SEAMEO School Network Program provided by CRICED-University of Tsukuba

Mathematics Education to Develop Students Agency: The Case of Fractions

Teaching Mathematics to Develop Mathematical Thinking as Higher Order Thinking:
How do you teach? Why? II

Lesson 2: Dividing and Operational Fraction, and Quantity Fraction

Lectured by: Masami Isoda, Prof/PhD., University of Tsukuba, Japan

With support of: Teh Kim Hong, Ms., SEAMEO-RECSAM, Malaysia

Raimundo Olfos, Prof/PhD., Pontificia Universidad Catolica de Valparaiso, Chile

With participation of: Ms. Laura Lopez Zarate, Colombia

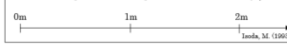
Ms. Mei Nakada, Japan

Mr. Diego Solis, Costa Rica

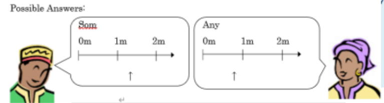
Chapter 1: What is fraction? **Lesson #1**

The professor asked the teachers:

There is 2m tape. Where is the position of ⅔ m? Show it by ↓



Possible Answers:



What do you want to do next? Isoda&Katagiri (2012)

Mathematical Values
Seeking -
- Generality and expandability
- Reasonableness and harmony
- Usefulness and efficiency
- Simpler and easier
- Beautifulness

Mathematical Attitude
Attempting to -
- See and think mathematically
- Pose questions and develop explanations
- Generalise and extend
- Appreciate others' ideas and change representations for meaningful elaborations

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Lesson #1

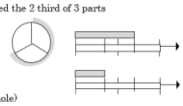
Professor: You are good teachers, aren't you? Because you already have the custom to ask why to others and discuss. It means that you usually engage in a similar activity by yourself. You already have the mind set for learning how to learn by yourself!

Discussion 1: Why do you think so?

Som: It is larger than 1m because 2/3 means:

Ano: No, the whole is divided by 3, then shaded the 2 third of 3 parts

Som: Yeah, I should draw like that.



Any: Wow, 2/3 is less than 1. It is like this:

Som & Ano: ..continue..(talking about part-whole)

Mathematical Attitude
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Any: ...continue... (talking about part-whole) **Lesson #1**

Prof: I could not understand well all of your explanations and diagrams. What is the original question?

Any: Where is the 2/3m?

Prof: Yes, then, the denomination 'meter' is missing in all of your explanations. We should write the meter on the number line. Can you explain 2/3m, again?

Any: It is 2/3 of 1m

Som: It is 2/3 of 2m

Prof: Now we can discuss what differences are there between these two ideas.

The denomination of quantity is important. When we explain something with the situation, we usually omit some words which are already known in the situation. In this case, the unit of meter 'm' itself is written in the task; however the meter was missing from their explanation and diagrams. In their discussion, if the denomination of quantity is missing, we are not sure which part you are explaining.

People who participated in the discussion; however, not sure which answer is appropriate.

Mathematical Ideas of:
Set, Unit, Comparison, Operation, Algorithm, Fundamental Principles, Permanence of Form, Various Representations and Translations.

Mathematical Ways of Thinking:
- Generalisation and Specialisation
- Extension and Integration
- Inductive, Analogical and Deductive Reasoning
- Abstracting, Concretising and Embodiment
- Objectifying by Representation and Symbolizing
- Relational and Functional Thinking
- Thinking Forward and Backward

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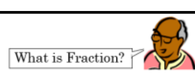
Lesson #1

Then, Professor asked.



Discussion 2: What is Fraction?

Ano: It is a part of the whole.

Prof: It is not clear. Could you explain it more exactly?

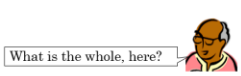
Ano: When the whole is divided into parts, fraction is the number of the parts in the whole.

Prof: Still are missing some important terms. Can anyone support?

Oth: When the whole is divided into 'm equally parts', 'fraction n/m' is 'n pieces of the parts in the whole'.

Any: The whole for 2/3m is 1m.

Som: The whole for 2/3m is 2m.



Mathematical Ideas of:
Set, Unit, Comparison, Operation, Algorithm, Fundamental Principles, Permanence of Form, Various Representations and Translations.

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Discussion 3: Which one should be the whole? **Lesson #1**

Any: 1m

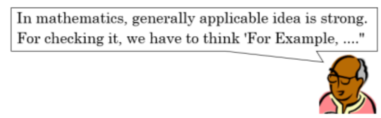
Som: 2m

Ano: 1m or 2m, which one shall we chose for the whole, in this task?

Any: The whole for 2/3m is 1m

Som: The whole for 2/3m is 2m.

Oth: How can we discuss?



Mathematical Values
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Discussion 4: For Example, if............. Lesson #1

Prof: For example, if I change the task from 2/3m to 1/2m what will happen?

Any: The 1/2m is 0.5m.

Som: The 1/2 of 2m is 1m.

Ano: No, what you are saying is that 1/2m is 1m. However, 1/2m is 0.5m, isn't it?

Pro: If I change the original question '2/3m' to '1/1m' what will happen?

Oth: If Som's idea, 1/1m is the 1/1 of 2m, thus 2m. 1/1m =2m. It is strange.

Prof: Yes, we can generalize Any's idea and not generalize Som's idea. When we say 2/3m, the whole is denominated by the 2/3 'meters': The unit of meters is 1m in any time.

What did you learn from this class? Let's write your own resume based on what you learned.

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Chapter 2: Dividing Fraction and Quantity Fraction

On the Fourth APEC-Tsukuba Lesson Study Conference, Gould. P.(2010), Inspector of NSW, Australia, lectured the roots of misconception of fraction in the case of Australian students with Australian textbook. He mentioned that the standard fraction notation itself encourages a 'count' interpretation of the regional 'parts of a whole' model (Figure 1). He quoted Hackenberg (2007) who asked the students to draw seven-fifths of a candy bar if the drawing of the rectangle on their paper represented a candy bar. Although both girls correctly created seven fifths of the rectangle, when asked about the size of the pieces in the bars they had drawn, the girls maintained that the pieces were sevenths. Depending on his report, the major root of misconception is just counting of the parts without considering the whole. On his lecture, he also mentioned Japanese better approach.

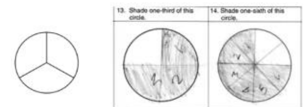

Figure 1 Circle Model and Answer of Grade 5 student (Gould, P. 2010)

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On the Fifth APEC-Tsukuba Lesson Study Conference, Lewis, C. (2011) lectured about her research for professional development which compared three groups: Lesson Study with Japanese Resource for fractions, Lesson Study without it, and Ordinal professional development programs. In her lecture, she mentioned two points: Firstly, Japanese approach for introducing fractions is better approach for understanding the meaning of fractions. Her recommended Japanese Approach is shown on the Figure 3a &3b.

Change in Teachers' Fraction Knowledge (N=213)

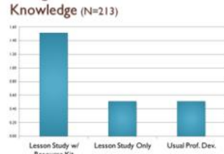

Figure 2 The effect of Lesson Study with Japanese Resource (Lewis, 2011)

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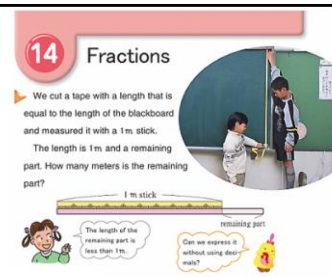

Terminology to explain the nature/objective of each task and task sequence.

Terminology to design Learning trajectory by using what students already learned.

Quantity Fraction
Operational Fraction
Quotative Division

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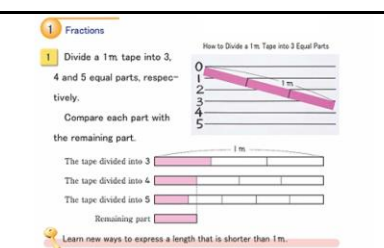

Figure 3a. Study with your friends: Mathematics for elementary school, Gakkotosho (2005, Grade 4, vol.2, pp.65-66; 2011, Grade 3, vol.2, pp88-89)

Dividing Fraction
Partitive Division (Learned at Grade 2 as half and quarter).

Quotative division
Remaining part

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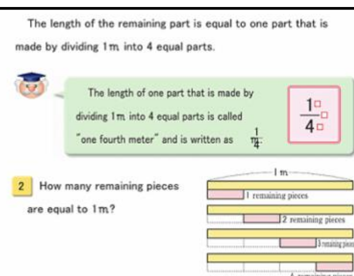

Here the definition is not the part-whole relationship.

Unit Fraction.

Quotative Division can extend more than whole by using unit fraction/remaining part as a scale.

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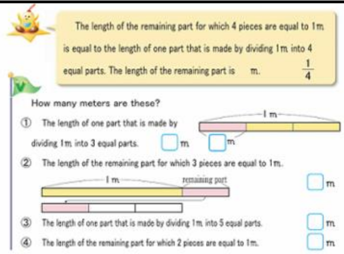


Figure 3b. Study with your friends: Mathematics for elementary school, Gakkotosho (2005, Grade 4, vol.2, pp.65-66; 2011, Grade 3, vol.2, pp88-89)

Secondly, only the case using Japanese Resource for fractions has improved teachers' fraction knowledge, significantly.

Both lectures recommended Japanese approach. One of their recommended approaches is shown in Figure 3a and 3b which uses the remainder for measuring the unit length 1m.

Questions for professional development 2

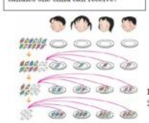
Q9. Let's analyze the ideas of Som and Ano in Chapter 1 from the view point of Peter, G.(2009).

Q10. Let's view Figure 3a and 3b. What is the common idea between Figure 3 and the suggestion of Professor in Chapter 1?

Q11. In division, there are two meanings as follows. Let's analyze the dividing activity in Figure 3a and 3b from the viewpoint of two different meanings.

Partitive division

12 candies are divided by 4 children, equally. How many candies one child can receive?



Quotative Division

12 candies are distributed by 4 candies for one child. How many children can receive candies?

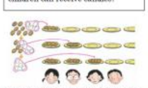


Figure 4. Gakko Tosho Grade 3 (Vol2. p4,p8, 2005; vol1, p60, p64, 2011)

Both activities for dividing are different for children however both activities include the repeated subtraction of the same amount from the total amount (Only see the left hand part of every picture, in Figure 4 left and right). Partitive division establishes the equally partition at first. Quotative division quotes the same amount recursively until we cannot quote, not sure the number of partitions at the beginning.

Q12. Please do the activity in the Figure 3 by yourself and explain the difference with your traditional approaches for teaching fraction based on Pizza model such as on the figure 1.

In mathematics, fraction n/m as for rational number is defined when n and m are integers (or rational number) and m is not equal 0.[1] If n or m are irrational or complex numbers, the fraction is not rational number. In mathematics, the concept is usually explained by the definition and the equivalent properties with exemplars. However, in school mathematics, especially for the elementary school, the meaning is explained using the different representations, model or situation with action (See Chapter 1, Q7,1). If we only teach number and calculation, children usually explain the ways of calculation as a rule (See Chapter 1, Q7, 3) when we ask them to explain 'Why?'. If readers are hoping children to explain the meaning, you have to assist children to use the further representations and models with situation. To meet these demands, Q12 on this chapter has asked you to engage in the same activity by yourself. When you use the 1m tape and you recognize the length of blackboard is 125cm, you can well prepare the activity on Figure 3.

Fraction is used in various contexts/situations with different meanings even if it is clearly defined mathematically. First part of this booklet explains how Japanese distinguish those meanings.[2] Second part explains how Japanese teach four operations of fraction.

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[1] Fraction, rational number, as a number system will be discussed at later chapters.

[2] Internationally, there are various technical terms to explain the meaning of fraction on situations such as Behr, M., et al. (1983) and Charalamous, C. Y & Pitta-Pantazi, D. (2007). However, it is un-usual to distinguish dividing fraction and operational fraction. Japanese technical terms are more precise to establish fraction in relation to what students learned on multiplication and division (see Isoda&Olfos, 2021, p.91).

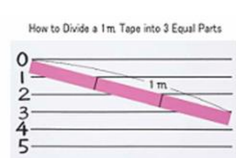
- Dividing Fraction

Many people believe that fraction is Dividing Fraction. It is the part-whole relationship which means that fraction n/m is dividing the whole into m equally parts and selected n parts from them. It is deeply related with the activity of partitive division situation. 2/3 means the part-whole relationship which shows the whole is divided equally into three parts and selected two parts. If the pizza is divided into three parts and take two parts, it means 2/3 of the pizza.

Paper folding, Origami, of the square paper includes the possible action for dividing the object equally such as half, and quarter. On this action, children usually think that dividing fraction cannot be larger than the one whole. However, it is shown in Chapter 1, where people lost the denomination, children usually lost the process dividing the whole into equally parts. As Gould mentioned, the misconception for dividing fraction is originated from giving children worksheet to shade the object already equally divided through counting. On this worksheet, children are not necessary to recognize the whole and to think how to divide it at first because they just count the number of parts on equally divided object. At the same time, on the ill-solution in Figure 1, it is a relationship to see the confusion of the fraction as ratio which allows to think the part-

part relationship instead of the part-whole relationship described in Chapter 5. The task itself expected to think part-whole relationship, thus, the major problem is related with their losing of whole for equal-dividing of the whole on the context. For developing better understanding, we have to begin the activity to set a whole as one unit by children and ask them to divide it into the number of equally parts. After dividing into equally parts, children chose the necessary number of parts from the whole.

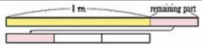
The any length of tape can be divided into equally if we use the parallel lines with same interval. Necessity of this diagram is related with dividing the whole to the equally parts.



Exercise:

Draw the 1/3 of Pisa on your notebook and compare the size of it with other's drawing.

Operational (measuring) Fraction



Operational Fraction (measuring fraction) can be seen as a kind of dividing fraction however it tries to measure the whole as a unit by the remaining part. From the viewpoint of division, it can be seen the activity of quotative division. In the following, three times of the remaining part is 1m. It means that 1/3 of 1m is the unit for counting however the whole, that should be divided, is 1m. Additionally, the original length is longer than the whole 1m. Mixed fractions and improper fractions have already existed even though children do not study them.

It is strange! We cannot always measure the 1m using the remaining part. What shall we do when we get another remaining when we measure the whole by the part? The operation does not work!

Yes, you usually give me the good question. Euclid (BC 3C) approved the way to find the unit (Greatest Common Divisor) for the measurement. It works with whole numbers (Integers) and called Euclidian Algorithm.

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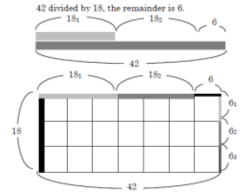
Euclidian Algorithm

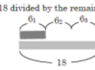
For example, Greatest Common Divisor of 18 and 42 is: 42=6x7, 18=3x6, 3 and 7 are prime numbers, then 6 is GCD of 18 and 42. It will be found using the remainders as follows.

42=2x18+6 → 18=3x6

42 divided by 18, the remainder is 6.

18 divided by the remainder 6.





The rectangular diagram is the better way to demonstrate Euclidian Algorithm.

Euclidean Algorithm is the way to find a unit measurement for given two numbers as greatest common divisor through measuring by remaining part.

Exercise:

Let's find the GCD of (41, 18) by the rectangular diagram.

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Quantity Fraction

Quantity fraction is the fraction with denomination. 1/3 of 1m is 1/3m in quantity fraction. The dialectic in Chapter 1 is done between quantity fraction (2/3m) and dividing fraction (2/3 of 2m). If we lost the denomination, we cannot distinguish quantity and dividing fraction like the communication in Chapter 1. It means that we cannot arrange the position of fraction on the number line if we do not have the denomination of the quantity. Quantity fraction allows us to compare the size of fraction in relation to the unit quantity. Dividing fractions are not easy to compare on the number line because the size of the whole unit is not clear and it looks always less than one.

In Figure 3a and 3b, all fractions are denominated with meter. All of them are quantity fraction based on the unit meter. In Japanese textbook, fraction is introduced by the quantity fraction with operational fraction and dividing fraction. It is better than the traditional approach introduced by dividing fraction for recognizing what is a unit (whole). Because it allows us to extend the fraction larger than the whole unit, and enables us to compare the size of fraction on the number line.

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Unit Fraction for measuring unit

Unit fraction is the fraction in which numerator is the one such as 1/3. Unit fraction is the unit for measuring up to improper fraction. Any fraction is represented as follows: (the specific number) x (the unit fraction). It is the necessary base to extend proper fraction to improper fraction. Thus, in fraction, there are two types of the unit. Firstly, the whole is the unit. Secondly, unit fraction: The unit fraction is the unit for counting the numerator.

The idea of unit fraction on base ten place value system is related with decimals. For example, 1 mm as unit quantity for length is usually introduced as dividing 1 cm by 10, equally. The relationship between 1cm and 1 mm is a base to introduce decimals such as 10 mm is 1 cm and 1 mm is 0.1 cm. 1 cm and 1 mm scales are given on the ruler and tape measure, it is the bases for number line which begins from 0 to +∞. On this context, operational fraction and unit fraction is the bases for extension of numbers.

In world known Japanese approach, fraction is didactically explained various technical terms and introduced based on quantity and operational fraction, the reminder as measurement, for preparing the unit fraction (Figure 3b). If children understand the quantity fraction and the unit fraction, they can easily represent the fraction on the number line such as in Chapter 1 and easier to think four operations of fractions.

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Tenoch Cedillo, Masami Isoda, et al. (2013) Matemáticas para la educación normal : guía para el aprendizaje y enseñanza de la aritmética, Estado de México : Pearson Educación de México.

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The textbook used in this course must be cited as shown below:

Masami Isoda (2013). Fraction for Teachers: Knowing What before Planning How to Teach. Tokyo: CRICED, University of Tsukuba.