SEAMEO School Network Program provided by CRICED-University of Tsukuba

Mathematics Education to Develop Students Agency: The Case of Fractions

Teaching Mathematics to Develop Mathematical Thinking as Higher Order Thinking: How do you teach? Why? II

Lesson 6: Multiplication and Division of Fractions (1)

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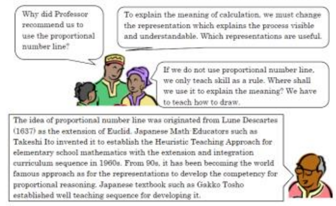
Chapter 6. Multiplication and Division of Fraction

For children who produce the multiplication and division of fraction by and for themselves, this chapter explains the necessary knowledge needed to produce the idea.



Number line

In chapter 1 and 2, we recognize that shading activities the parts for counting is the root for the misconceptions of fractions for missing the idea of whole as for a unit. For developing children who learn mathematics for and by themselves (see Chapter 1), children are necessary to draw and use appropriate diagram for explaining their ideas. On the previous chapters, appropriate diagram needs to show original unit with quantity fraction and the unit fraction for measuring its number like measuring by using the reminder as for operational fraction. The tape and number line with quantities are appropriate diagram on this condition. For developing children who will draw such a diagram, firstly, we have to develop children who draw the number line by and for themselves.

Number line which shows the position of number on the line is introduced by taking the same intervals by the unit of measurement recursively for comparing the size of number. At this moment, it looks a line the discrete number is given because it is given

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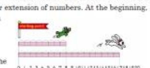
by **interval** as for the scale on the line and there is only one interval but no number between two numbers. On the process of extension of numbers, when students rescale it by using smaller unit or larger unit, it begins to function as number lines which shows the position of various numbers and used for extension of numbers. At the beginning, children learn the '0' is the starting point on the line instead of 'nothing.' On the number line, 0 shows the origin of **position** as for measuring by the interval (unit). The difference of positions shows the distance (the number of intervals: cardinal number).



If teacher does not teach the measuring by the unit from the starting point 0, children may confuse one on counting intervals as scale number 0 on the left instead of the scale number 1 on the right. Children learn the number line as for comparing the size and ordering of numbers and recognize the number on the base ten systems. Taking interval is the preparations for the multiplication and division, too.

5 Write the correct numbers in the ☐.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14		16	17	18	19
20	21		23		25	26		28	
30	31	32		34	35		37	38	39
	41	42		44		46	47		
50	51		53		55		57	58	59

If your children do not know how to draw the number line, let's give them the opportunity to draw it by themselves. It is the activity of measurement by using arbitrary unit.

Multiplication

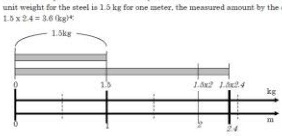
(Gakko Tosho, Grade 2, vol.2 pp2-3, 2005; pp6-7,2011)

In the case for lower elementary school mathematics, children study how to use daily language mathematically on the situation. Definition of arithmetic operations is usually done on the situation in daily context because we have to develop children to use four arithmetic on their daily life.

On this issue, multiplication is introduced at the following situations: the number of dishes and the number of objects for each dish.

From the viewpoint of measurement, the situation is used to explain multiplication that the multiplication is measuring the amount of the quantity by the unit of quantity when the unit for the amount is known by the quantity (Ministry of Education, Japan, 1960; Freudenthal, 1983). For example, when the amount is 8 dishes and the unit for the amount is 2 cakes in each dish, the measured amount by the quantity is $8 \times 2 = 16$ (cakes).

This definition works for multiplication of decimal numbers and fractions as well as the situation of repeated addition. For example, when the amount of steel is 2.4 m and the unit weight for the steel is 1.5 kg for one meter, the measured amount by the quantity is $1.5 \times 2.4 = 3.6$ (kg)*:



In relation to how to calculate, multiplication is explained with the repeated addition, however, multiplication of decimal numbers and fractions is not explained by the repeated addition but explained by multiplication table with distribution law on base ten place value system. For explaining the multiplication of decimal numbers and fractions, we can use the proportional number line which represents the meaning of multiplication by the measuring.

How to find the expression from the situated problem

At the end of last chapter, you learned how to draw the proportional number line by the task below:

[4] This textbook using Japanese notation: 1.5 (kg/m) x 2.4 (m) = 3.6 (kg). In English notation, it should be 2.4 (m) x 1.5 (kg/m) when '2.4 x 1.5' is read as '2.4 times 1.5'. In English, '2.4 times' implicates 'multiplied by 2.4'. Thus, as long as you read '1.5 x 2.4' as '1.5 multiplied by 2.4,' Japanese notation of multiplication is understandable. Indeed, 'a x b' can be read as 'a multiplied by b' even in English. English usage has inconsistency.

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1 The price of the ribbon is 80 yen per meter. Let's find out how much it would cost for ☐m.

The box (in blanket) is 2.4 at the end of last chapter. The children who have not yet learned the multiplication of decimal numbers and fractions cannot easily recognize that this task is multiplication. On the other hands, if we put the whole number such as 2 into the box, children who learned the multiplication of whole number could easily understand that this task is multiplication because multiplication is introduced in daily situations on the whole number.

On the context of extension of numbers and operations, Japanese teachers usually prefer this problem posing strategy like this form and ask children to put any number they want into the box and discuss how. Through putting into the simple number, children recognize this task as multiplication and in the case of whole number, they already learned and in the case of fraction and decimal number, they did not yet learn. When the class begins this way, children recognize this task as the task for multiplication and they would like to inquire how to find the answer using what they already learned.

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Exercise

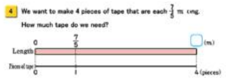
Let's draw the proportional number line when the box (blanket) is 2 (m), 2.3 (m) or $\frac{3}{2}$ (m) on 1.

In this exercise, for answering in the case of 2.3 m, we have to change the unit from 1m to 0.1m as well as the case of 2.4m. In the case of fraction, we usually change the unit from 1m to the unit fraction: this case $\frac{1}{2}$ m is the unit for measuring. If 1m is 80 yen, $\frac{1}{2}$ m is 80 ÷ 2 yen. If $\frac{1}{2}$ m is 80 ÷ 2 yen, $\frac{3}{2}$ m is 80 ÷ 2 × 3 yen. For considering like this, children needs to draw proportional number line and apply multiplication and division on the number line.

Fraction × Whole Number

Please explain the following:

4 We want to make 4 pieces of tape that are each $\frac{7}{5}$ m long. How much tape do we need?



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If 1 piece is $\frac{7}{5}$ m and addition of fractions is known, 4 pieces are $\frac{7}{5} \times 4 = \frac{7}{5} + \frac{7}{5} + \frac{7}{5} + \frac{7}{5} = \frac{28}{5}$.
After the explanation, we should ask children as follows: Is it possible to find an easier or faster way? Then, $\frac{7 \times 4}{5}$ is recognized as a simple way for $\frac{7}{5} \times 4$.

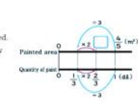
In mathematics, we usually produce shorter and simple ways. Seeking simplicity is a basic value of mathematics.

Exercise

In Gakko Tosho textbooks, proportional number line changes from the tape diagram and number line to two number lines at Grade 6. Let's put the number in the box and answer: "We can cover an area of $\frac{4}{5}$ m^2 with 1dl paint. How many m^2 can we cover with ☐ dl of the paint?"

Fraction × Fraction

If the box is a whole number, we already learned. If the box is a fraction such as $\frac{2}{3}$ dl, we can draw the proportional number line:



If we develop the way of calculation as well as multiplication of decimal numbers, it can be calculated as follows: $\frac{4}{5} \div 3 \times 2$.

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After the explanation, we ask children as follows: Is it possible to find easier or faster way? Then, $\frac{4}{5} \div 3 \times 2 = \frac{4\times 2}{5\times 3}$ is recognized as the simple way for $\frac{4}{5} \times \frac{2}{3}$.

In mathematics, we usually try to produce shorter and simple way.

Area Diagram

In school mathematics, area diagram is usually recommended for use in explaining (a+b)(c+d)=ac+ad+bc+bd. On the area diagram, multiplication is the two dimensional idea and it functions as for the model of commutativity. Some people strongly believe that the area diagram is the best way for explaining multiplication and division because it provides the wall painting/shading metaphor based on two dimensions. The misconception will appear if students do not feel the necessity to draw the same size diagram.

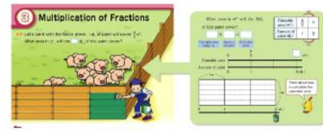
As long as teachers try to explain fraction as dividing fraction it might be true, however, shading activities of the area diagram itself is a major source of the misconception if children recognize fraction only by dividing fraction (see Chapter 2).

What is necessary to develop students by and for themselves is that students are able to draw the area diagram by and themselves as for the tool for reasoning as well as the proportional number lines.

Historically, Euclid produced the theory under the dimension and Descartes overcame the wall between dimensions by the proportional number line which define multiplication by the measurements.

Exercise

1) Let's solve the following task by three different methods.



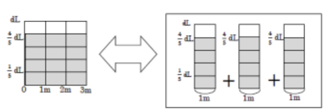
1) Let's compare three methods. Which one do you recommend? Why do we need more?

2) There are a number of students who get the answer using the area diagram such as $\frac{4}{5} \times 3 = \frac{12}{15}$. Why does the area diagram produce such answer?

There are two types of area diagram. The first type is already shown in the textbook. It shows the area of the wall itself for showing the painting. Another type of area diagram represents the following situation:

There is a wall which we use the paint $\frac{4}{5}$dL for painting 1 mater of wall. How much litter do we need for painting 3 meters?

For this task, we draw the following two diagrams for the same meaning.



In both diagrams, the denominations of quantities are necessary because children might develop misunderstanding such as $\frac{4}{5} \times 3 = \frac{12}{15}$ if there are no denominations.

Area diagram is fine to explain meaning. However, the children who still keep the dividing fraction misunderstand the meaning of whole. Indeed, the left hand side of the above diagram can be read as $\frac{4}{5}$ if the square is the whole even though it shows quantity. Here, the key is a unit fraction $\frac{1}{5}$ dL (quantity fraction!) as well as the whole 1 L. If teachers ask students just shading without considering these two units, we are not sure students understand well or not even though teachers felt success to explain for him/herself.

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Masami Isoda, Raimundo Olfos. Edited. (2021). *Teaching Multiplication with Lesson Study*. Cham: Springer.

The textbook used in this course must be cited as shown below:

Masami Isoda (2013). Fraction for Teachers: Knowing What before Planning How to Teach. Tokyo: CRICED, University of Tsukuba.