Mathematical Thinking through investigation in multiplication For Primary Teacher Education programme in Hong Kong

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Introduction

Multiplication is an important component in primary mathematics learning. However, most teachers taught according to the textbooks and there is little room for investigation in this topic. The set of materials introduced her wish to complement mathematical thinking for primary schools students.

In Hong Kong, primary school teachers can go to Institutions to do an in-service release course, ranging from 5-week to a year, to refresh their teaching. This 5-week course may have different focus, such as small class teaching, children with special needs.

The following is the list of material that was used in the teachers' course, so that teachers can try them in their classroom. The materials have been tried out in schools before they are introduced to primary teachers. In total, there are 5 topics of materials introduced in this programme.

Topic A: Finishing the multiplication table
Topic B: Filling in numbers represented by English Letters
Topic C: Finding maximum / minimum product of multiplication
Topic D: Using factor and common factor
Topic E: Finding Equal product

Topic A: Finishing the multiplication table

Filling in table, with no specific requirement. For example, fill in the boxes for the following question so that multiplication established.

□ □ × □ 2 6 4	There are many answers to this question, and children are asked to find out answer and also the number of answers. Children can use trail and error to solve the problem with a calculator. Also, children can use their knowledge of divisibility to shorten their calculation.

$$\begin{array}{r} 88 \\ \times\ \ 3 \\ \hline 264 \end{array} \qquad \begin{array}{r} 66 \\ \times\ \ 4 \\ \hline 264 \end{array} \qquad \begin{array}{r} 44 \\ \times\ \ 6 \\ \hline 264 \end{array} \qquad \begin{array}{r} 33 \\ \times\ \ 8 \\ \hline 264 \end{array}$$

Apart from the working, a set of problems with progressive requirement of numbers are set so that children may be able to obtain the structure of mathematics. Answers to the above questions may not be easy to establish, so the following easier question are set.

Question： fill in integers so that the calculation is correct.	$\begin{array}{r} \square\square \\ \times\ \ \square \\ \hline \square\square 7 \end{array}$

Students at different ability can get their answer. There are 16 possible answers (answer in appendix A).

Question Using the numbers 1、2、3、4、5 once to fill in the boxes so that multiplication established. $\begin{array}{r} \square\square \\ \times\ \ \square \\ \hline \square\square \end{array}$	$\begin{array}{r} AB \\ \times\ \ C \\ \hline DE \end{array} \qquad \begin{array}{r} 13 \\ \times\ \ 4 \\ \hline 52 \end{array}$ First, C could not be 1 or 5. Similarly, B could not be 1 or 5. Hence 5 could be A or D. Finally D = 5, and A = 1. Trial and error give 13×4 = 52。

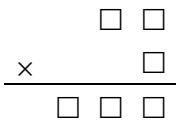
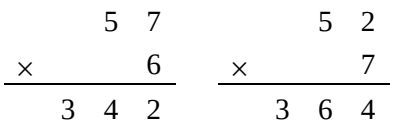
Question：	Answer：
Using the numbers 1、2、3、4、5、6 once to fill in the boxes so that multiplication established. $\begin{array}{r} \square\square \\ \times\ \ \square \\ \hline \square\square\square \end{array}$	$\begin{array}{r} 54 \\ \times\ \ 3 \\ \hline 162 \end{array}$

There are two strategy used by students:

By using trial and error and obtain 53×2 = 106.
This is a close answer and a few tries gives 54×3 = 162.

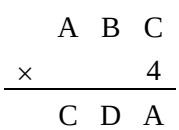
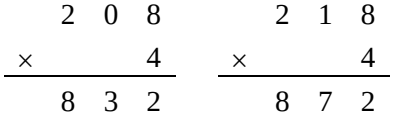
The second one is by using logical deduction
The number 1or 5could not be placed on the unit digit (marked by a cross ✕), since it will gives an answer 5 or 0, and the number 1 will give a same answer. After some trial, they get the same answer.

Also, posing question helps to learn the structure of mathematics. The following questions are posted by students and solved by their classmates.

Posing Question：	Answer：
Using the numbers 2、3、4、5、6、7 once to fill in the boxes so that multiplication established. 	

Topic B: Filling in numbers represented by English Letters
The next topic is similar to the topic A, except that students need to reason with the digits which are written in letters.

Logical deduction for multiplication

Question：	Answer：
Each letter represents a different integer; find the answer to the multiplication. 	

The process of reasoning are discussed during classes. The following is the sequence of discussion.

As $\overline{ABC}$ multiply by 4，the unit digit A of $\overline{CDA}$ must be one of 4，8，2，6，0.

The digit A could not be 0 or 1. A could only be 2 (otherwise the product is 4 digits)

$$\begin{array}{r} 2\ B\ C \\ \times\quad\ 4 \\ \hline C\ D\ 2 \end{array}$$

$\overline{CD2}$ is a multiple of 4,

$\overline{D2}$ must be a multiple of 4.

Hence D could be 3 or 7.

Suppose D = 3, then C ≠ 3, C = 8.

$$\begin{array}{r} 2\ B\ C \\ \times\quad\ 4 \\ \hline C\ 3\ 2 \end{array} \qquad \begin{array}{r} 2\ B\ 8 \\ \times\quad\ 4 \\ \hline 8\ 3\ 2 \end{array} \qquad \begin{array}{r} 2\ 0\ 8 \\ \times\quad\ 4 \\ \hline 8\ 3\ 2 \end{array}$$

Suppose D = 7, then C = 8, B = 1.

$$\begin{array}{r} 2\ B\ C \\ \times\quad\ 4 \\ \hline C\ 7\ 2 \end{array} \qquad \begin{array}{r} 2\ B\ 8 \\ \times\quad\ 4 \\ \hline 8\ 7\ 2 \end{array} \qquad \begin{array}{r} 2\ 1\ 8 \\ \times\quad\ 4 \\ \hline 8\ 7\ 2 \end{array}$$

After that more questions are posed. More materials is attached in appendix B.

Question：	Answer：
Fill in suitable integers $\begin{array}{r} A\ B\ C\ D \\ \times\quad\ 4 \\ \hline D\ C\ B\ A \end{array}$	$\begin{array}{r} 2\ 1\ 7\ 8 \\ \times\quad\ 4 \\ \hline 8\ 7\ 1\ 2 \end{array}$

Process of thinking：

$\begin{array}{r} A\ B\ C\ D \\ \times\quad\ 4 \\ \hline D\ C\ B\ A \end{array}$ $\begin{array}{r} 2\ B\ C\ D \\ \times\quad\ 4 \\ \hline D\ C\ B\ 2 \end{array}$ $\begin{array}{r} 2\ B\ C\ 8 \\ \times\quad\ 4 \\ \hline 8\ C\ B\ 2 \end{array}$	
A = 1 or A = 2, but A ≠1, and A = 2. D = 3 or D = 8, but D ≠3. C = 2，7. B = 1。	

Topic C: Finding maximum / minimum product of multiplication

Students can use calculator to find out the answers for the following questions. It is meant for them to discover the structure of multiplication and sought the pattern for the questions.

Question：

Using 1, 2, 3, 4 to fill in the boxes to make the greatest product

□□×□□.

The answer is not difficult, 41×32

The process is extended to the following questions.

Question 1：	Question 2：	Question 3：
Using 1，2，3，4，5，6，7 to form the greatest product. □ □ □ □ □ □ × □ Answer： 654321×7。	Using 1，2，3，4，5，6，7 to form the greatest product. □ □ □ □ □ × □ □ Answer： 75321×64。	Using 1，2，3，4，5，6，7 to form the greatest product. □ □ □ □ × □ □ □ Answer： 7531×642。

Students may not be able to work out the solutions and they are encouraged to identify the greatest product and verified with a calculator.

Learning Activities：

Try to identify the greatest product from the followings.

6 4 2 1	7 3 2 1	7 5 3 1	6 5 3 1
× 7 5 3	× 6 5 4	× 6 4 2	× 7 4 2

Answer：

6421	7321	7531	6531
× 753	× 654	× 642	× 742
4835013	4787934	4834902	4846002

These questions are extended to more complicated situations for class discussion. More

materials are attached in Appendix C.

Another type of finding greatest product is shown in the following examples.

> Question：
> From a set of 4 numbers, select 3 number to form a sum and multiply this sum with the fourth numbers, so that the product is greatest。
> For example, the 4 numbers are 2、3、5、7.

The following is a list of possibilities.

2×(3 + 5 + 7) = 30，

3×(2 + 5 + 7) = 42，

5×(2 + 3 + 7) = 60，

7×(2 + 3 + 5) = 70，

And the greatest product is 70, having the three smaller numbers add together and multiple with the largest number.

Topic D: Using factor and common factor

The following material required students to use their knowledge on factors and common factors to solve some problem. It is conducted in two steps. Step (1), doing multiplication, Step (2), reverse the thinking, given the product and find the numbers.

Question (Step 1): Write down the product of the numbers.

Question		
8	15	
18	12	

Answer		
8	15	120
18	12	216
144	180	

3×7

For the following table, fill in numbers so that the product equal to the number of the grids

For example, 「A×B = 35」, 「C×D = 12」,「A×C = 21」,「B×D = 20」.

Question		
A	B	35
C	D	12
21	20	

Answer			
7	5	35	5×7
3	4	12	2×6, 3×4
21	20		

3×7

Another Example:

Question			
			15

……			
3	1	5	15

Answer			
3	1	5	15

			108
			224
144	8	315	

	2	9	108
	4	7	224
144	8	315	

6	2	9	108
8	4	7	224
144	8	315	

Another kind of material also builds on using factors and common factors.

Starting activities：

From a set of 4 numbers, select 3 numbers to form a sum and multiply this sum with the fourth numbers.

For example, the 4 numbers are 3、5、7.

The following is a list of possibilities.

3×(5 + 7) = 36,

5×(3 + 7) = 50,

7×(3 + 5) = 56,

The above is a preparation for the following “reverse questions”.

Reverse Question：

There are 4 numbers a、b、c、d (2 to 9).

If

a×(b + c + d) = 69

b×(a + c + d)= 105

c×(a + b + d) = 133

d×(a + b + c) = 165。

Find the values of a、b、c、d.

Listing the number with their factors:

69 = 3×23

105 = 5×21 = 7×15

133 = 7×19

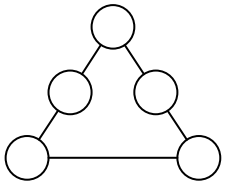
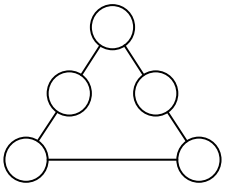
165 = 11×15

We can decide that a < b < c < d. and judging by the expression, the answer are 3、5、7、11.

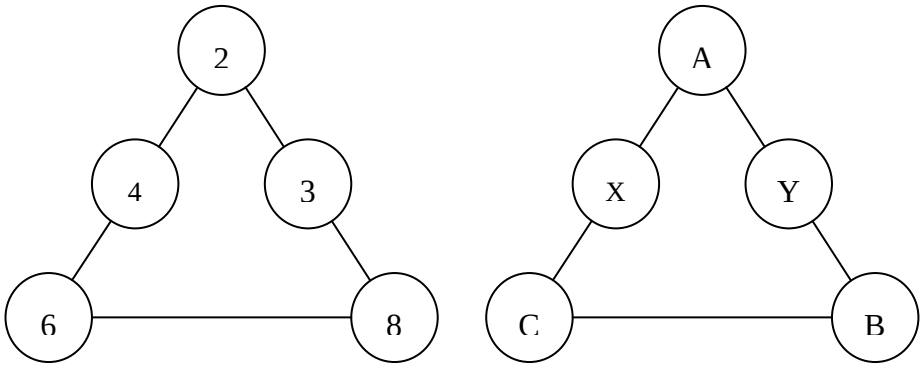
Topic E: Finding Equal product

This set of material can be used for fun and also for finding the structure of indices. The logic of the work is explained to class.

Question 1：	Question 2：

Using the numbers from 1 to 9, fill in the circle so that the product on the three sides is equal. 	Using the numbers 2、3、4、6、8, fill in the circle so that the product on the three sides are equal. 

The following is the answer to the question. The product is 48.



Though student form the product by trial and error, the question is analyed so that the knowledge can be transferred to a new situation.

Suppose the product on each side is T.

Then (A×Y×B)×(B×C)×(A×X×C) = T×T×T

$\Rightarrow$ A×B×C ×X× Y×A×B ×C .= T×T×T

$\Rightarrow$ (2×3×4×6×8)×A×B ×C = T×T×T

$\Rightarrow$ ($2^7 \times 3^2$)×A×B×C = T×T×T

Hence, the least value of A×B×C is $2^2 \times 3^1$ and T×T×T = $2^9 \times 3^3$

That is, the least value of T= $2^3 \times 3^1$ = 24。

As T= 24 $\Rightarrow$ B×C = 24, which is not possible.

The next possible value of T= $2^4 \times 3^1$ = 48 $\Rightarrow$ B×C = 48。

Hence, B and C are 6 and 8. The rest is not difficult.

A set of material is attached in appendix E.

Summary

A trail teaching is included for the topic B, filling in numbers represented by English Letters.

Students are encouraged to trial out the answer and pose their question. It is found that students can abstract their thinking and work on these tasks which do not have physical context.

The programme have been introduced for some years and the result are positive. Not only that students understand multiplication more deeply, their interest in mathematics also grows stronger.

Appendix A Materials used in multiplication tables

Question: fill in integers so that the calculation is correct.

$$\begin{array}{r} \square\,\square \\ \times\quad\square \\ \hline \square\,\square\,7 \end{array}$$

Answer:

$$\begin{array}{r} 89 \\ \times\quad 3 \\ \hline 267 \end{array} \qquad \begin{array}{r} 79 \\ \times\quad 3 \\ \hline 237 \end{array} \qquad \begin{array}{r} 69 \\ \times\quad 3 \\ \hline 207 \end{array} \qquad \begin{array}{r} 59 \\ \times\quad 3 \\ \hline 177 \end{array}$$

$$\begin{array}{r} 49 \\ \times\quad 3 \\ \hline 147 \end{array} \qquad \begin{array}{r} 39 \\ \times\quad 3 \\ \hline 117 \end{array} \qquad \begin{array}{r} 21 \\ \times\quad 7 \\ \hline 147 \end{array} \qquad \begin{array}{r} 31 \\ \times\quad 7 \\ \hline 217 \end{array}$$

$$\begin{array}{r} 21 \\ \times\quad 7 \\ \hline 147 \end{array} \qquad \begin{array}{r} 31 \\ \times\quad 7 \\ \hline 217 \end{array} \qquad \begin{array}{r} 41 \\ \times\quad 7 \\ \hline 287 \end{array} \qquad \begin{array}{r} 51 \\ \times\quad 7 \\ \hline 357 \end{array}$$

$$\begin{array}{r} 61 \\ \times\quad 7 \\ \hline 427 \end{array} \qquad \begin{array}{r} 71 \\ \times\quad 7 \\ \hline 497 \end{array} \qquad \begin{array}{r} 81 \\ \times\quad 7 \\ \hline 567 \end{array} \qquad \begin{array}{r} 91 \\ \times\quad 7 \\ \hline 637 \end{array}$$

Question: Fill in 5 different integers so that the calculation works.

Question	Answer			
$\begin{array}{r} \square\,\square \\ \times\quad\square \\ \hline \square\,\square \end{array}$	$\begin{array}{r} 43 \\ \times\quad 2 \\ \hline 86 \end{array}$	$\begin{array}{r} 34 \\ \times\quad 2 \\ \hline 68 \end{array}$	$\begin{array}{r} 38 \\ \times\quad 2 \\ \hline 76 \end{array}$	$\begin{array}{r} 39 \\ \times\quad 2 \\ \hline 78 \end{array}$

Posing Question：	Answer：
Using the numbers4、5、6、7、8、9 once to fill in the boxes so that multiplication established. □ □ × □ □ □ □	9 4 × 7 6 5 8 8 4 × 9 7 5 6

Using the technique of division

Question: Fill in suitable integers	There are many answers. 3456 ÷ 2 = 1728 3456 ÷ 3 = 1152 3456 ÷ 4 = 864 3456 ÷ 6 = 576 3456 ÷ 8 = 432 3456 ÷ 9 = 384
□ □ □ × □ 3 4 5 6	

Question：	Answer：
Fill in suitable integers □ □ □ × □ 2 3 4 5	4 6 9 × 5 2 3 4 5 3 3 5 × 7 2 3 4 5

Appendix B Materials used in deduction thinking in multiplication

Question	Answer
A B C × C D B C	1 2 5 × 5 = 6 2 5 1 7 5 × 5 = 8 7 5

Question

```
  A B
×   B
-----
1 C B
```

Answer

```
  3 5
×   5
-----
1 7 5
```

```
  2 6
×   6
-----
1 5 6
```

Question

```
  A B
×   8
-----
C C 6
```

Answer

```
  9 7
×   8
-----
7 7 6
```

```
  4 2
×   8
-----
3 3 6
```

Question

```
  A B
×   2
-----
C D A
```

Answer

```
  6 8
×   2
-----
1 3 6
```

Question

```
  A B C 4
×       3
---------
5 A B C
```

Answer

```
  1 7 2 4
×       3
---------
5 1 7 2
```

Question

```
  A B C D E
×         4
-----------
E D C B A
```

Answer

```
  2 1 9 7 8
×         4
-----------
8 7 9 1 2
```

Appendix C　Materials used in finding maximum / minimum product of multiplication

Question：	Question：
Using 1，2，3，4 to form the greatest product. □ □　　2 1 □　　3 × □　　× 4 2 5 2	Using 1，2，3，4，5 to form the greatest product. □ □　　2 1 □　　3 □　　4 × □　　× 5 1 2 6 0

Using trial and error, students got the following product and they figure out the pattern for obtaining greatest product.

3 2	4 1	2 1	5 1
1	2	3	2
4	3	4	3
× 5	× 5	× 5	× 4
6 4 0	1 2 3 0	1 2 6 0	1 2 2 4

It seems that 21×3×4×5 = 1260 has the greatest product. Such thinking is deduced to the following question.

Question：

Using 1, 2, 3, 4, 5,6 to fill in the boxes to make the greatest product

□□×□□×□ ×□.

3 2	6 1	3 1
4 1	5 2	4 2
5	3	5
× 6	× 4	× 6
3 9 3 6 0	3 8 0 6 4	3 9 0 6 0

Appendix D Materials used in multiplication (factors and common factors)

Reverse Question：

There are 3 numbers a、b、c (2 to 9).

If

a×(b + c) = 56，

b×(a + c) = 72

c×(a + b) = 80

Find the values of a、b、c.

Finding the factors

56 = 2×28 = 4×16 = 8×7

72 = 2×36 = 3×24 = 4×18 = 6×12 = 8×9

80 = 2×40 = 4×20 = 5×16 = 8×10

Answer a < b < c and a = 4，b = 6，c = 8。

Generalise:

Two numbers are selected from 4 numbers a、b、c、d (2 to 9) and added.

The sum will multiply the reminding two numbers so that the product is greatest.

Suppose the four numbers are 2、3、5、7。

2×3×(5 + 7) = 72

2×5×(3 + 7) = 100

2×7×(3 + 5) = 112

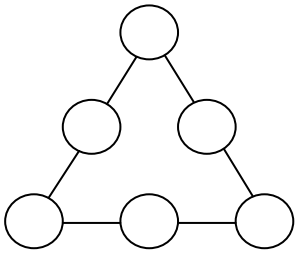
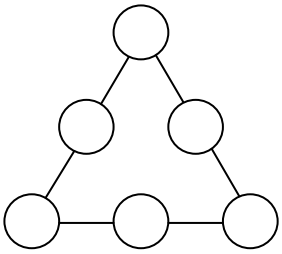
3×5×(2 + 7) = 135

3×7×(2 + 5) = 147

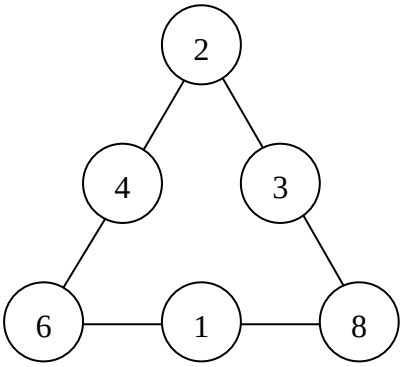
5×7×(2 + 3) = 175。

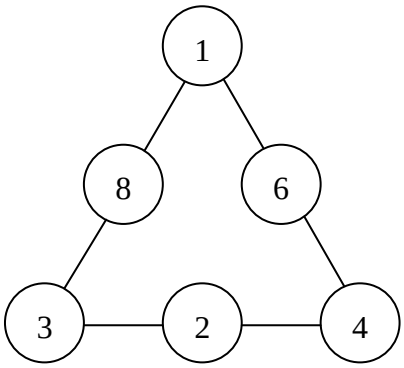
The greatest answer is selection of the two largest number and multiply together with the um of the reminding two.

Appendix E　Materials used in multiplication (finding equal products)

Question1：	Question2：
Select numbers from 1 to 10 and fill in the circles, so that the product on each side are equal. 	Using numbers 1、2、3、4、6、8 and fill in the circles, so that the product on each side are equal. 

Answer to the questions





Question1：	Answer：
Using numbers 1，2，3，4，6，8，9 and fill in the circles, so that the product on each side are equal.