


SEAMEO School Network Program provided by CRICED-University of Tsukuba

Mathematics Education to Develop Students Agency: The Case of Fractions

Teaching Mathematics to Develop Mathematical Thinking as Higher Order Thinking: How do you teach? Why? II

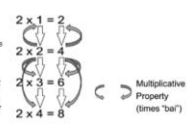
Lesson 7: Multiplication and Division of Fractions (2)

Lectured by: Masami Iwada, Prof/PhD, University of Tsukuba, Japan
 With support of: Wahid Yunianto, Dr., SEAMEO-QITEP, Indonesia
 With participation of:
 Ms. Mei Nakada, Japan
 Mr. Diego Solis, Costa Rica



Chapter 7. Multiplication and Division of Fraction (2)

The reasoning for ratio and proportion is already begun from the multiplication at the second grade. Proportional number lines and Rules of three on the table are the representation of the proportionality. Children should study them before learning ratio and proportion for multiplication and division of fraction by and for themselves by using what they already learned.

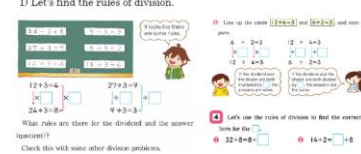


Rules of Multiplication and Division
 Rules of Multiplication and Rules of Division which appears the comparison of expressions for multiplication and division can be seen as the representation of the proportionality. Both rules are also useful for the extension of multiplication and division into decimals and fractions from the whole number.

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Exercise:
 The followings are the samples of the rules of multiplication and division:

1) Let's find the rules of division.



Gakkotosho G4, vol.1, pp21-24 (2011)

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2) Let's find the rules of multiplication:

2) Let's compare two mathematical sentences to find rules about multiplication.

| | |
|----------------------|----------------------|
| $40 \times 6 = 240$ | $80 \times 3 = 240$ |
| $80 \times 3 = 240$ | $40 \times 6 = 240$ |
| $40 \times 6 = 240$ | $80 \times 6 = 480$ |
| $80 \times 6 = 480$ | $40 \times 6 = 240$ |
| $40 \times 6 = 240$ | $40 \times 12 = 480$ |
| $40 \times 12 = 480$ | $40 \times 6 = 240$ |

Gakkotosho G4, vol.1, p89 (2011)

3) Let's explain those rules by using four number table for the Rule of three.

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4) Rules of multiplication are easier ways to find the answer of multiplication of fractions. Let's compare the following two approaches and explain how to use the rule of multiplication below (see page 49).

Yuki's idea
 Parallel area with $\frac{1}{3}$ dl. is $\frac{4}{3} \times 3 = 4$ (ml).
 $\frac{4}{3}$ dl. is twice of $\frac{1}{3}$ dl.
 $\frac{4}{3} \times 3 = 2 \times 2 = 4$
 Amount of juice: $\frac{4}{3} \times 3 = 4$

Mio's idea
 Calculate by changing fraction into integers, just as we did with decimals.
 $\frac{4}{3} \times 3 = \frac{4}{1} \times \frac{3}{1} = 12$

Gakkotosho, G6, vol.1, p36 (2011)

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Exercise: Fraction divided by Fractions

1) Let's find the answer of the following task by the proportional number line, rule of three, rule of division, and area diagram.

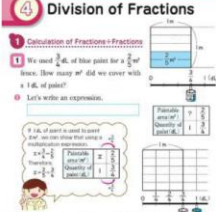
4) Division of Fractions

1) Calculation of Fractions = Fractions

1) We used $\frac{1}{4}$ dl. of blue paint for a $\frac{1}{2}$ ml of paint. How many ml did we cover with $\frac{1}{4}$ dl. of paint?

2) Let's write an expression.

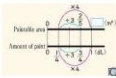
3) How many ml can be covered by 1 dl. of paint?

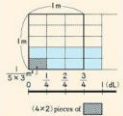


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2) Explain the following approaches to obtain the answers.

a) Rule of Division
 $\frac{2}{5} \div \frac{3}{4} = \left(\frac{2}{5} \times \frac{4}{3}\right) = \left(\frac{8}{15} \times \frac{4}{3}\right)$

b) Proportional Number Line


c) Area Diagram


3) Let's compare the three approaches. Which approach uses the unit fraction for changing the measurement scale? Which approach directly shows the way for the multiplication of inverse fraction? Which approach has the possibility to produce the wrong answer $\frac{2}{15}$?

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Teaching Mathematics to Develop Mathematical Thinking as Higher Order Thinking: How do you teach? Why? II

Summary

Lectured by Masami Isoeda, Prof./PhD., University of Tsukuba, Japan

Reference:
 Masami Isoeda (2013) Fraction for Teachers: Knowing What before Planning How to Teach, (2nd edition) Tokyo: CRICED, University of Tsukuba

***To appear as a book with contributors.



Various Meanings of Fraction

Problematic caused by Dividing Fraction

Missing of the whole, difficulty of equally dividing, a pair of numbers, and less than one

Dividing fraction is discussed as actions of equally dividing. It is good for learning the half and quarter. However, dividing into three, five, seven, ... we have to use the tools for equally dividing such as a set of parallel lines with the same distance.

Dividing fraction enhances the shading activity with counting numbers of numerator and denominators. Then, Dividing fraction produces the source of various misconceptions because there is no discussion of whole, and it is a pair of numbers not the number itself. It is a kind of ordered pair of numbers. For seeing it as a number, we have to discuss equivalence class as **Equivalent fraction**.

Dividing fractions can not compare if the whole is not the same. To compare, we have to fix the whole and add the information of quantity or denominator. Then, it is **Quantity Fraction**.

Dividing fraction can not extend more than one without fixing the whole because it is a part-whole model.

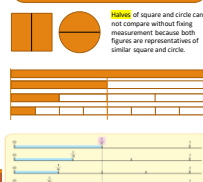
Area model and shading model are usually used as for explaining procedures of operations by teachers however many teachers lack these understanding and missing to notice necessary ideas. It supports students understanding of procedure momentary however we are not sure these models/representations to produce the operations by and for themselves.

Number and its extension

- Existence and Necessity (in which verification of well defined for integers is for Algebraic construction of the number system)
- Compare and Equivalence

Extension of Four Arithmetic operations

See Previous Course: Teaching Mathematics to Develop Mathematical Thinking as Higher Order Thinking: How do you teach? Why? II



Various Meanings of Fraction:

Beyond Dividing Fraction

Operational Fraction, Fraction as Quotient (Value of Division), Fraction as Ratio

Dividing Fraction is based on **Partitive Division**.

Partitive Division: 12 candies are divided by 4 children, equally. How many candies one child can receive?
 Ans. 3 candies for one child

Quotative Division: 12 candies are distributed by 4 children for one child. How many candies one child can receive candies?
 Ans. 3 persons

Operational Fraction is based on **Quotative Division**.

- Operational Fraction requires **Unit Fraction**.
- Unit Fraction is necessary for **Improper Fraction and Mixed Fraction**.
- Unit Fraction is necessary for considering **Equivalent Fraction**.

Fraction is the Value of Division which is called **Fraction as Quotient**. Fraction as Quotient is necessary to recognize fraction as number.

14 Fractions

We cut a tape with a length that is equal to the length of the blackboard and measured it with a 1m stick.

The length is 1m and a remaining part. How many meters is the remaining part?

9 m sticks

1 m sticks

Compare each part with the remaining part.

The tape divided into 3

The tape divided into 4

The tape divided into 5

Remaining part

Learn new ways to express a length that is shorter than 1m.

Task sequence by using what already learned and Progressive renovation of the Meanings of Fractions for Students Agency: learned how to learn

| Multiplication | Division | Quantities | Ratio | Fraction as Ratio |
|---|---|----------------------------------|--|-------------------|
| (how many children) × (how many candies for each child) | Partitive Division Situations To find how many candies for each child | Division of Different Quantities | Ratio (Rate) for different quantity Speed (km) : (hour) Population Density (person): (km ²) | Fraction as Ratio |
| Minutes | Dividing Fraction (Part-Whole relationship, Equally dividing a whole but it is action and not the number) | Related | Ratio (times of base) for same quantity Part to Whole (Boys) : (Human) Likely dividing fraction Part to Part in Whole (Boys) : (Girls) | |
| Distance divided by time | Quotative Division Situations To find how many children | Division of Same Quantities | | |
| | Operational Fraction (Based on Unit Fraction: Scaling number line by the unit fraction, it can compare) | No Related! | | |

Rational Number

Proportionality

Grade 2

If multiplicands increase by 1, products increase by the number of multiplier

$2 \times 1 = 2$
 $2 \times 2 = 4$
 $2 \times 3 = 6$
 $2 \times 4 = 8$
 $2 \times 5 = 10$

Grade 5 Proportion

$2 \times 1 = 2$
 $2 \times 2 = 4$
 $2 \times 3 = 6$
 $2 \times 4 = 8$

Grade 7 Y=AX

Grade 3 Making Tapes

1. Let's make a tape.
 Make a tape which length is 2 sets of 4cm.
 $4 \times 2 = \square$

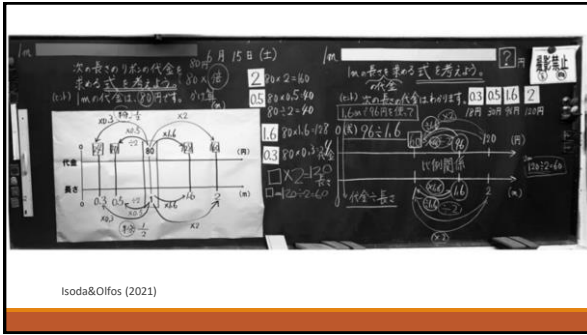
2. Let's find 4 times the following length.
 $3 \times 4 = \square$

3. A cinema seats holds 8 times the amount of water in a cup. A cup holds 26 of water. How many dl of water can be poured into the cinema tank?

4. A tank has 15cm of red tape and 3cm of blue tape. How many times the length of the blue tape is equal to the length of the red tape?

5. How many times of tape is equal to tape 35?

6. The fish tank in the science room holds 24L of water. The tank in the third grade classroom holds 6L of water. How many times the water in the third grade classroom tank can be held in the science room tank?



Isoda&Offos (2021)

Various Representations or Models for Fraction

Proportionality is a key for understanding multiplication and division.

Let's Develop Students who Draw Diagram by and for Themselves

Proportional number line with Tape Diagram is the tool for students who develop multiplication and division by and for themselves if they are able to draw.

Proportional number line **makes** visible Rule of Three on Table.

Proportionality is also represented by relationship between the expressions of multiplication and division.

$12 \div 3 = 4$
 $\frac{12}{3} = 4$

$27 \div 3 = 9$
 $\frac{27}{3} = 9$

To show proportionality on the proportional number line, the rule of three and the relationship between expressions, **arrows** are necessary!

Area diagram is also useful if it is drawn with quantity or denomination. If not, it provide the source for misconceptions based on dividing fraction such as which part is the whole?

Number and its extension

- Existence and Necessity in Math: verification of well defined for defining set for Algebraic construction of the number system)
- Compare and Equivalence
- Establish Four Arithmetic operations

See Previous Course:
Teaching Mathematics to Develop Mathematical Thinking as Higher Order Thinking: How do you teach? Why?

References

Masami Isoda et al. (2019). Mathematics Education Theories for Lesson Study: Problem Solving Approach and the Curriculum through Extension and Integration. *Journal of Japan Society of Mathematical Education*, 92(3), 1-18.

Sho Hottomatsu et al. (2005). Study with your friends. *Mathematics for Elementary School* (12 vols). Tokyo: Gakko-tosyo.

Masami Isoda et al. (2010). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakko-tosyo.

Masami Isoda (2012). Introductory Chapter: Problem Solving Approach to Develop Mathematical Thinking. In Isoda, M. & Katagiri, S. (ed) (2012). *Mathematical Thinking: How to Develop it in the Classroom*. Singapore: World Scientific. 1-28.

Masami Isoda, Terech Eusebio Cedillo Awiloli, et al. (2012). Matemáticas para la educación normal (12vols). Estado de México : Pearson Educación de México.

Terech Cedillo, Masami Isoda, et al. (2013). Matemáticas para la educación normal : guía para el aprendizaje y enseñanza de la aritmética, Estado de México : Pearson Educación de México.

Masami Isoda (2013). The Science of Lesson Study in the Problem Solving Approach. In Inprasitha, M. et al. (ed) *Lesson Study: Challenges in Mathematics Education*. Singapore: World Scientific. pp.81-108.

Masami Isoda (2015). *Quadratic on the Problem Solving Approach: Illustrating Hermeneutics as the Ground Theory for Lesson Study in Mathematics Education*. In Song, J. (ed). *Selected Regular Lectures from the 12th International Congress on Mathematical Education*. Cham: Springer. pp.355-381.

Masami Isoda & Aki Murata (2011). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakko-tosyo.

Masami Isoda, Aki Murata, Aida Ichino Yae (2015). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakko-tosyo.

Masami Isoda & Aki Murata (2021). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakko-tosyo.

Masami Isoda, Ramundo Offo. (2021). *Teaching Multiplication with Lesson Study*. Cham: Springer. DOI: 10.1007/978-3-030-28561-6

The textbook used in this course must be cited as shown below:

Masami Isoda (2013). *Fraction for Teachers: Knowing What before Planning How to Teach*. Tokyo: CRICED, University of Tsukuba.