to map one method into the other, and, instead, to turn their full attention to uncovering the 'other's perspective' from within. However, for some participants, such exercise emphasized the power, generality and immediacy of the algebraic symbolism, in contrast to the lengthy numerical Egyptian method. Thus, the intention to nurture attentive listening by creating a gap between our method and the Egyptian had a slightly counterproductive effect for some: the symbolic method is so powerful and efficient, why bother to even consider a complicated alternative so difficult to understand.

It may well be that in interactions of this type (reader–text or teacher–student), judgment and evaluation are inescapable. Comparisons (e.g., between methods), expressions of opinion in a class discussion, and decision making (e.g., what course of action to adopt) are impregnated with evaluation and judgment. Is this undesirable? Perhaps not always. We ultimately strive for our students to develop sophisticated mathematical methods and understandings; therefore it is very important to re-appreciate the power of algebra over other methods.

One implication of this finding may be that we need to conciliate between positive and adverse effects of the 'evaluative' stance. How can we do that? We propose to consider appropriate timings. Educating towards attentive listening does not need to imply fully discarding evaluation and judgment. Instead it should aim at deferring them to later stages in which we have an understanding of where the students (or the original texts) stand. In the workshop, many participants expressed surprise at and appreciation of the richness of the mathematics which can be done with 'restricted means' – that in itself is a productive evaluative realization, which was produced only *after* fully understanding what they read and heard. Therefore, the challenge consists of walking the fine line between dismissing the immediate judgment of student ideas (instead of trying to understand them) and benefiting from an evaluation of them. Whereas our workshop helped bringing these nuances to the fore, more research is needed to further elucidate this issue.

8.5 Interpreting texts – listening to students

As a result of the workshop experience, we consider that the linkage between interpreting texts and interpreting students is promising: participants practiced the interpretation and understanding of initially cryptic texts and some aspects of the experience seemed to transfer to the understanding of students' mathematical products. One of the challenges is to find further appropriate historical texts around which such linkage can be strengthened. We relied on a source-work collection on the history of linear and quadratic equations (Arcavi, 1985, unpublished data). More appropriate texts may be needed, also including sources presenting conceptual doubts, possible contradictions between ideas, and so on (see, for example, Arcavi & Bruckheimer, 2000).

Another challenge would be to find an appropriate collection of interesting idiosyncratic student solutions or "student products" suitable for interpretation and discussion, as a follow up of the reading of the sources. Finally, the ultimate challenge would be that learning to listen attentively is achieved not only at the in vitro intellectual level, but also properly and productively during in vivo classroom interactions. Practicing attentive listening in a classroom alongside the multiple educational tasks to which teachers are required to attend can be an extremely demanding task (Lampert, 2001).

8.6 What about in-service teachers?

Would more experienced teachers gain from the experiences described above? In order to attempt an answer to this question, a second workshop with the same plan and leader was conducted for in-service teachers at Joetsu University of Education, Japan. Time constraints allowed only for shorter periods of individual work and whole class discussions. However, the participants seemed to work at a faster pace. This second workshop, followed by a presentation by the workshop leader, totaled five hours.

The atmosphere of engagement during the historical component was very similar to that of the first workshop. However, there were differences in the pedagogical component: the analyses and discussions were quicker and more efficient. The participants' previous experiences discussing student's perspectives (in their practices of lesson study) made this part a simple exercise for them. By their own account, these in-service teachers found the historical part much more significant and instructive, and they could spontaneously bridge between the historical and the pedagogical components.

It would seem that working around carefully chosen historical materials still makes this workshop meaningful and supportive even for teachers used to the practice of attentive listening.

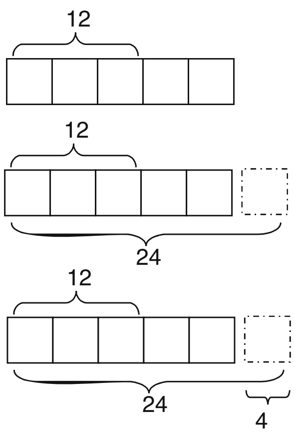
8.7 Tools for listening

Both pre-service and in-service teachers used similar graphical models to make sense of Ron's solution: in-service teachers used blocks (see Fig. 5) and pre-service teachers used the number line.

The participants engaged in understanding Ron's reasoning rather than evaluating it against the expected 'classical' solution. However, in the process they brought to bear their own tools not explicitly present in Ron's recorded solution. This implies that decentering towards attentive listening in order to understand others' solutions may be mediated by our own repertoire of tools. Although the 'spirit' behind the drawings may faithfully reflect Ron's reasoning, participants could not know (from the data provided) whether he thought in this way. Nevertheless, they used the representation in order to make sense of Ron's solution.

One may argue about the relevance of knowing whether this was Ron's model. In any case, a cautionary note is in place. A mathematician, who audited the second workshop, commented that one way of making sense of the Egyptian method is to think of linear functions of the form y=ax. Linear functions helped him to make sense of the Egyptian method; however, this is a clear imposition of a modern view into a primary source, rather than engaging in attentive listening of what the source is telling.

Fig. 5 Blocks used by in-service teachers



The graphical model teachers used to understand Ron's solution and the linear functions brought by the mathematician to understand the Egyptian method are different types of tools. These tools differ not only because the former is a graphical illustration and the latter a conceptually sophisticated formalization. The graphical model, as an illustration to represent the thinking process, could have been part of Ron's repertoire, whereas linear functions were absent in Egyptian mathematics.

These findings illustrate that:

(a) Decentering aimed at making sense of the other's perspective may rely on tools we, as interpreters, bring to the situation, in order to mediate the construction of our understandings; and
(b) The kinds of tools we bring can be of very different types, ranging from those which can be attributable to the students to those clearly beyond their reach – in both cases, tools may be helpful to the interpreter. However, some tools may be mistakenly misattributed to students.

8.8 In sum

The materials from the history of mathematics designed with the purpose of learning to understand the other's perspective can be engaging and meaningful. Prospective and in-service teachers may learn that teaching practice includes their understanding of idiosyncratic ways to grasp and do mathematics, and that such understanding can be pursued by means of ad hoc ways of questioning – different from the traditional questions most teachers are so used to asking. Also, the challenges of listening can be even more subtle than expected – for example, the evaluative stance may play complex roles in the development of attentive listening. Finally, the enactment of attentive listening may be mediated by ad hoc tools brought by the listener, and these tools can be of very different kinds – some of which, even if useful, may lead to over-interpretation. The results suggest that crafting such scenarios for learning to listen is a promising and fascinating avenue for research and development which should be further explored in all its complexity.

We believe we have answered our questions (of Section 3 above): We have described an example of an activity designed to promote attentive listening, its rationale, its implementation and some of the results thereof. However, we stress that in order to learn to listen we need much more than the good use of ad hoc and appropriate curriculum materials (only exemplified here – more work is needed towards full-fledged curricula). Even when implementation of these materials is faithful to their rationale, it may not be enough. The complex process of listening is deeply linked to an intricate collection of deep rooted beliefs about knowledge, about students, about learning and about the teacher's role. One cannot become a good listener unless one is genuinely convinced (much more deeply than at the mere declarative level) that:

- Students are sense makers in idiosyncratically sophisticated ways,
- Learning is a long-winded multifarious process in which social interactions (especially those between teacher and students) should respect ideas and incorporate them to the process, and that
- The teaching role rests heavily in setting up and nurturing dialogues which are central to the ongoing reshaping of knowledge.

Thus, we suggest that a curriculum of the kind proposed should be embedded within a conglomerate of teacher development activities, discussions and reflections coherently geared towards the development of a web of interconnected beliefs capable of sustaining attentive listening. The work described in this paper can be a small contribution to such a comprehensive effort.

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