Fraction for Teachers

Knowing What before Planning How to Teach

Masami Isoda, PhD/Prof. Director of CRICED, University of Tsukuba, Japan

First Edition for the SMASE-INSET project by JICA, Japan and ME, Nigeria

Electronically Published from CRICED, University of Tsukuba, May 8, 2013

Revised September 21, 2019

2nd Edition prepared for SEAMEO School Network program

Mathematics Education to Develop Students Agency: The Case of Fractions

October 23-December 4, 2021.

https://www.criced.tsukuba.ac.jp/seameo_online-program-fractions.html

Preface for First Edition

Education is the work to prepare for the future. Developing children who learn mathematics by and for themselves is one of the major issues on mathematics education reforms in the world (See such as Isoda & Katagiri, 2012). After the comparative study of mathematics classroom such as TIMSS video study in 90s, Japanese lesson study is the world-shared methodology as for the tools for professional development because the study indirectly demonstrated the quality of Japanese mathematics teaching and it is established by the lesson study. However, people often misunderstand the lesson study as for the talking about the class rather than studying subject matter. They enjoy the classroom observation likely listening to the music or watching the theatre. However, through listening to the music, and even if we enjoy talking about actors, we cannot prepare the good player ourselves. In Japanese lesson study, most efforts are done for the preparation of the class. The misunderstanding originated due to the limitation of the content guidebook to refer in English. On this reason, I have developed several resources which show the theory for the purpose to improve mathematics education with researches in the world.

For the workshop of SMASE-INSET project under Japan International Cooperation Agency (JICA), Japan and Federal Ministry of Education (FME), Nigeria, this booklet includes the essential theory for enabling teachers to plan the class for developing children who learn mathematics by and for themselves. It focused on the innovation of elementary school mathematics based on the content which is well written in the textbooks in each country and known by teachers. The workshop done in Nigeria was based on the author's experience in Central and South America, South East Asia and Pacific as well as in Japan.

May 7, 2013

Masami Isoda, PhD

CRICED, University of Tsukuba, Japan

Acknowledgment

Author Masami Isoda, PhD, acknowledges Gakko-Tosho which allowed the author to use the textbooks 'Study with Your Friends MATHEMATICS for Elementary School (2005) for this booklet. The content of this booklet was adopted for the workshop in SMASE-INSET by the author. Author acknowledges the following colleagues who contributed to the workshop in Abuja, Nigeria in May, 2013.

National Trainers

Olasinde Kudrat, Hadizat Garba, Zainab Muhammad Shuaibu, Abubakar Ibrahim

State Trainers

Ahmed Eneji, Mohammed Kabir Falalu,Monday Kurah

Japanese Staffs of JICA SMASE-INSET

Kazuhiro Okamoto, Tsuyoshi Ito, Hiroko Miura, Masako Tsuzuki, Hiromi Miyao

JICA Nigeria

Tatsuo Seki

Additionally, the author deeply acknowledges a number of colleagues who has been working with or supporting the author to develop the materials in the world.

Maitree Inprasitha (Thailand), Peter Gould (Australia), Catherine Lewis (USA), Raimundo Olfos (Chile), Tenoch Cediro (Mexico), Ui Hock Cheah (Malaysia), Suhaidah Tahir (Malaysia), Kaye Stacey (Australia), Max Stephens (Australia), Abraham Arcavi (Israel), Fadja (Indonesia) and all colleagues of APEC Lesson Study project: http://www.criced.tsukuba.ac.jp/math/apec/

Akinori Ito (JICA, PNG project)

About the Author

Affiliation and Address:

CRICED/Graduate School of Education, University of Tsukuba, 305-8572, Japan.

Website:

http://math-info.criced.tsukuba.ac.jp/

Title:

PhD (Education), Waseda University;

Honorary PhD (Mathematics Education), Khon Kaen University, Thailand;

Associate Professor, University of Tsukuba (1998-).

Education:

Master Program of Education, University of Tsukuba (1984).

Academic Membership:

Advisory Board Member, History and Pedagogy of Mathematics (2002-);

Project Overseer (Director), APEC Lesson Study Project in Mathematics (2006-);

Council Member, World Association of Lesson Study (2011-);

Chief Editor, Journal of Japan Society of Mathematical Education (2010-).

Awarded by:

Minister of Education (MEXT), Japan, as for the Best Educational Software in the Year 2005;

President of the Japan Publishers Association as for the Most Beautiful Book of the Year 2010 in the area of natural science;

Honorary PhD (Mathematics Education) by Khon Kaen University, Thailand as for his contribution to Khon Kaen University and Thailand, 2011.

Book for Acquisitions Program of Chilean Authors, Selected by National Committee of Culture and Art, Government of Chile. 2012.

President of the University of Tsukuba, as for the Best Faculty Member of the Year 2012. 2013.

Further CV and Publications:

Japanese Full

http://www.trios.tsukuba.ac.jp/Profiles/0006/0000997/profile.html

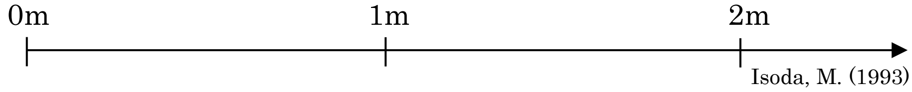
English Part
http://www.trios.tsukuba.ac.jp/Profiles/0006/0000997/prof_e.html

Table of Content

Chapter 1: What is fraction?

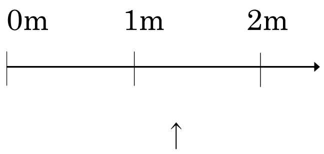
The professor asked the teachers:

There is 2m tape. Where is the position of ⅔ m? Show it by ↓

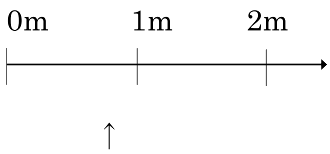


Possible Answers:

Som



Any



What do you want to do next? Isoda&Katagiri (2012)

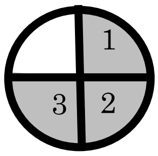
I would like to ask why?

Yes, we would like to discuss!

Professor: You are good teachers, aren't you? Because you already have the custom to ask why to others and discuss. It means that you usually engage in a similar activity by yourself. You already have the mind set for learning how to learn by yourself!

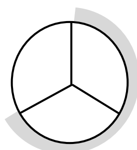
Discussion 1: Why do you think so?

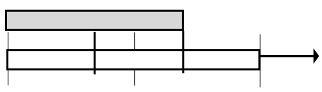
Som: It is larger than 1m because 2/3 means:



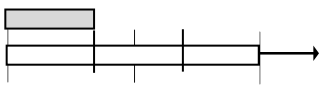
Ano: No, the whole is divided by 3, then shaded the 2 third of 3 parts

Som: Yeah, I should draw like that.





Any: Wow, 2/3 is less than 1. It is like this:



Som & Ano: ..continue..(talking about part-whole)

Any: ...continue... (talking about part-whole)

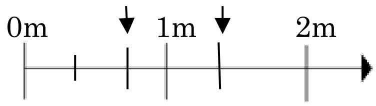
Prof: I could not understand well all of your explanations and diagrams. What is the original question?

Any: Where is the 2/3m?

Prof: Yes, then, the denomination 'meter' is missing in all of your explanations. We should write the meter on the number line. Can you explain 2/3m, again?

Any: It is 2/3 of 1m

Som: It is 2/3 of 2m

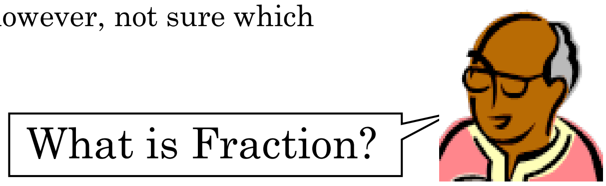


Prof: Now we can discuss what differences are there between these two ideas.

The denomination of quantity is important. When we explain something with the situation, we usually omit some words which are already known in the situation. In this case, the unit of meter 'm' itself is written in the task; however the meter was missing from their explanation and diagrams. In their discussion, if the denomination of quantity is missing, we are not sure which part you are explaining.

People who participated in the discussion; however, not sure which answer is appropriate.

Then, Professor asked.



Discussion 2: What is Fraction?

Ano: It is a part of the whole.

Prof: It is not clear. Could you explain it more exactly?

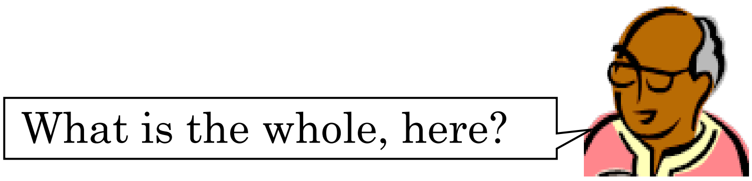
Ano: When the whole is divided into parts, fraction is the number of the parts in the whole.

Prof: Still are missing some important terms. Can anyone support?

Oth: When the whole is divided into 'm equally parts', 'fraction n/m' is 'n pieces of the parts in the whole'.

Any: The whole for 2/3m is 1m.

Som: The whole for 2/3m is 2m.



Discussion 3: Which one should be the whole?

Any: 1m

Som: 2m

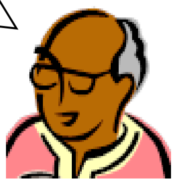
Ano: 1m or 2m, which one shall we chose for the whole, in this task?

Any: The whole for 2/3m is 1m

Som: The whole for 2/3m is 2m.

Oth: How can we discuss?

In mathematics, generally applicable idea is strong. For checking it, we have to think 'For Example, ..."



Discussion 4: For Example, if..............

Prof: For example, if I change the task from 2/3m to 1/2m what will happen?

Any: The 1/2m is 0.5m.

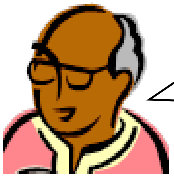
Som: The 1/2 of 2m is 1m.

Ano: No, what you are saying is that 1/2m is 1m. However, 1/2m is 0.5m, isn't it?

Pro: If I change the original question '2/3m' to '1/1m' what will happen?

Oth: If Som's idea, 1/1m is the 1/1 of 2m, thus 2m. 1/1m =2m. It is strange.

Prof: Yes, we can generalize Any's idea and not generalize Som's idea. When we say 2/3m, the whole is denominated by the 2/3 'meters': The unit of meters is 1m in any time.



What did you learn from this class? Let's write your own resume based on what you learned.

Questions for professional development 1

Q1. Why did the author choose this story as for the introductory chapter?

Q2. Professor asked several questions in the class. Which question is most important in this class? Why do you think so?

Q3. Do you think the definition of fraction in the class is appropriate? Why do you think so?

Q4. Explain the class using the term of appreciation.

Q5. If you conduct this task in your class or your teacher training program, what is your objective?

Q6. In your class, do you ask your students or teachers 'what do you want to do next?'. If you do ask, when do you ask it? If you do not ask, why?

Q7. Please explain the professor's questioning and values from the viewpoint of the following.

Three Major Objectives for Education as for Future Preparation

1. Human Character Formation ∋ {Developing Mindset, Attitude, Value} *appreciation*

2. Learning How to Learn ∋ {Knowing how to develop and reconstruct} *reflection*

3. Knowledge and Skills ∋ {Understanding and Proficiency} *acquisition*

Dizon, D., Ahmad, J., Isoda, M. (2017)

Q8. What is the explanation in mathematics at elementary school level? Using the discussion in class, please explain what it is from the following three perspectives.

Explanation of:

1. Meaning, such as the base for the reasoning using different representation;

2. Significance or Objective, why I would like to think such a way;

3. Procedure, how I did.

Isoda, M.(2008, 2009)

Q9. In the argumentation, for progressive dialectic, the way of discussion below is known as meaningful: 'If your saying is true, what will happen?'

In the class, Professor has used this dialectic method. Where did he use it?

Major Reference and Further readings 1

Isoda, M. (1993). The logic for Argumentation in Mathematics Classroom. (written in Japanese:礒田正美(1993).算数授業における説得の論理を探る.北海道教育大学教科教育学研究図書編集委員会編.教科と子どもとことば：言語で探る教科教育.東京：東京書籍) English translation: Isoda, M. (2008). Getting Others' Perspectives through the Hermeneutic Effort: Paper presented at TSG 26, ICME11 on *http://tsg.icme11.org/document/get/803*

Isoda, M. (2008). Teaching Mathematics through Argumentation for Junior High School. *(written in Japanese: 礒田正美(2008). 思考・判断・表現による『学び直し』を求める数学の授業改善：新学習指導要領が求める対話:アーギュメンテーションによる学び方学習.東京：明治図書出版)* Original theory for dialectic communication to develop mathematics knowledge and thinking was in English: Isoda, M. (2007). Developing Mathematical Thinking in Classroom, APEC Project report on *http://www.criced.tsukuba.ac.jp/math/apec/apec2007/progress_report/specialists_session/Masami_Isoda.pdf*

Isoda, M. (2009). Teaching Mathematics through Argumentation for Elementary School *(written in Japanese: 礒田正美(2009). 思考・判断・表現による「学び直し」を求める算数の授業改善：新学習指導要領が求める言語活動：アーギュメンテーションの実現. 東京：明治図書出版)* Original theory for dialectic communication to develop mathematics knowledge and thinking was in English: Isoda, M. (2007). Developing Mathematical Thinking in Classroom, APEC Project Report on *http://www.criced.tsukuba.ac.jp/math/apec/apec2007/progress_report/specialists_session/Masami_Isoda.pdf*

Isoda, M. & Katagiri,S. (2012). Mathematical Thinking: How to Develop it in the classroom. NJ: World Scientific. The part of 'What do you want to do next' can be downloaded:
http://www.worldscientific.com/doi/suppl/10.1142/8163/suppl_file/8163_chap01.pdf

Dizon, D., Ahmad, J., Isoda, M. (2017). SEAMEO Basic Education Standards (SEA-BES): Common Core Regional Learning Standards (CCRLS) in Mathematics and Science. Penang, Malaysia: SEAMEO RECSAM
https://www.researchgate.net/publication/335465253_SEAMEO_Basic_Education_Standards_SEA-BES_Common_Core_Regional_Learning_Standards_CCRLS_in_Mathematics_and_Science

Chapter 2: Dividing Fraction and Quantity Fraction

On the Fourth APEC-Tsukuba Lesson Study Conference, Gould. P.(2010), Inspector of NSW, Australia, lectured the roots of misconception of fraction in the case of Australian students with Australian textbook. He mentioned that the standard fraction notation itself encourages a 'count' interpretation of the regional 'parts of a whole' model (Figure 1). He quoted Hackenberg (2007) who asked the students to draw seven-fifths of a candy bar if the drawing of the rectangle on their paper represented a candy bar. Although both girls correctly created seven-fifths of the rectangle, when asked about the size of the pieces in the bars they had drawn, the girls maintained that the pieces were sevenths. Depending on his report, the major root of misconception is just counting of the parts without considering the whole. On his lecture, he also mentioned Japanese better approach.

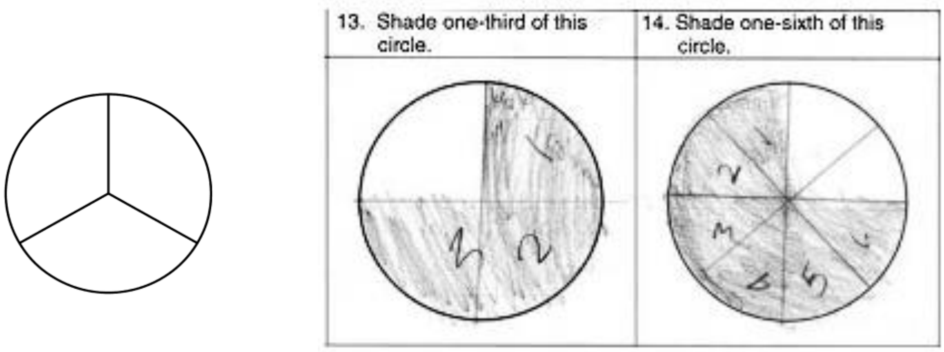


Figure 1 Circle Model and Answer of Grade 5 student (Gould, P. 2010)

On the Fifth APEC-Tsukuba Lesson Study Conference, Lewis, C. (2011) lectured about her research for professional development which compared three groups: Lesson Study with Japanese Resource for fractions, Lesson Study without it, and Ordinal professional development programs. In her lecture, she mentioned two points: Firstly, Japanese approach for introducing fractions is better approach for understanding the meaning of fractions. Her recommended Japanese Approach is shown on the Figure 3a &3b.

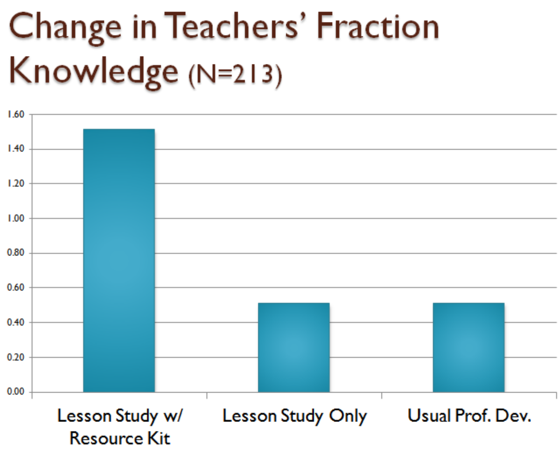
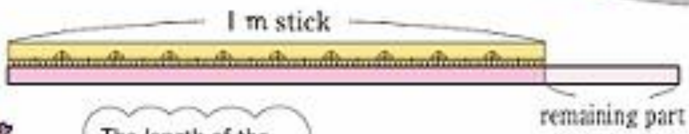


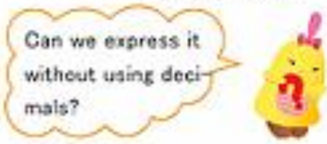
Figure 2 The effect of Lesson Study with Japanese Resource (Lewis, 2011)

14 Fractions

We cut a tape with a length that is equal to the length of the blackboard and measured it with a 1 m stick.

The length is 1 m and a remaining part. How many meters is the remaining part?

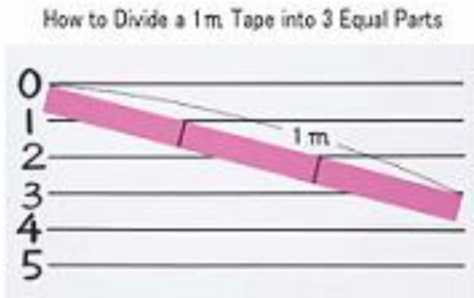


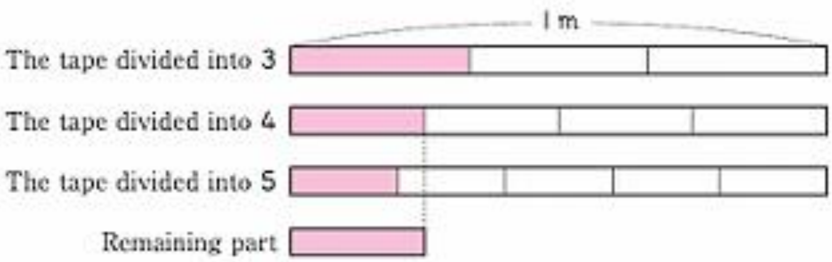


1 Fractions

1 Divide a 1 m tape into 3, 4 and 5 equal parts, respectively.

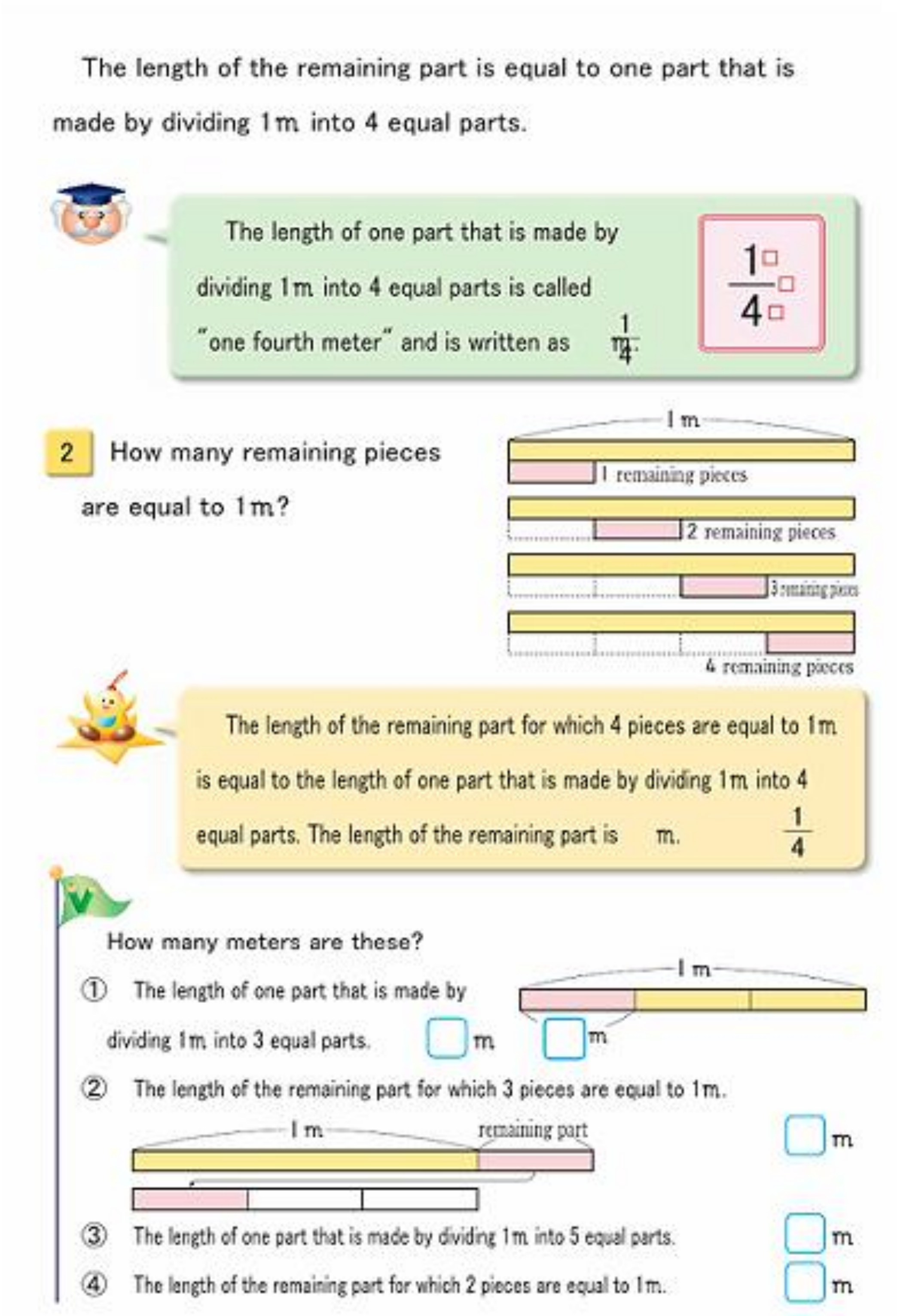
Compare each part with the remaining part.





Learn new ways to express a length that is shorter than 1 m.

Figure 3a. Study with your friends: Mathematics for elementary school, Gakkotosho (2005, Grade 4, vol.2, pp.65-66; 2011, Grade 3, vol.2, pp88-89)



The length of the remaining part is equal to one part that is made by dividing 1m into 4 equal parts.

The length of one part that is made by dividing 1m into 4 equal parts is called "one fourth meter" and is written as $\frac{1}{4}$m.

2 How many remaining pieces are equal to 1m?

The length of the remaining part for which 4 pieces are equal to 1m is equal to the length of one part that is made by dividing 1m into 4 equal parts. The length of the remaining part is $\frac{1}{4}$ m.

How many meters are these?

① The length of one part that is made by dividing 1m into 3 equal parts. □ m

② The length of the remaining part for which 3 pieces are equal to 1m. □ m

③ The length of one part that is made by dividing 1m into 5 equal parts. □ m

④ The length of the remaining part for which 2 pieces are equal to 1m. □ m

Figure 3b. Study with your friends: Mathematics for elementary school, Gakkotosho (2005, Grade 4, vol.2, pp.65-66; 2011, Grade 3, vol.2, pp88-89)

Secondly, only the case using Japanese Resource for fractions has improved teachers' fraction knowledge, significantly.

Both lectures recommended Japanese approach. One of their recommended approaches is shown in Figure 3a and 3b which uses the remainder for measuring the unit length 1m.

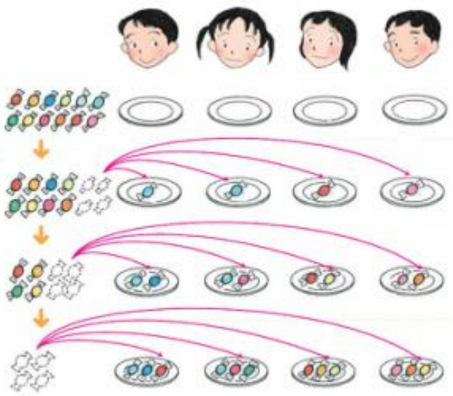
Questions for professional development 2

Q10. Let's analyze the ideas of Som and Ano in Chapter 1 from the view point of Gould, P. (2009).

Q11. Let's view Figure 3a and 3b. What is the common idea between Figure 3 and the suggestion of Professor in Chapter 1?

Q12. In division, there are two meanings as follows. Let's analyze the dividing activity in Figure 3a and 3b from the viewpoint of two different meanings.

Partitive division	Quotative Division
12 candies are divided by 4 children, equally. How many candies one child can receive?	12 candies are distributed by 4 candies for one child. How many children can receive candies?



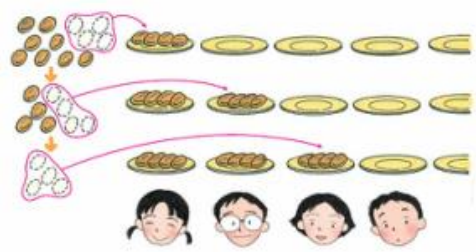
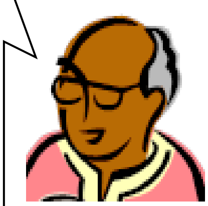


Figure 4. Gakko Tosho Grade 3 (Vol2. p4,p8, 2005; vol1, p60, p64, 2011)

Both activities for dividing are different for children however both activities include the repeated subtraction of the same amount from the total amount (Only see the left-hand-part of every picture, in Figure 4 left and right). Partitive division establishes the equally partition at first. Quotative division quotes the same amount recursively until we cannot quote, not sure the number of partitions at the beginning.



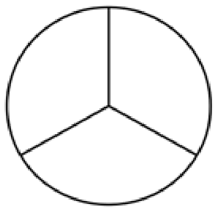
Q13. Please do the activity in the Figure 3 by yourself and explain the difference with your traditional approaches for teaching fraction based on Pizza model such as on the figure 1.

In mathematics, fraction n/m as for rational number is defined when n and m are integers (or rational number) and m is not equal 0.[1] If n or m are irrational or complex numbers, the fraction is not rational number. In mathematics, the concept is usually explained by the definition and the equivalent properties with exemplars. However, in school mathematics, especially for the elementary school, the meaning is explained using the different representations, model or situation with action (See Chapter 1, Q7,1). If we only teach number and calculation, children usually explain the ways of calculation as a rule (See Chapter 1, Q7, 3) when we ask them to explain 'Why?'. If readers are hoping children to explain the meaning, you have to assist children to use the further representations and models with situation. To meet these demands, Q12 on this chapter has asked you to engage in the same activity by yourself. When you use the 1m tape and you recognize the length of blackboard is 125cm, you can well prepare the activity on Figure 3.

Fraction is used in various contexts/situations with different meanings even if it is clearly defined mathematically. First part of this booklet explains how Japanese distinguish those meanings.[2] Second part explains how Japanese teach four operations of fraction.

Dividing Fraction

Many people believe that fraction is Dividing Fraction. It is the part-whole relationship which means that fraction n/m is dividing the whole into m equal parts and selecting n parts from them. It is deeply related with the activity of partitive division. 2/3 means the part-whole relationship which shows the whole is equally divided into three parts and selecting two parts. If the pizza is divided into three parts and take two parts, it means 2/3 of the pizza.



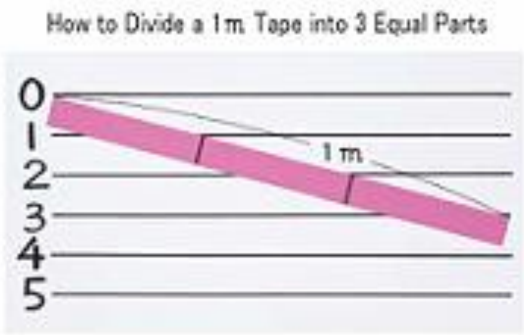
Paper folding, Origami, of a square paper includes the possible action for dividing the object equally, such as half and quarter. On this action, children usually think that dividing fraction cannot be larger than the one whole. However, as shown in Chapter 1 where people lost the denomination, children usually lose the process dividing the whole into equal parts. As Gould mentioned, the misconception for dividing fraction is originated from giving children worksheets to shade the object already equally divided through counting. On these worksheets, children are not needed to recognize the whole and to think how to divide it at first, because they just count the number of parts on an equally divided object. At the same time, on the ill-solution in Figure 1, it is a relationship to see the confusion of the fraction as ratio which allows to think the part-part relationship instead of the part-whole relationship described in Chapter 5. The task

[1] Fraction, rational number, as a number system will be discussed at later chapters.

[2] Internationally, there are various technical terms to explain the meaning of fraction on situations such as Behr, M., et al. (1983) and Charalamous, C. Y & Pitta-Pantazi, D. (2007). However, it is un-usual to distinguish dividing fraction and operational fraction. Japanese technical terms are more precise to establish fraction in relation to what students learned on multiplication and division (see Isoda&Olfos, 2021, p.91).

itself expected to think part-whole relationship, thus, the major problem is related with their losing of whole for equally-dividing the whole on the context. For developing better understanding, we have to begin the activity to set a whole as one unit by children and ask them to divide it into the number of equal parts. After dividing into equal parts, children chose the necessary number of parts from the whole.

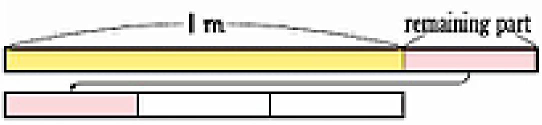
Any length of tape can be divided equally if we use the parallel lines with same interval. Necessity of this diagram is related with dividing the whole to the equal parts.



Exercise:

Draw 1/3 of Pizza on your notebook and compare the size of it with other's drawing.

Operational (measuring) Fraction



Operational Fraction (measuring fraction) can be seen as a kind of dividing fraction however it tries to measure the whole as a unit by the remaining part. From the viewpoint of division, it can be seen the activity of quotative division. In the following, three times of the remaining part is 1m. It means that 1/3 of 1m is the unit for counting however the whole, that should be divided, is 1m. Additionally, the original length is longer than the whole 1m. Mixed fractions and improper fractions have already existed even though children do not study them.

It is strange! We cannot always measure the 1m using the remaining part. What shall we do when we get another remaining when we measure the whole by the part? The operation does not work!

Yes, you usually give me the good question. Euclid (BC 3C) approved the way to find the unit (Greatest Common Divisor) for the measurement. It works with whole numbers (Integers) and called Euclidian Algorithm.

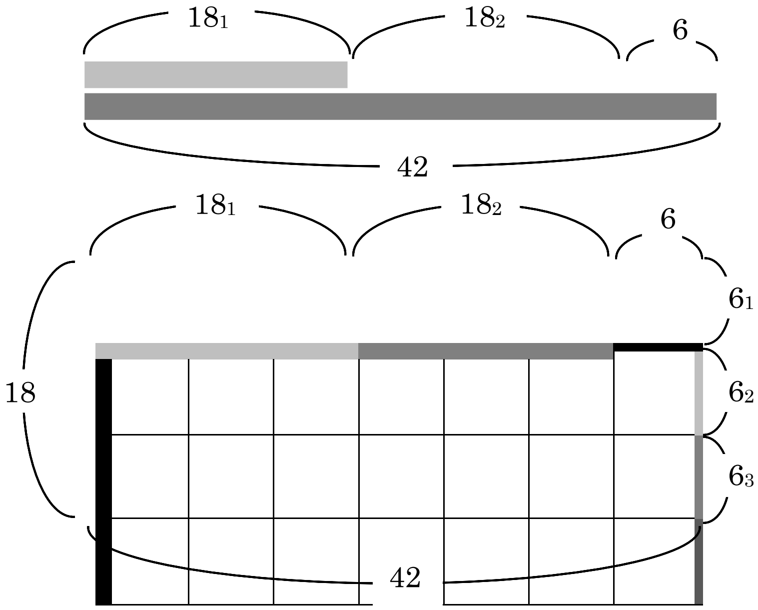
Euclidian Algorithm

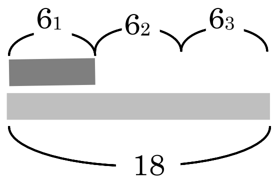
For example, Greatest Common Divisor of 18 and 42 is: 42=6x7, 18=3x6, 3 and 7 are prime numbers, then 6 is GCD of 18 and 42. It will be found using the remainders as follows.

42=2x18+6 → 18=3x6

42 divided by 18, the remainder is 6.

18 divided by the remainder 6.





The rectangular diagram is the better way to demonstrate Euclidian Algorithm.

Exercise:

Let's find the GCD of (41, 18) by the rectangular diagram.

Quantity Fraction

Quantity fraction is the fraction with denomination. 1/3 of 1m is 1/3m in quantity fraction. The dialectic in Chapter 1 is done between quantity fraction (2/3m) and dividing fraction (2/3 of 2m). If we lose the denomination, we cannot distinguish quantity and dividing fraction like the communication in Chapter 1. It means that we cannot arrange the position of fraction on the number line if we do not have the denomination of the quantity. Quantity fraction allows us to compare the size of fraction in relation to the unit quantity. Dividing fractions are not easy to compare on the number line because the size of the whole unit is not clear and it looks always less than one.

In Figure 3a and 3b, all fractions are denominated with meter. All of them are quantity fraction based on the unit meter. In Japanese textbook, fraction is introduced by the quantity fraction with operational fraction and dividing fraction. It is better than the traditional approach introduced by dividing fraction for recognizing what is a unit (whole). Because it allows us to extend the fraction larger than the whole unit, and enables us to compare the size of fraction on the number line.

Unit Fraction for measuring unit

Unit fraction is the fraction in which numerator is one such as 1/3. Unit fraction is the unit for measuring up to improper fraction. Any fraction is represented as follows: (the specific number) x (the unit fraction). It is the necessary base to extend proper fraction to improper fraction. Thus, in fractions, there are two types of unit. Firstly, the whole is the unit. Secondly, unit fraction. The unit fraction is the unit for counting the numerator.

The idea of unit fraction on base ten place value system is related with decimals. For example, 1 mm as unit quantity for length is usually introduced as dividing 1 cm by 10, equally. The relationship between 1cm and 1 mm is a base to introduce decimals such as 10 mm is 1 cm and 1 mm is 0.1 cm. 1 cm and 1 mm scales are given on the ruler and tape measure, it is the base for number line which begins from 0 to $+\infty$. On this context, operational fraction and unit fraction are the bases for extension of numbers.

In world known Japanese approach, fraction is didactically explained using various technical terms and introduced based on quantity and operational fraction, the remainder as measurement, for preparing the unit fraction (Figure 3b). If children understand the quantity fraction and the unit fraction, they can easily represent the fraction on the number line such as in Chapter 1, and easier to think four operations of fractions.

Major Reference and Further Readings 2

Behr, M., Lesh, R., Post, T. and Silver, E.: 1983, Rational number concepts. *in R. Lesh and M. Landau (eds.), Acquisition of Mathematics Concepts and Processes, Academic Press*, New York, pp. 91–125.

Charalabous, C. Y., Pitta-Pantaze, D. (2007). Drawing a theoretical model to study student's understanding of fractions. *Educational Studies in Mathematics.* 64, 239-316.

Gould. P.(2010). Fraction notation and textbooks in Australia. 4th APEC-Tsukuba Conference. Sited on http://www.criced.tsukuba.ac.jp/math/apec/apec2009/index_en.php

Lewis, C. (2011). Lesson Study with Japanese Curriculum Materials: A Randomized Controlled Trial. Sited on http://www.criced.tsukuba.ac.jp/math/apec/apec2011/index_en.php

Masami Isoda et al. (2010). Mathematics Education Theories for Lesson Study: Problem Solving Approach and the Curriculum through Extension and Integration, Journal of Japan Society of Mathematical Education, 92(11). 1-158.

Shin Hitotsumatsu et al. (2005). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda et al. (2010). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda, Tenoch Esaú Cedillo Ávalos, et al. (2012). Matemáticas para la educación normal (11vols). Estado de México : Pearson Educación de México.

Tenoch Cedillo, Masami Isoda, et al. (2013) Matemáticas para la educación normal : guía para el aprendizaje y enseñanza de la aritmética, Estado de México : Pearson Educación de México.

Masami Isoda, Raimundo Olfos. Edited. (2021). Teaching Multiplication with Lesson Study. Cham: Springer.

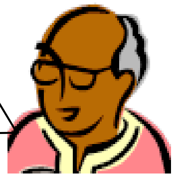
Chapter 3: Addition of Fraction with Different Denominators

The professor asked the teachers:

In addition of the fractions, we teach the case of the same denominator at first and then we introduce the case of different denominator.

In the case of $\frac{1}{2}+\frac{1}{3}$:

We have two bottles with $\frac{1}{2}$L and $\frac{1}{3}$L of milk. How much L in total? Can you imagine the student's answer? Isoda (1996)



Possible Answers:

Som

Yes, we have children who calculate as follows:

$\frac{1}{2}+\frac{1}{3}=\frac{1+1}{2+3}=\frac{2}{5}$

Any

Yes, we have. However we have to teach:

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

How can you plan their argument?

I would like to ask why?

I would like to ask to draw diagrams to explain the meaning.

Professor: Yes, we would like to listen their ideas. Can you imagine their diagrams for explanation? How do you teach to find the common denominator?

Exercise

Exercise1: Let's plan the process of dialectic discussion between two different ideas through the drawing diagrams which support each of answers.

Exercise 2: When you plan, what are the expected knowledge for children before the class and what knowledge do they have, actually, for those two ideas? What shall they learn before this class for enabling them such a discussion? What shall we teach before you plan your class?

A Sample of Arguments:

Teacher: Good, now we have different answers. What shall we do?

C1: How did they get their answers?

C2: Yes.

Teacher: Then, we would like to ask: how do you get it?

C3: $\frac{1}{2}+\frac{1}{3}=\frac{1+1}{2+3}=\frac{2}{5}$, I added numerators and denominators each.

C4: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$, I tried to find the common denominator.

Teacher: Do they explain in the same way?

C: (No, answer).

Teacher: Do you have any questions for them?

C1: I have a question for C3, why you added the numerators?

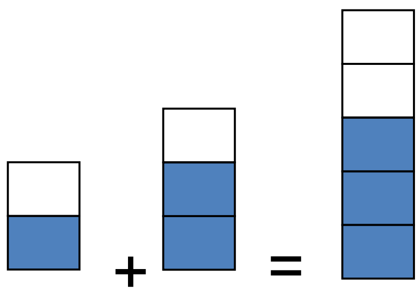
C3: As I explained, $\frac{1}{2}+\frac{1}{3}=\frac{1+1}{2+3}=\frac{2}{5}$, I added numerators and denominators.

C4: The total,$\frac{2}{5}$ is smaller than $\frac{1}{2}$. If $\frac{1}{3}$L of milk is added to $\frac{1}{2}$L of milk is less than $\frac{1}{2}$ L. It is very much strange, isn't it?

C3: Yes, it's strange...then I cannot add denominators?

C5: No, it is not strange. I will explain the bottles like this:

In the case of fractions, it is true.



C3: Aha, yes, it is!

Teacher: Then, you say in the case of fractions, it is possible to say that the total from $\frac{1}{2}$L of milk and $\frac{1}{3}$L of milk is less than $\frac{1}{2}$L.

C: No! Yes!

Teacher: Yes or No? It looks C5 is less supported. Why?

C4: The milk bottle of C5, the size is not the same.

C5: Yes, the shape of the milk bottles are not usually the same.

C4: ....

Teacher: Why did you use the common denominator, C4?

C4: Because we added when the denominators are the same.

We learned the addition of fractions in the case when the denominators are the same.

C2: Yes, we know the addition of the same denominator. In that case, we added the numerator. This task is not the same.

Teacher: Did you draw a diagram, like C5?

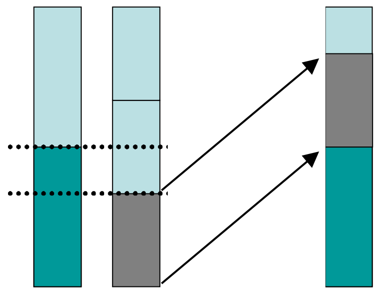
C4: No, I did not. I think that it is the rule.

C5: Oh, the rule? I could explain my idea by the diagram.

Please count in my diagram.

The way of C3 will be the right rule, not C4.

C6: I draw a similar diagram, not the same as C5:



It shows that the value is larger than the parts.

Teacher: Now, we have two drawings. Which diagram is reasonable for establishing the way of calculation?

C3: I do not think C6 is explaining C4: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$.

It may explain the part of $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, however it does not explain $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}$.

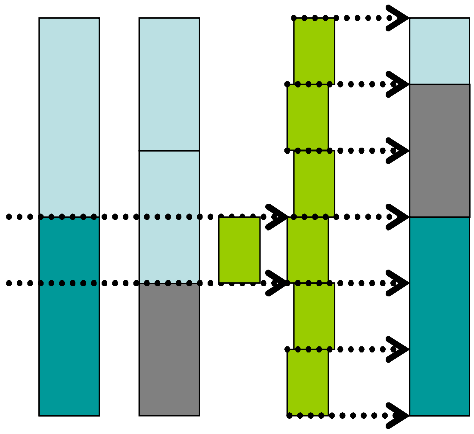
Why you have to change the fraction on $\frac{1}{6}$?

Teacher: It means that why C4 has to change the denominator, right?

C3: Is that a rule, also? If it is the rule, my own is better because we can explain the idea by counting like C5.

C6: If we have to explain it by counting, we have to share the unit at first. If we set the unit for counting from the difference between $\frac{1}{2}$ and $\frac{1}{3}$:

.



I could count six times of the unit.

Thus, the unit for counting is $\frac{1}{6}$.

C5: Aha, I got it. However, why do you use the unit for counting by the difference? Is it occasionally? Can you do that every time? Or, you already knew the difference in fractions with different denominator?

C6: Not sure, however, if you consider every time, do you think your idea also works every time?

Teacher: Yes, in mathematics, we develop generally applicable ideas. It must be important point for discussion.

C5: Yes.

C4: Wow, did you add the denominators in the case that the denominators are the same?

C5: What?

C4: For example, how do you think in the case of $\frac{1}{2}+\frac{1}{2}$.

We already learned it is $\frac{1}{2}+\frac{1}{2}=\frac{1+1}{2}=1$, right?

But if your idea is true, $\frac{1}{2}+\frac{1}{2}=\frac{1+1}{2+2}=\frac{2}{4}=\frac{1}{2}$. It is strange.

C3,C5: Wow, yes, it is strange. Is it the reason, if the denominators are the same, we add the numerators?

C4: Yes, now I understand why I did it. That is the objective why I changed the denominators into common denominators! However, how can we produce the common denominator for the addition of fractions in the case of different denominators. I am still not sure of the ways of calculation.

Teacher: Aha, for considering generally, we checked each proposed way by C3 and C4 on the different denominators by the known case of the same denominator. Then, the way of C4 works and the way of C3 does not. However, the diagram by C5 still looks fine. Why did it produce the inappropriate answer?

C6: The size of the bottle should be the same. In this case, if we draw the diagram of a 1L milk bottle, it is fine.

Teacher: Yes, we need to write the size of quantity on the diagram. In this case, we should draw 1L as for the whole bottle size in every diagram. We should use the same size bottle for explaining the addition and subtraction of fractions. In the diagram of C5, we counted different size of fractions. $\frac{1}{2}$L is counted one and $\frac{1}{3}$L is counted one. We cannot count one, two because the size is not the same. The diagram by C6 used $\frac{1}{6}$ L for the same counting unit.

C4: Wow, this is the reason why I used the common denominator. Still not sure of the ways.

Teacher: Then, from now, we would like to find the shorter and simple way of calculation by considering the common denominator.

Questions for professional development 3

Q14. In the argument, you may see the same discussion which you read in Chapter 1 and it will be explained by the several terminologies as mentioned in Chapter 2. Let's explain the argument by the terminologies. And explain your appreciation about the terminologies for understanding the difficulty of fraction and what is the necessary content for teaching.

Q15. In the argument, what kind of explanation did you find? Please read each explanation from the viewpoint of Chapter 1, Q8. Why do we need the diagram? What C4 wished to say? For developing children who learn mathematics by/for themselves, what type of argument you would like to establish in your classroom.

Q16. If you are not used to draw a diagram for explaining fractions by yourself, the argument might be difficult for you to understand. The difficulty that you recognized is based on the unknown. It means that it is the chance for learning. At first, let's discuss about which part do you feel a difficulty and then, talk about what you shall learn.

Q17. When you feel a difficulty, your children also feel the same. Please ask your children who knows addition of the same denominator and equivalent fraction, but not yet learned the case of different denominators to read this discussion. And ask them, how do they read? It is also a good chance to learn from children for knowing what is the task for their learning and what is necessary for your preparations.

Q18. The argument itself was implemented in a class (see Isoda, 1996) and not unusual in Japan. When we compared the classes, major difference is the teaching before the class. What kinds of content shall we teach before this class?

Major Reference and Further readings 3

Isoda, M. (1996). Problem Solving Approach with Dialectic. *(written in Japanese:多様な考えを生み練り合う問題解決授業：意味とやり方のずれによる葛藤と納得の授業づくり. 東京：明治図書出版)* in English: Isoda, M. (2007). Developing Mathematical Thinking in Classroom, APEC Project report on *http://www.criced.tsukuba.ac.jp/math/apec/apec2007/progress_report/specialists_session/Masami_Isoda.pdf*

Masami Isoda et al. (2010). Mathematics Education Theories for Lesson Study: Problem Solving Approach and the Curriculum through Extension and Integration, Journal of Japan Society of Mathematical Education, 92(11). 1-158.

Shin Hitotsumatsu et al. (2005). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda et al. (2010). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda, Tenoch Esaú Cedillo Ávalos, et al. (2012). Matemáticas para la educación normal (11vols). Estado de México : Pearson Educación de México.

Tenoch Cedillo, Masami Isoda, et al. (2013) Matemáticas para la educación normal : guía para el aprendizaje y enseñanza de la aritmética, Estado de México : Pearson Educación de México.

Masami Isoda, Raimundo Olfos. Edited. (2021). *Teaching Multiplication with Lesson Study*. Cham: Springer.

Chapter 4: When does Fraction become Number?

Number system is the set with mathematical structures such as the relationship of equivalence, greater/less and operations (addition and multiplication). Mathematically, fraction n/m is an element of rational numbers, if n and m are integers and m≠ 0. If n or m are irrational or complex, it becomes a part of a larger number set.

In school mathematics, when we extend numbers we usually discuss existence/necessity/significance, equivalence/larger/smaller, and four arithmetic operations. If fraction is completely learned, we can recognize the fraction as a part of number set: It is a representation of rational numbers which are bridged with decimals. In other words, fraction is not the number until it has been fully completed.

On this context, every teacher should know that the dividing fraction is not the number. It represents the part-whole-relationship actions on the object such as half of a Pizza. Quantity fraction can be arranged on the number line for quantity as well as other numbers such as decimal numbers. Dividing fraction usually begins from the whole and it is not easy to represent on the number line as long as what is the whole which we discussed on Chapter 1.

Equivalent Fraction

$\frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\frac{4}{8}=\frac{5}{10}$... is the equivalent fraction of $\frac{1}{2}$. At the lower grades the dividing fraction, which is represented by paper folding activity, is learned for knowing the half and the quarter, two quarters, and three quarters. It is a necessary activity for knowing the procedure to get the half and the quarter. However, the paper folding activity for dividing fraction is usually missing $\frac{3}{6}$, because it cannot be represented by the folding procedure into half.

Number Line

Number line shows the magnitude/size of numbers up to real numbers as position which is useful for comparison. Number line is scaled by using equally dividing of 0-1 span as dividing fraction and by the unit fraction as operational fraction. As shown on the Figure 3a in Chapter 2, we can divide any string into the number of equal parts. On this basis, we can draw the diagram of number lines which demonstrates the equivalence, larger and smaller (Figure 5). Through showing the position of fraction on the number line, it supports to see the fraction as number as well as others.

The number line includes more than 1 on Figure 5 like a ray. If it is represented by a segment between 0 and 1 without any extra part more than 1, it is just a representation of dividing fraction which shows a part-whole relationship like folding a tape. Here, the whole is 1 without considering the possibility of extension. As discussed in Chapter 2, operational fraction is majored by a unit fraction (remaining part). The number of unit fraction shows a numerator: then extended to improper and mixed fraction more than 1.

Quantity fraction $\frac{2}{3}$m implicates that the unit-quantity is 1m. If quantity is indicated on the number line, the number line shows quantity.

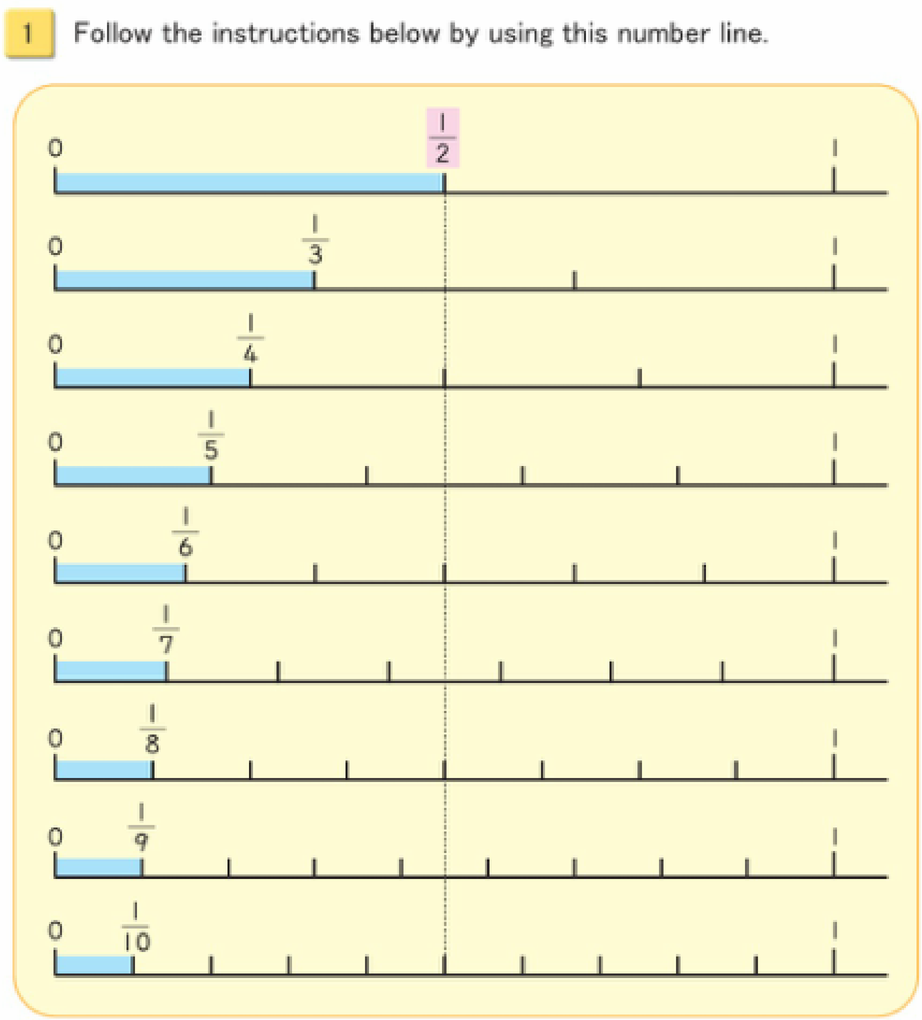


Figure 5. Equivalent fraction on the number line. GakkoTosho (Grade 5, vol2, p24, 2005; Grade4,Vol2, p78, 2011)

Fraction is Expression：Fraction as Quotient

The answer of division is known as a number. When a fraction can be seen as a division expression, it supports to see fraction as a part of number.

Fraction is related with division which includes both meanings of quotative and dividing fraction on situations (see Chapter 2). In both situations, fraction represents the value of division such as $1 \div 3 = \frac{1}{3}$. We call it **fraction as quotient**. On this definition, fraction is connected with a number as well as the whole number (See figure 6a and 6b) because the answer of division should be a number on the perspective of the permanence of form. In relation to two division situations, $1 \div 3 = \frac{1}{3}$ implicates equally dividing like dividing fractions. 1 ÷ 3=0.3333333… in action related with operational fraction by focusing on remainders. However, if we write $0.\overline{3}$ ("—" comes to the top of 3) it is $\frac{1}{3}$.

Exercise. Let's read and answer.

3 Fractions, Decimal Numbers and Whole Numbers

The Quotients of Divisions and Fractions

1 When we divide 2 ℓ of milk among ☐ students equally, how many liters will each student receive?

2÷ ☐

① Enter the numbers from 1 to 5 in the ☐ and calculate the answers.

2÷ ☐, 2÷ ☐, 2÷ ☐, 2÷ ☐, 2÷ ☐

② Divide the above equations into 3 groups based on the answers.

ⓐ Answers that are whole numbers.

( )

ⓑ Answers that are expressed exactly as decimal numbers.

( )

ⓒ Answers that are not expressed exactly as decimal numbers.

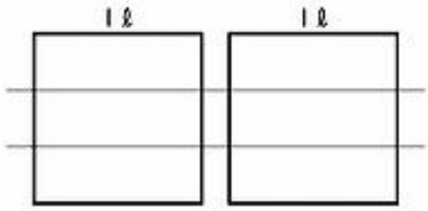
( )

2÷3 is 0.666..., so this cannot be expressed exactly as a decimal number because there is no end.

③ When 2 ℓ is divided equally among 3 students, how many liters does each student receive?

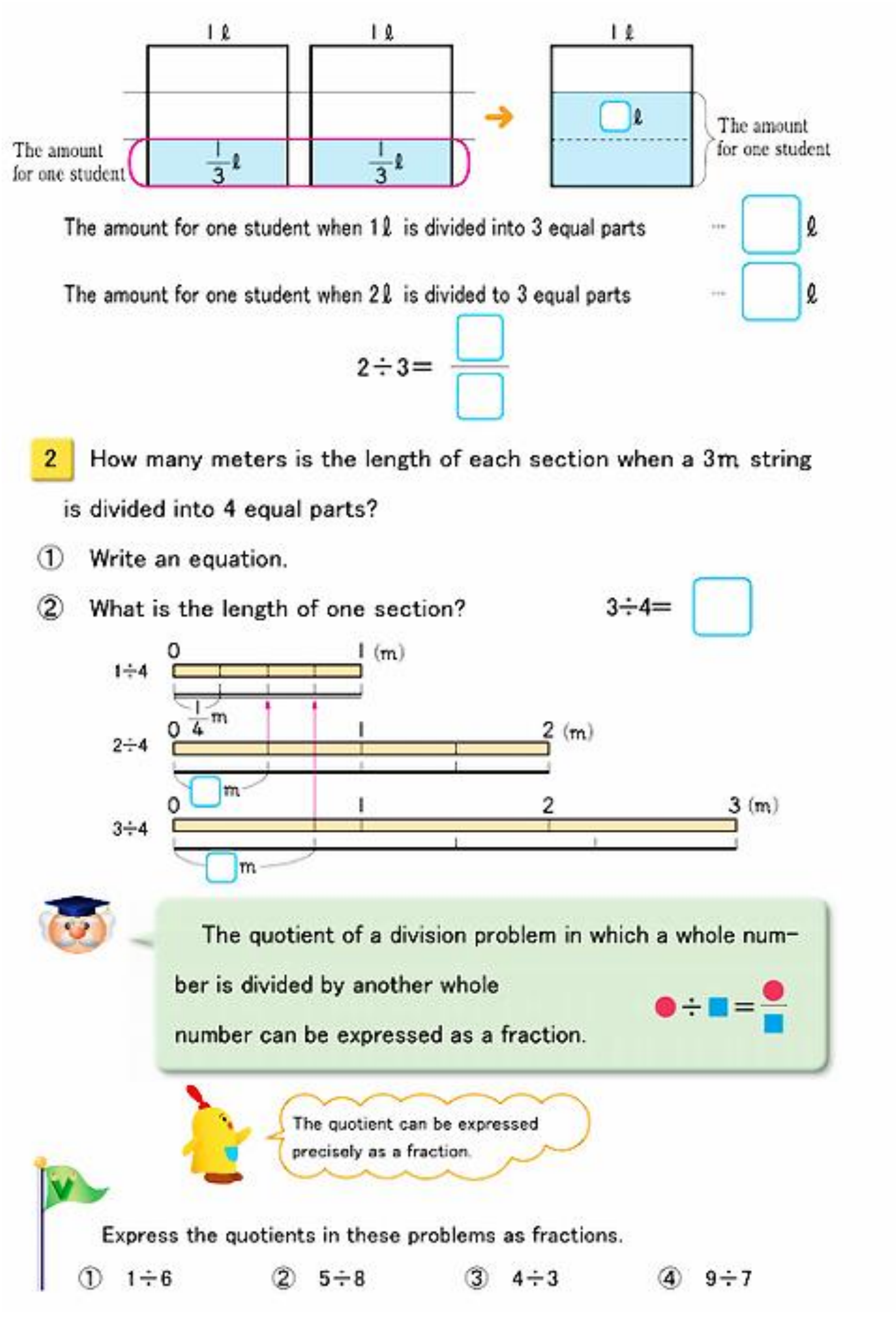
ⓐ Color in the portion for one student.

ⓑ How many liters is one portion?



Let's see how to express the quotient of a division problem when it cannot be expressed exactly as a decimal number.

Figure 6a. Gakko Tosho Grade 5. (vol.2. p29, 2005; vol.1. p128, 2011)



The amount for one student when 1ℓ is divided into 3 equal parts … ℓ

The amount for one student when 2ℓ is divided to 3 equal parts … ℓ

2÷3=

2 How many meters is the length of each section when a 3m string is divided into 4 equal parts?

① Write an equation.

② What is the length of one section? 3÷4=

The quotient of a division problem in which a whole number is divided by another whole number can be expressed as a fraction.

●÷■=$\frac{●}{■}$

Express the quotients in these problems as fractions.

① 1÷6 ② 5÷8 ③ 4÷3 ④ 9÷7

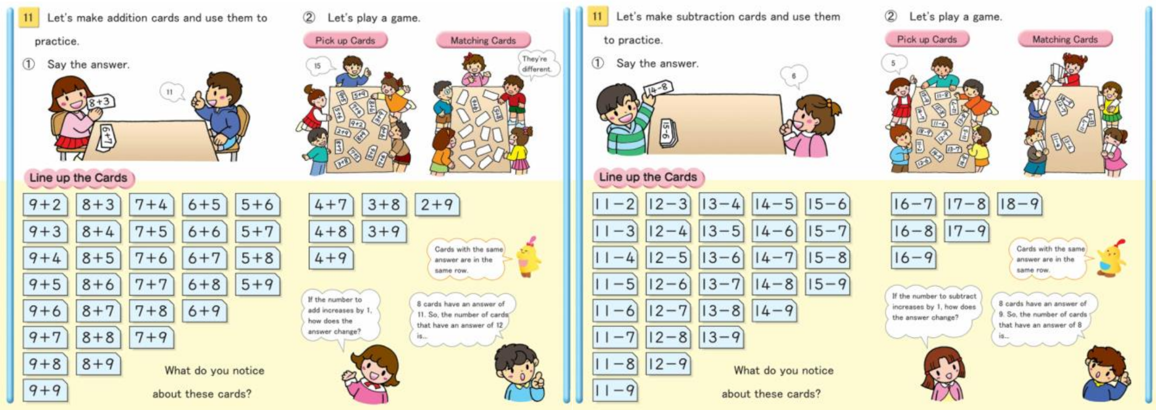
Figure 6b. Gakko Tosho Grade 5. (vol.2. p30, 2005; vol.1. p129, 2011)

If children can see the fraction as quotient, now fraction is a part of number which can be seen as the expression of division such as $\frac{1}{3} = 1 \div 3$. If fraction is given by the expression of division, the meaning of equivalence fraction is also understandable because it is not a mysterious property that only appeared on the fraction. In any arithmetic operation, it has similar properties. When the equivalence of fraction is recognized on the basis of the fraction as quotient, all four operations complete the properties of equivalence on their expressions.

On mathematics, it is the discussion of equivalent class.

Questions for professional development 4

Q19. Let's find the equivalent property of addition and subtraction: Gakkotosho Grade 1.



(pp82-83, pp92-93,2005; pp96-97, pp106-107, 2011)

Q20. In Chapter 2, Q11, there are two meanings of dividing activity. Which activity can you see on the figure 6a and 6b and explain why.

Multiplication Table

multiplicand \ multiplier		1	2	3	4	5	6	7	8	9
row of 1	1	1	2	3	4	5	6	7	8	9
row of 2	2	2	4	6	8	10	12	14	16	18
row of 3	3	3	6	9	12	15	18	21	24	27
row of 4	4	4	8	12	16	20	24	28	32	36
row of 5	5	5	10	15	20	25	30	35	40	45
row of 6	6	6	12	18	24	30	36	42	48	54
row of 7	7	7	14	21	28	35	42	49	56	63
row of 8	8	8	16	24	32	40	48	56	64	72
row of 9	9	9	18	27	36	45	54	63	72	81

(Gakkotosho Grade vol.2, p.92, 2005)

Q21. Let's find the same product in the multiplication table.

Major Reference and Further readings 4

Masami Isoda et al. (2010). Mathematics Education Theories for Lesson Study: Problem Solving Approach and the Curriculum through Extension and Integration, Journal of Japan Society of Mathematical Education, 92(11). 1-158.

Shin Hitotsumatsu et al. (2005). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda et al. (2010). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda (2012). Introductory Chapter: Problem Solving Approach to Develop Mathematical Thinking. *In Isoda, M. & Katagiri, S. edited (2012). Mathematical Thinking: How to Develop it in the Classroom.* Singapore: World Scientific. 1-28.

https://www.criced.tsukuba.ac.jp/math/apec/ICME12/Lesson_Study_set/MATHEMATICAL%20THINKING%20(c)World%20Scientific/MathematicalThinking_chap01(c)WorldScientific.pdf

https://doi.org/10.1142/9789814350853_0001

Masami Isoda, Tenoch Esaú Cedillo Ávalos, et al. (2012). Matemáticas para la educación normal (11vols). Estado de México : Pearson Educación de México.

Tenoch Cedillo, Masami Isoda, et al. (2013) Matemáticas para la educación normal : guía para el aprendizaje y enseñanza de la aritmética, Estado de México : Pearson Educación de México.

Masami Isoda (2015). The Science of Lesson Study in the Problem Solving Approach. *In Inprasitha, M. et.al. edited. Lesson Study: Challenges in Mathematics Education.* Singapore: World Scientific. pp.81-108. https://doi.org/10.1142/9789812835420_0006

Masami Isoda (2015). Dialectic on the Problem Solving Approach: Illustrating Hermeneutics as the Ground Theory for Lesson Study in Mathematics Education. *In Sung, J. edit. Selected Regular Lectures from the 12th International Congress on Mathematical Education*. Cham: Springer. pp.355-381. https://doi.org/10.1007/978-3-319-17187-6_21

Masami Isoda & Aki Murata edited (2011). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.

Masami Isoda, Aki Murata, Aida Istino Yap edited (2015). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.

Masami Isoda & Aki Murata edited (2021). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo. https://support.gakuto.co.jp/mathematics-textbook/

Masami Isoda, Raimundo Olfos. Edited. (2021). *Teaching Multiplication with Lesson Study.* Cham: Springer. DOI: 10.1007/978-3-030-28561-6

Chapter 5. Fraction in relation to ratio and proportion

In relation to the fraction as quotient, fraction is used for representing the ratio ' : ' as (Ratio) = (Comparing Quantity) ÷ (Base Quantity for the Unit of the Ratio). There are several contexts to use ratio. Japanese approach consistently uses Tape diagrams, Number lines, Area diagrams and Tables, as representations for thinking.

Firstly, ratio (value of ratio) is the result of division. There are two contexts in relation to the meaning of division. The following two types of dividing activity: partitive division and quotative division (Chapter 2, Q11).

Partitive Division	Quotative Division
12 candies are divided by 4 children, equally. How many candies one child can receive? Ans. 3 candies for one child	12 candies are distributed by 4 candies for one child. How many children can receive candies? Ans. 3 persons

When we see the underlined words in both tasks, the partitive division treats different quantity/denomination such as candies and children, and quotative division treats the same quantity/denomination.

On terminology, rate is the ratio for different quantities. Division by the different quantity (partitive division) is the rate which is unknown quantity represented by the quantity per the different quantity such as (speed (km/h)) = (distance (km)) ÷ (duration (hour)). One person runs 10km/h and another person runs 8km/h. Even if two people run together, they do not run at 18km/h. The quantity such as speed which cannot be added is called connotative quantity. It is usually given specific name such as speed.

Division by the same quantity (quotative division) establishes the ratio of the same quantity which produces the rate to show coefficient/multiple (how many times). If we carefully read the task for quotative division situation, it is not exactly the same quantity/denomination. Because, contextually, it can be represented by (number of children) = (12 (candies)) ÷ (4 (candies for one child)). On the quantity expression, it is $\frac{(Candies)}{(\frac{Candies}{a\ child})} = (Children)$. Both numbers 12 and 4 are number of candies for students. Thus, it is unnecessary to see them as different quantities.

There are countries that use the ratio symbol ' : ' as the for division symbol '÷'. It means that a : b is the same as a÷b and $\frac{a}{b}$. To distinguish the ratio of different quantities (usually with special name as for new quantity) and the ratio of the same quantity (just in case, a<b and it is the narrow usage of WARIAI in Japanese) is reasonable. On the context of ratio a : b, $\frac{a}{b}$ is called value of ratio.

Questions for professional development 5

The following tasks show the various context of ratio. Let's answer the following questions.

Q22.

1) Solve the following task. Provide the answer using percentage.

2) Explain the feature of the task using the following terms: Partitive division, Different Quantities.

3 Some students checked the number of passengers on airlines one day. Which plane is more crowded?

Number of Passengers and Seats

	Small plane	Large plane
Number of passengers	117	442
Number of seats	130	520

A number that is expressed by the quantity being compared when the basic quantity is made 1, like a shooting record or crowding, is called "a ratio."

Ratio = Quantity being compared ÷ Basic quantity

(Gakko Tosho, Grade 5, vol.2, pp57-58, 2005; pp85-86, 2011)

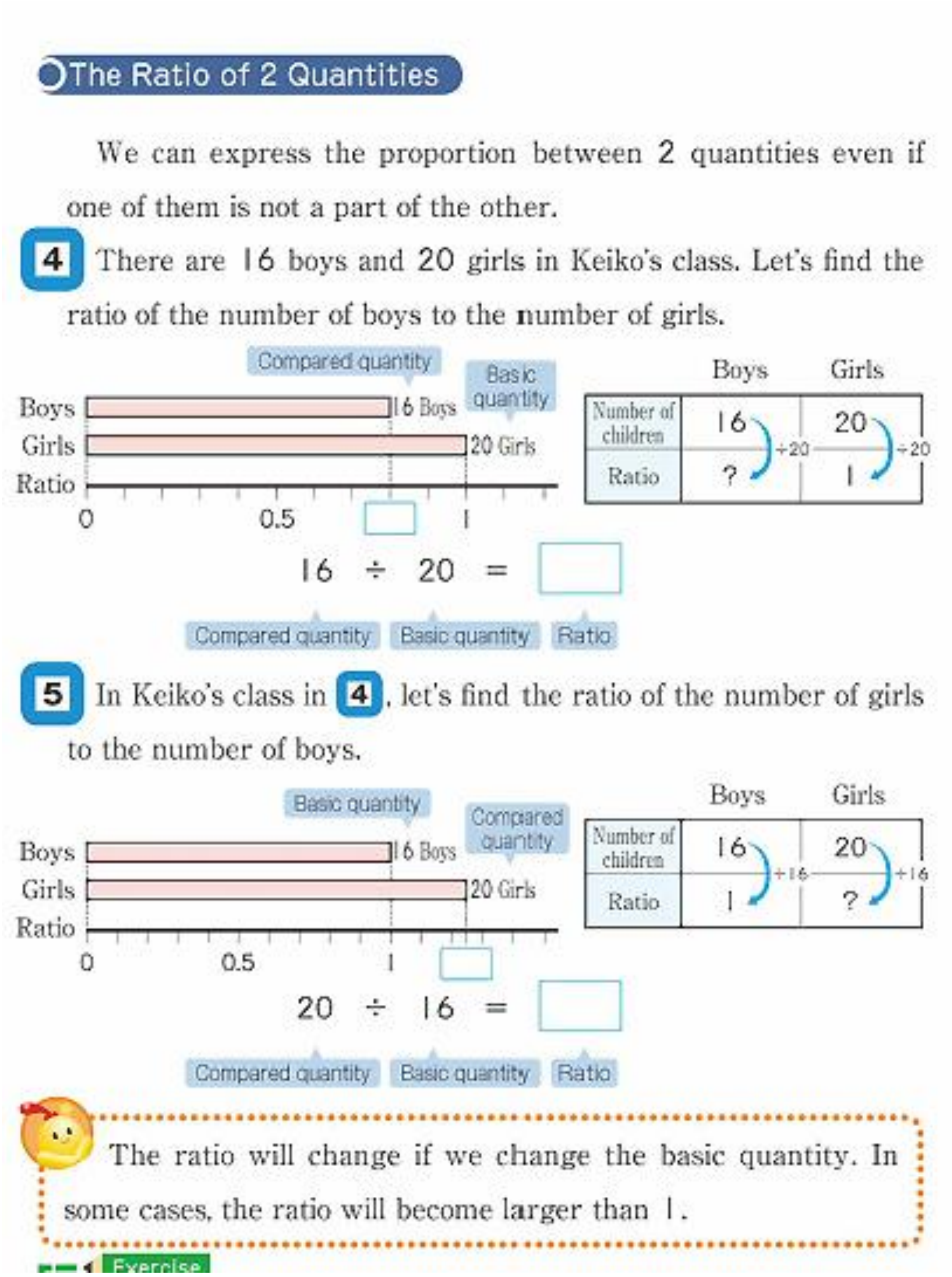
Q23.

1) Solve the following tasks and explain the base quantity for each task.

2) Explain the role of tape and number-line diagram.

3) Explain the situation using the quotative division and the same quantity.

4) $16 \div 20 = \frac{16}{20} = \frac{4}{5}$. Can we explain the task by the part-whole relationship?



The Ratio of 2 Quantities

We can express the proportion between 2 quantities even if one of them is not a part of the other.

4 There are 16 boys and 20 girls in Keiko's class. Let's find the ratio of the number of boys to the number of girls.

	Boys	Girls
Number of children	16	20
Ratio	?	1

(÷20, ÷20)

16 ÷ 20 = □

Compared quantity, Basic quantity, Ratio

5 In Keiko's class in **4**, let's find the ratio of the number of girls to the number of boys.

	Boys	Girls
Number of children	16	20
Ratio	1	?

(÷16, ÷16)

20 ÷ 16 = □

Compared quantity, Basic quantity, Ratio

The ratio will change if we change the basic quantity. In some cases, the ratio will become larger than 1.

Exercise

(Gakko Tosho, Grade 5, vol.2, pp59, 2005; pp87, 2011)

Q24.

1) Solve the task using ratio.

2) If we consider the case of 6 mats for comparison, we do not need to use the idea of ratio.

2 Measurement per Unit

1 Some students are standing on mats.

Which one, ⓐ, ⓑ or ⓒ, is the most crowded?

ⓐ 2 mats,	12 students
ⓑ 3 mats,	12 students
ⓒ 3 mats,	15 students

ⓐ 2 mats, 12 students

ⓑ 3 mats, 12 students

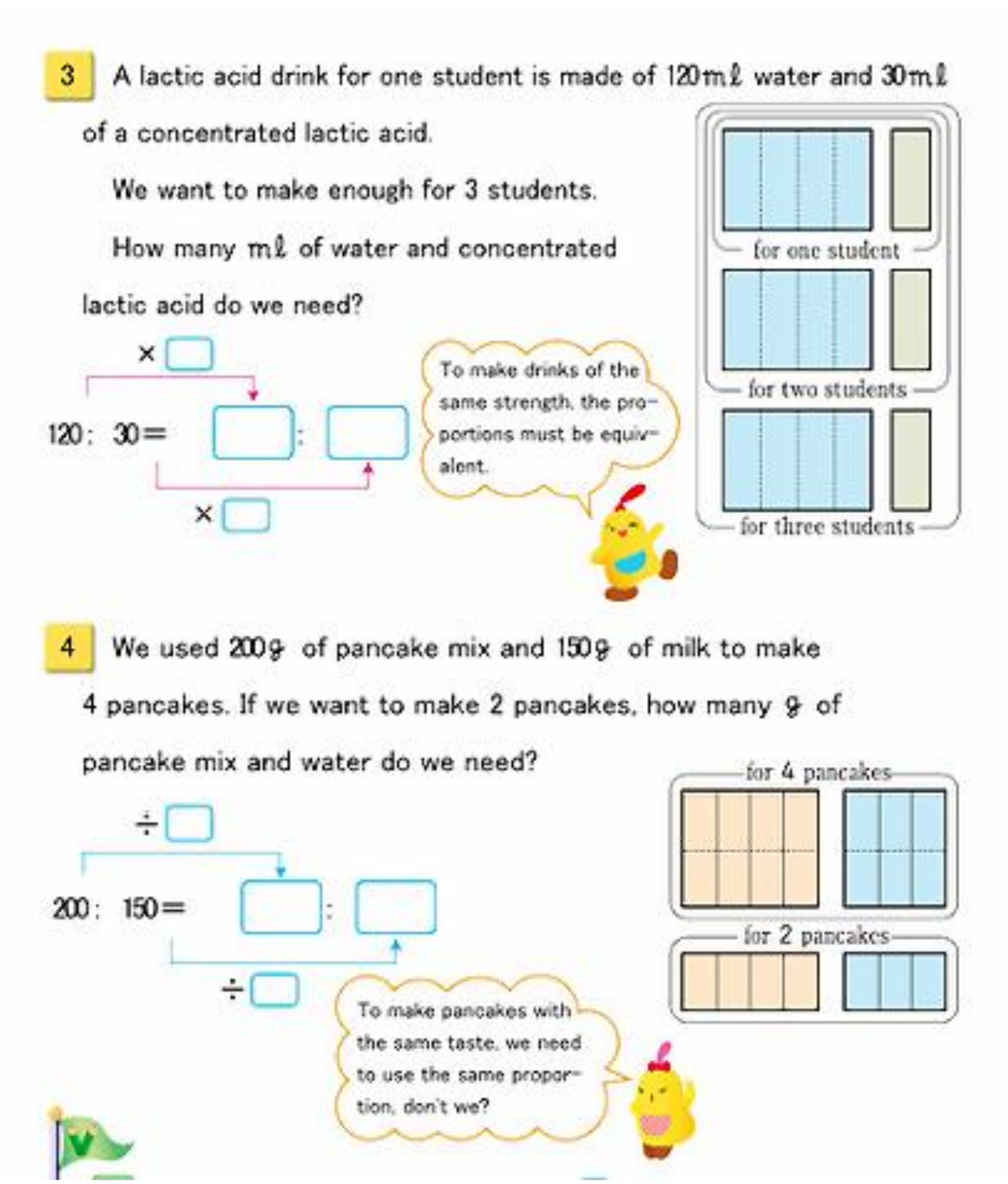
ⓒ 3 mats, 15 students

(Gakko Tosho, Grade 6, vol.1, p74; Grade 5, vol.1, p19)

Q25.

1) Let's solve the following tasks.

2) If you change the ratio to fraction, is it representing the part-whole relationship?



3 A lactic acid drink for one student is made of 120mℓ water and 30mℓ of a concentrated lactic acid.

We want to make enough for 3 students.

How many mℓ of water and concentrated lactic acid do we need?

120 : 30 = ☐ : ☐ (×☐)

4 We used 200g of pancake mix and 150g of milk to make 4 pancakes. If we want to make 2 pancakes, how many g of pancake mix and water do we need?

200 : 150 = ☐ : ☐ (÷☐)

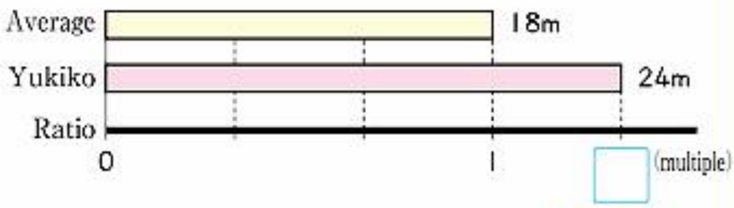
(Grade 6, vol2,p35,2005; p11, 2011)

Q26. Compare the following task with Q23. Can we see the fraction by the part-whole relationship?

Ratio Represented by a Fraction

2 Yukiko and her friends played a game by comparing how far they could throw a softball. The average was 18m.

❶ Yukiko's record is 24m. How many times is her record to the average? Show it by a fraction.



24 ÷ 18 = ☐

Compared quantity / Basic quantity / Multiple

Suppose her record is x times the average.

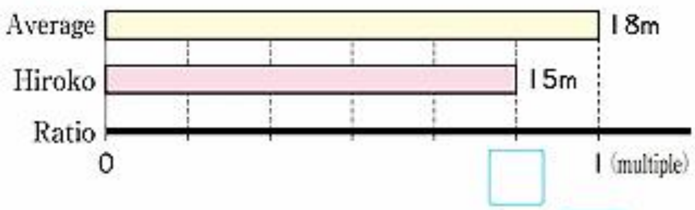
Distance (m)	18	24
Ratio (multiple)	1	x

$18 \times x = 24$

$x = 24 \div 18$

Ratio is sometimes expressed as fractions.

❷ Hiroko's record was 15m. Her record is how many times the average?



15 ÷ 18 = ☐

Compared quantity / Basic quantity / Multiple

Suppose her record is x times the average.

Distance (m)	18	15
Ratio (multiple)	1	x

$18 \times x = 15$

$x = 15 \div 18$

Exercise

Let's fill the ☐ with fractions.

① 15m is ☐ times of 9 m. ② 35kg is ☐ times of 42kg.

(Gakko Tosyo,Grade6, vol2,p27,2005; Grade 6, vol.1,p53,2011)

Fraction as Ratio

On the context of ratio a : b, $\frac{a}{b}$ is called value of ratio. However, in some countries, fraction is usually explained on the context of the part-whole relationship. Contextually, it is correct, however, in the previous chapter, the author already explained that this narrow understanding of fraction has developed huge misconceptions of fraction itself. Fraction $\frac{n}{m}$ $(m \neq 0)$ in mathematics is a representation of rational numbers. It is the value of ratio n:m which cannot be defined by the part-whole relationship. For understanding fraction, we should know the various meanings and must use suitable representations for every meaning.

Indeed, on the context of usage on the ratio, we cannot generally explain the fraction by the part-whole relationship. For example, speed is the quantity such as km per hour. Distance is not the part of time duration. The ratio of boy and girl is 2:3. In fraction, it is $\frac{2}{3}$. It means that if there are three girls, the boys must be two: $\frac{2}{3} \times 3 = 2$. Here, the boy is not the part of the girl. The boy is the part of the whole human, not the girl. The ratio of boy and human 2:5 can be seen as a part-whole relationship.

Thus, through the learning sequence, children relearn all meanings or contexts of using fraction, and at the end of it. They can be re-presented/integrated by the idea of ratio.

Fraction as ratio is usually used in the context of multiplication. The ratio of boys and girls is 2:3. The value of ratio is $\frac{2}{3}$. $\frac{2}{3}$ is used for knowing the number of boys when the number of the girls is given: (the number of boys)= $\frac{2}{3}$ x (the number of girls). This is the proportion. On the equation to represent the proportion, the ratio is used for the coefficient. Until students well understand the proportionality[3], we cannot say that the dividing fraction is the fraction as ratio.

Let's think about the various representation of proportionality. Firstly, proportionality appeared from each row of the multiplication table. However, on the multiplication table, children do not clearly think co-variable. It is difficult to distinguish the co-variable with the term of multiples from the co-variable with the term of difference/increase at the introduction of rows because they are developing each row by adding multiplier based on accumulation.

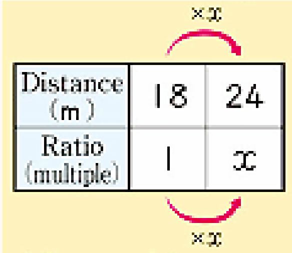
Exercise

If there is proportionality between the length of ribbon and its price, please fill in the ?.

Price (Cost)	80	?
Length of ribbon (m)	1	2.4

[3] Here, proportionality means properties to correspond proportion such as if x value become 2x, 3x, and so on, corresponding y value become 2y, 2y and so on.

Rule of Three: two by two matrix table



In the situation of the proportionality, if three numbers are given on the table, we can get the one in the remaining part of the table. Historically, this rule is called 'Rule of Three' which was the ancient's method for ratio and proportional reasoning. The idea itself is well known at the age of Egyptians (BC17C). It has existed before the invention of expressions and equations for algebra. Historically, the Rule of Three in 17th century is represented the arrangement of table with only four numbers.

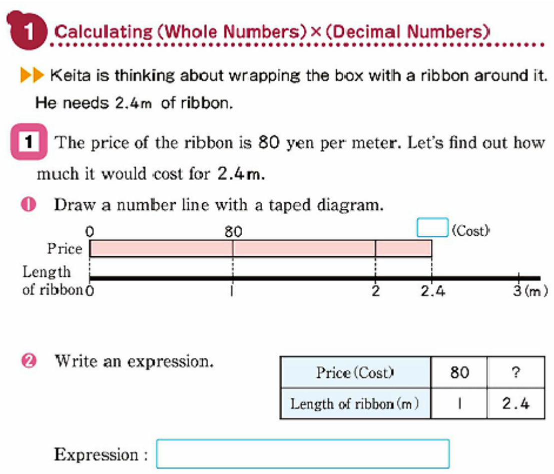
In school mathematics, the rule of three was taught just as a rule in relation to ratio and proportion. Later, the rule of three is represented by the three different expressions or formula, however we should know that the original rule of three itself was used without current expressions. It was used to be the number-arrangement on the paper to consider the ratio and proportion. Currently, we distinguish it by different formula of multiplication and division depending on the arithmetic situations. However, rule of three is based on the ratio and we do not need to care to distinguish the multiplication or division expressions depending on the situations. If we array numbers and consider the relationship with idea of ratio and proportion by using multiplication and division, it is enough. On this meaning, the person who explains the rule of three as the three different types of formulas is not using the rule of three but thinking based on current arithmetic representation oriented to algebra which should be represented by the different expressions. Likely using rule of three in ancient era, table and arrow representation for representing ratio must be easier for everyone and closer to the ancients approach because rule of three does not need to distinguish the situations by the three different types of expressions. On the algebraic representation of situation, if given quantities on ratio x to y, the value of x/y is ratio and y is the base. The formulas, x/y=a, x=ya and y=x/a, correspond to three types of situations. On the rule of three, if the situations are written on the two by two matrix table as one unknown on three known, we do need to distinguish the place of given numbers and multiples,un-necessary to memorize three formulas. If we use the rule of three meaningfully on ratio and proportion, we do not need to distinguish these three types of formulas.

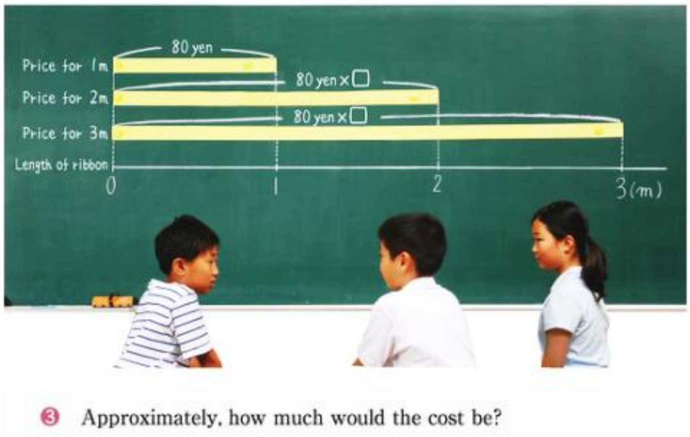
Tape and Number-Line diagram: Proportional number line

Proportionality or Rule of Three are represented on the tape and number-line diagram as well as table. Japanese has been using these representations since 1960s.

If the rule of three on the table is just used for the rule, relatively, the tape and number-line diagram are better to represent the meaning of proportionality because it works to represent co-variable with the images of the size of number on the line. Even if children do not know the term of proportion it supports to establish the way to develop expression and its calculation. We call it proportional number line. If we use the rule with tape diagram instead of table, it shows meaningful magnitude.

For explanation of meaning, we usually use various representations. Expressions can be derived from them. Proportional number line is also useful for studying fractions. The diagram of operational fraction with the remainder in Chapter 2, Figure 3, is also seen as a kind of proportional number line. Children draw table and diagram if their teacher clearly taught how to draw and use them. Teachers have to learn how to draw the proportional number lines.



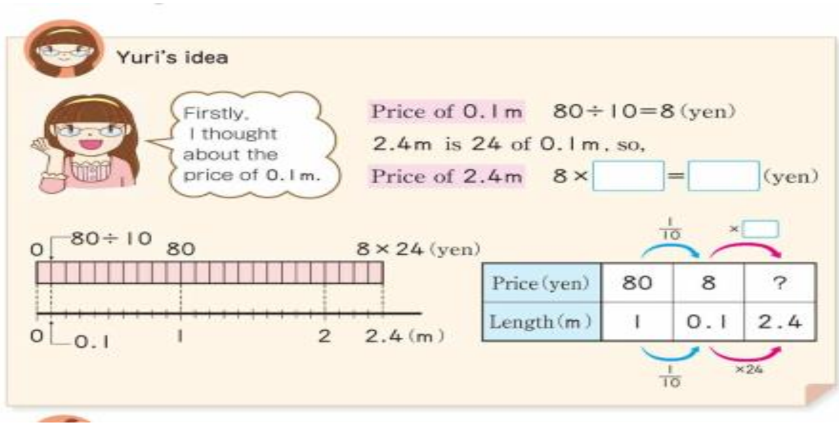


(Gakko Tosho Grade 5, Vol.1, pp29-30,2005;pp30-31,2011)

Exercise

Let's get the price for 2.4m by the proportional number line.

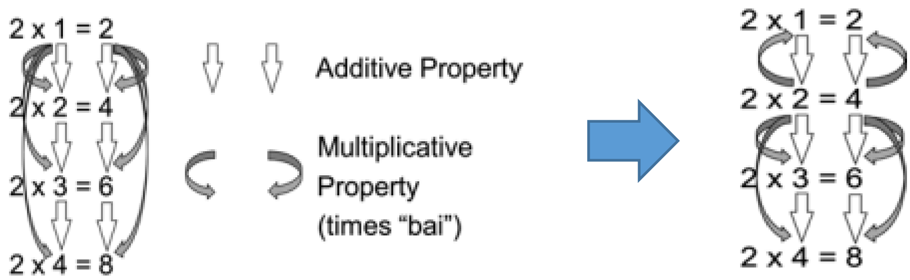
For knowing how and what, please explain the way to use the proportional number line by yourself.



(Gakko Tosho Grade 5, Vol.1, pp30, 2005;pp32,2011)

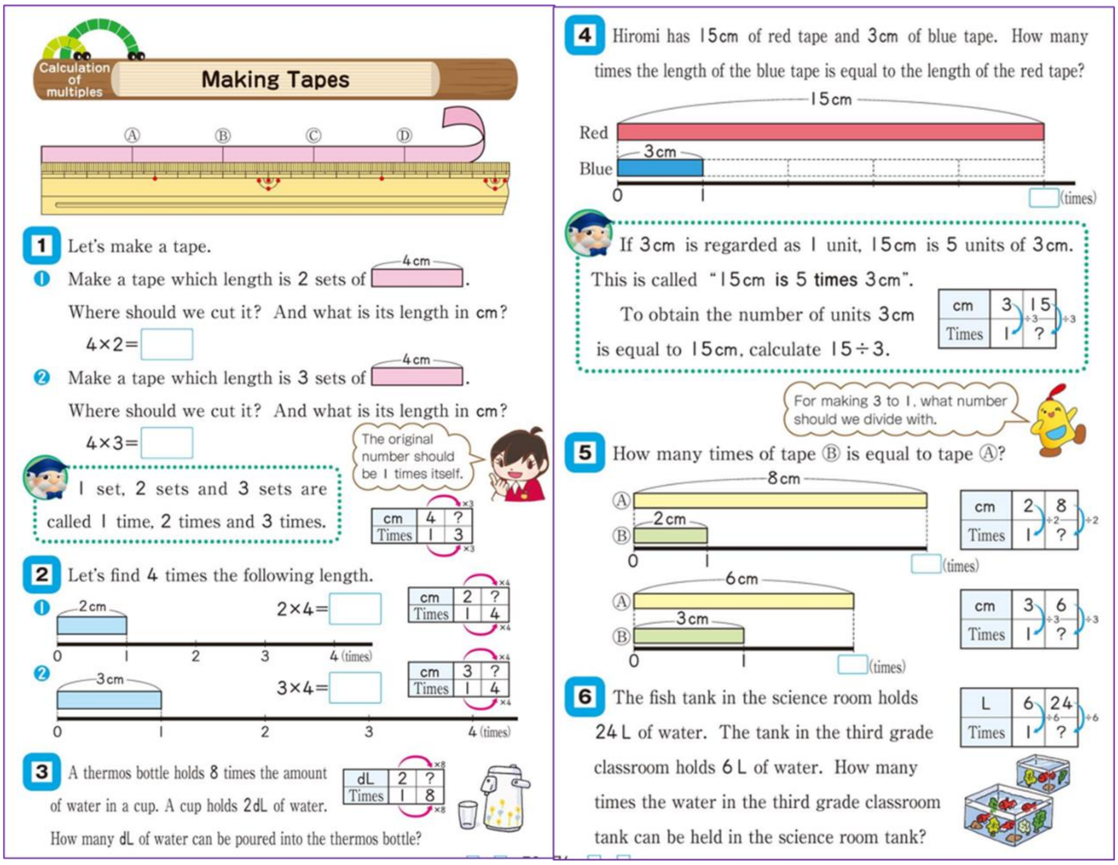
Multiplication as for the Origin of Proportionality

Origin of proportionality is multiplication table. At the introduction of multiplication table, every row can be seen by additive property and later, we re-see it for multiplicative property. It is the origin of proportionality.



(Isoda&Olfos, 2021. p.96)

For developing the proportionality, Gakko Tosho textbook (2021, Grade 3) introduces proportional number line as follows after students learned multiplication table.



Calculation of multiples — Making Tapes

1 Let's make a tape.

1. Make a tape which length is 2 sets of [4 cm]. Where should we cut it? And what is its length in cm?
   4×2=☐
2. Make a tape which length is 3 sets of [4 cm]. Where should we cut it? And what is its length in cm?
   4×3=☐

1 set, 2 sets and 3 sets are called 1 time, 2 times and 3 times.

The original number should be 1 times itself.

2 Let's find 4 times the following length.

1. 2 cm — 2×4=☐
2. 3 cm — 3×4=☐

3 A thermos bottle holds 8 times the amount of water in a cup. A cup holds 2 dL of water. How many dL of water can be poured into the thermos bottle?

4 Hiromi has 15 cm of red tape and 3 cm of blue tape. How many times the length of the blue tape is equal to the length of the red tape?

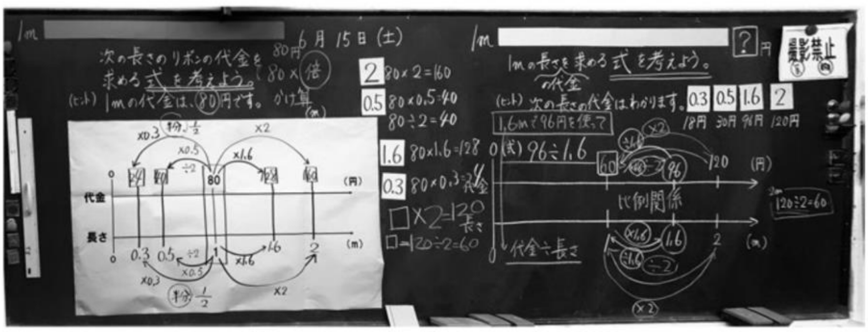
If 3 cm is regarded as 1 unit, 15 cm is 5 units of 3 cm. This is called “15 cm is 5 times 3 cm”. To obtain the number of units 3 cm is equal to 15 cm, calculate 15÷3.

For making 3 to 1, what number should we divide with.

5 How many times of tape Ⓑ is equal to tape Ⓐ?

6 The fish tank in the science room holds 24 L of water. The tank in the third grade classroom holds 6 L of water. How many times the water in the third grade classroom tank can be held in the science room tank?

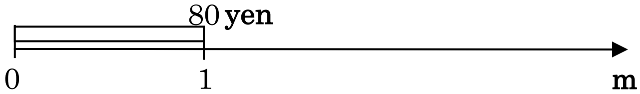
After students learned proportion in grade 5, it is drawn as follows (Isoda, Olfos 2021).



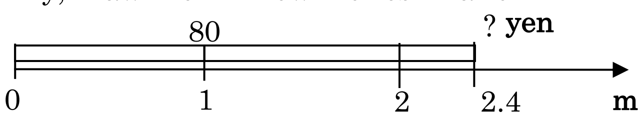
How to draw the proportional number line

One of the objectives to draw the proportional number line is searching the way of calculation. Its procedure for drawing is like this. Arrows show the calculation based on the proportionality.

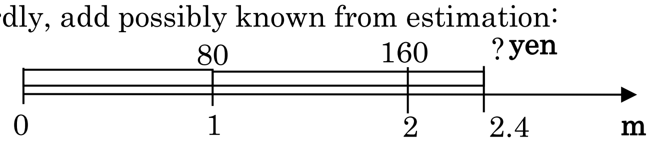
Firstly, draw the amount for unit which is known:



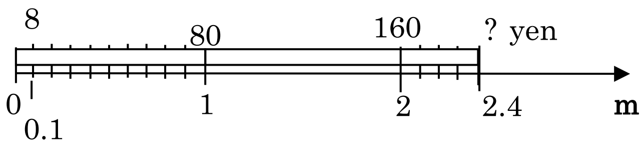
Secondly, draw the unknown for estimation:



Thirdly, add possibly known from estimation:

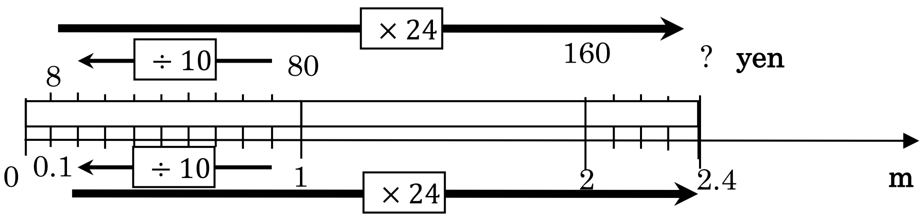


If not clear, let's **change the unit for measuring the unknown**:



The price of 2.4m is not sure. If I **change the unit** 'm' to 'cm,' it is 80 yen for 100cm. How much in 240cm. Aha, 8 yen for 10cm: it means 0.1m!

Finally, search the way of calculation under the proportionality:



80 × 2.4 =? The length corresponds the price. **The re-measuring process is represented by the arrows**. The arrow sequence represents 80 ÷ 10 × 24. Aha!!

Major Reference and Further readings 5

Masami Isoda et al. (2010). Mathematics Education Theories for Lesson Study: Problem Solving Approach and the Curriculum through Extension and Integration, Journal of Japan Society of Mathematical Education, 92(11). 1-158.

Shin Hitotsumatsu et al. (2005). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda et al. (2010). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda (2012). Introductory Chapter: Problem Solving Approach to Develop Mathematical Thinking. *In Isoda, M. & Katagiri, S. edited (2012). Mathematical Thinking: How to Develop it in the Classroom.* Singapore: World Scientific. 1-28.

https://www.criced.tsukuba.ac.jp/math/apec/ICME12/Lesson_Study_set/MATHEMATICAL%20THINKING%20(c)World%20Scientific/MathematicalThinking_chap01(c)WorldScientific.pdf

https://doi.org/10.1142/9789814350853_0001

Masami Isoda, Tenoch Esaú Cedillo Ávalos, et al. (2012). Matemáticas para la educación normal (11vols). Estado de México : Pearson Educación de México.

Tenoch Cedillo, Masami Isoda, et al. (2013) Matemáticas para la educación normal : guía para el aprendizaje y enseñanza de la aritmética, Estado de México : Pearson Educación de México.

Masami Isoda (2015). The Science of Lesson Study in the Problem Solving Approach. *In Inprasitha, M. et.al. edited. Lesson Study: Challenges in Mathematics Education.* Singapore: World Scientific. pp.81-108. https://doi.org/10.1142/9789812835420_0006

Masami Isoda (2015). Dialectic on the Problem Solving Approach: Illustrating Hermeneutics as the Ground Theory for Lesson Study in Mathematics Education. *In Sung, J. edit. Selected Regular Lectures from the 12th International Congress on Mathematical Education*. Cham: Springer. pp.355-381. https://doi.org/10.1007/978-3-319-17187-6_21

Masami Isoda & Aki Murata edited (2011). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.

Masami Isoda, Aki Murata, Aida Istino Yap edited (2015). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.

Masami Isoda & Aki Murata edited (2021). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.
https://support.gakuto.co.jp/mathematics-textbook/

Masami Isoda, Raimundo Olfos. Edited. (2021).
Teaching Multiplication with Lesson Study.
Cham: Springer. DOI: 10.1007/978-3-030-28561-6

Chapter 6. Multiplication and Division of Fraction (1)

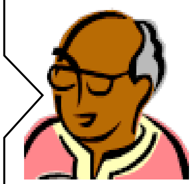
For developing the multiplication and division of fractions, this chapter mainly explains the necessary knowledge needed to produce the idea for it.

Why did Professor recommend us to use the proportional number line?

To explain the meaning of calculation, we must change the representation which explains the process in a visible and understandable way.

If we do not use it, we only teach skill as a rule. Where shall we use it to explain the meaning? We have to teach how to draw.

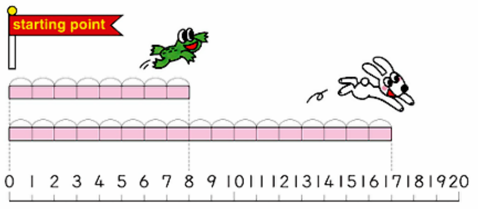
The idea of proportional number line was originated from Rene Descartes (1637). Japanese Math-Educators such as Takeshi Ito invented it to establish the Heuristic Teaching Approach for elementary school mathematics with the extension and integration curriculum sequence in the 1960s. From the 1990s, it became the world famous approach as for the representations to develop the competency for proportional reasoning. Japanese textbooks such as Gakko Tosho established well teaching sequence for developing it.



Number line

In chapter 1 and 2, we recognize that shading activities, the parts for counting, is the root for the misconceptions of fractions for missing the idea of whole as for a unit. For developing children who learn mathematics for and by themselves (see Chapter 1), children are necessary to draw and use appropriate diagrams for explaining their ideas. On the previous chapters, appropriate diagrams need to show original unit with quantity fraction and the unit fraction for measuring its number, like measuring by using the remainder as for operational fraction. The tape and number line with quantities are appropriate diagrams on this condition. For developing children who will draw such a diagram, firstly, we have to develop children who draw the number line by and for themselves.

Number line which shows the position of number on the line is introduced by taking the same intervals by the unit of measurement recursively for comparing the size of number. At this moment, it looks like a line of discrete numbers because it is given by **interval** as for the scale on the line and there is only one interval but no number between two numbers. On the process of extension of numbers, when students re-scale it by using smaller units or larger units, it begins to function as number lines which shows the position of various numbers and used for extension of numbers. At the beginning, children learn the '0' is the starting point on the line instead of 'nothing.' On the number line, 0 shows the origin of **position** as for measuring by the interval (unit). The difference of positions shows the distance (the number of intervals: cardinal number).



If teacher does not teach the measuring by the unit from the starting point 0, children may confuse one on counting intervals as scale number 0 on the left instead of the scale number 1 on the right. Children learn the number line as for comparing the size and ordering of numbers, and recognize the number on the base ten system. Taking interval is the preparations for the multiplication and division, too.

5 Write the correct numbers in the ☐.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14		16	17	18	19
20	21		23		25	26		28	
30	31	32		34	35		37	38	39
	41	42		44		46	47		
50	51		53		55		57	58	59

(Gakko Tosho, Grade 1, pp71-72, 2005; pp82-83,2011)

If your children do not know how to draw the number line, let's give them the opportunity to draw it by themselves. It is the activity of measurement by using arbitrary unit.

Multiplication

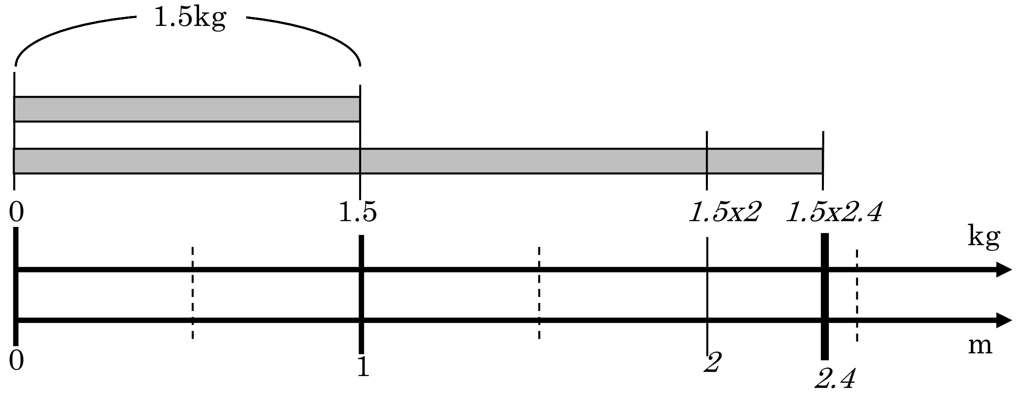
(Gakko Tosho, Grade 2, vol.2 pp2-3, 2005; pp6-7,2011)

In the case for lower elementary school mathematics, children study how to use daily language mathematically on the situation. Definition of arithmetic operations is usually done on the situation in daily context because we have to develop children to use four arithmetic operations on their daily life.

On this issue, multiplication is introduced at the following situations: the number of dishes and the number of objects for each dish.

From the viewpoint of measurement, the situation is used to explain multiplication that the multiplication is measuring the amount of the quantity by the unit of quantity when the unit for the amount is known by the quantity (Ministry of Education, Japan, 1960; Freudenthal, 1983). For example, when the amount is 8 dishes and the unit for the amount is 2 cakes in each dish, the measured amount by the quantity is 8 x 2 =16 (cakes).

This definition works for multiplication of decimal numbers and fractions as well as the situation of repeated addition. For example, when the amount of steel is 2.4 m and the unit weight for the steel is 1.5 kg for one meter, the measured amount by the quantity is 1.5 x 2.4 = 3.6 (kg)[4]:



In relation to how to calculate, multiplication is explained with the repeated addition, however, multiplication of decimal numbers and fractions is not explained by the repeated addition but explained by multiplication table with distribution law on base ten place value system. For explaining the multiplication of decimal numbers and fractions, we can use the proportional number line which represents the meaning of multiplication by measuring.

How to find the expression from the situated problem

At the end of last chapter, you learned how to draw the proportional number line by the task below:

---

[4] This textbook is using Japanese notation: 1.5 (kg/m) x 2.4 (m) = 3.6 (kg). In English notation, it should be 2.4 (m) x 1.5 (kg/m) when '2.4 x 1.5' is read as '2.4 times 1.5'. In English, '2.4 times' implicates 'multiplied by 2.4'. Thus, as long as you read '1.5 x 2.4' as '1.5 multiplied by 2.4,' Japanese notation of multiplication is understandable. Indeed, 'a x b' can be read as 'a multiplied by b' even in English. English usage has inconsistency.

1 The price of the ribbon is 80 yen per meter. Let's find out how much it would cost for ☐ m.

The box (in blank) is 2.4 at the end of last chapter. The children who have not yet learned the multiplication of decimal numbers and fractions cannot easily recognize that this task is multiplication. On the other hand, if we put the whole number such as 2 into the box, children who learned the multiplication of whole numbers could easily understand that this task is multiplication because multiplication is introduced in daily situations on the whole numbers.

On the context of extension of numbers and operations, Japanese teachers usually prefer this problem posing strategy like this form and ask children to put any number they want into the box and discuss how. Through putting into a simple number, children recognize this task as multiplication and in the case of whole numbers, they already learned, and in the case of fraction and decimal numbers, they did not yet learn. When the class begins this way, children recognize this task as the task for multiplication and they would like to inquire how to find the answer using what they already learned.

Exercise

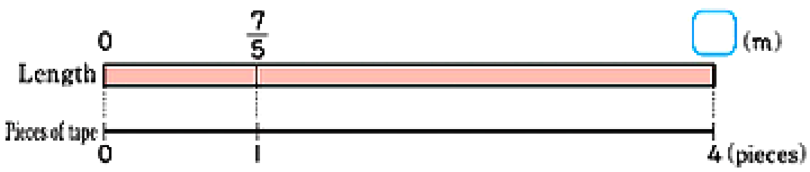
Let's draw the proportional number line when the box (blank) is 2 (m), 2.3 (m) or $\frac{3}{2}$ (m) on [1].

In this exercise, for answering in the case of 2.3 m, we have to change the unit from 1m to 0.1m as well as the case of 2.4m. In the case of fraction, we usually change the unit from 1m to the unit fraction: this case $\frac{1}{2}$ m is the unit for measuring. If 1m is 80 yen, $\frac{1}{2}$ m is 80 ÷ 2 yen. If $\frac{1}{2}$ m is 80 ÷ 2 yen, $\frac{3}{2}$ m is 80 ÷ 2 × 3 yen. For considering like this, children need to draw a proportional number line and apply multiplication and division on the number line.

Fraction × Whole Number

Please explain the following:

4 We want to make 4 pieces of tape that are each $\frac{7}{5}$ m long. How much tape do we need?



If 1 piece is $\frac{7}{5}$ m and addition of fractions is known, 4 pieces are $\frac{7}{5} \times 4 = \frac{7}{5} + \frac{7}{5} + \frac{7}{5} + \frac{7}{5} = \frac{28}{5}$. After the explanation, we should ask children as follows: Is it possible to find an easier or faster way? Then, $\frac{7 \times 4}{5}$ is recognized as a simple way for $\frac{7}{5} \times 4$.

In mathematics, we usually produce shorter and simple ways. Seeking simplicity is a basic value of mathematics.

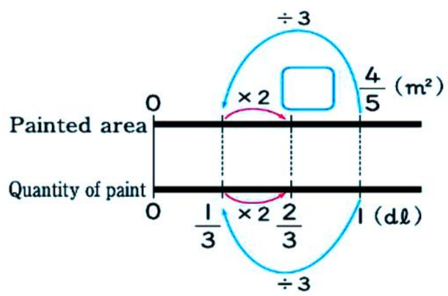
Exercise

In Gakko Tosho textbooks, the proportional number line changes from the tape diagram and number line to two number lines at Grade 5. Let's put the number in the box and answer: "We can cover an area of $\frac{4}{5}$ m^2 with 1dl paint. How many m^2 can we cover with

☐ dl of the paint?"

Fraction × Fraction

If the box is a whole number, we already learned. If the box is a fraction such as $\frac{2}{3}$ dl, we can draw the proportional number line:



If we develop the way of calculation as well as multiplication of decimal numbers, it can be calculated as follows: $\frac{4}{5} \div 3 \times 2$.

After the explanation, we should ask children as follows: Is it possible to find an easier or faster way? Then, $\frac{4}{5} \div 3 \times 2 = \frac{4 \times 2}{5 \times 3}$ is recognized as the simple way for $\frac{4}{5} \times \frac{2}{3}$.

In mathematics, we usually produce shorter and simple ways.

Area Diagram

In school mathematics, area diagram is usually recommended for use in explaining (a+b)(c+d)=ac+ad+bc+bd. On the area diagram, multiplication is a two-dimensional idea and it functions as for the model of commutativity. Some people strongly believe that the area diagram is the best way for explaining multiplication and division because it provides the wall painting/shading metaphor based on two dimensions. The misconception will appear if students do not feel the necessity to draw the same size diagram.

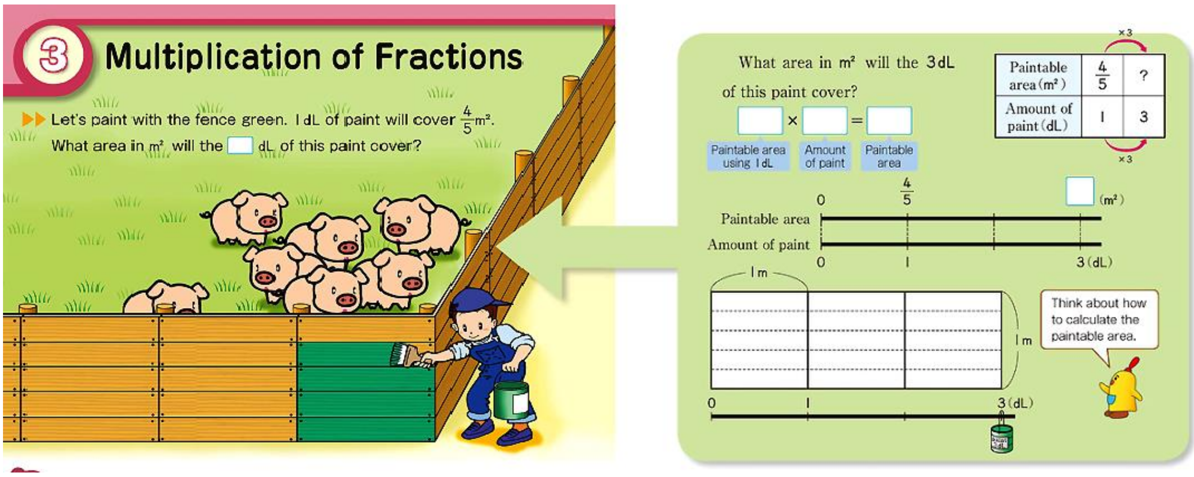
As long as teachers try to explain fraction as dividing fraction it might be true, however, shading activities of the area diagram itself is a major source of the misconceptions if children recognize fraction only by dividing fraction (see Chapter 2).

What is necessary to develop in students by and for themselves is that students are able to draw the area diagram by and themselves as for the tool for reasoning as well as the proportional number lines.

Historically, Euclid produced the theory under the dimension and Descartes overcame the wall between dimensions by the proportional number line which defined multiplication by the measurements.

Exercise

1) Let's solve the following task by three different methods.



Gakkotosho G6, vol.2, p3, 2011; G6,vol.1. pp34-35.

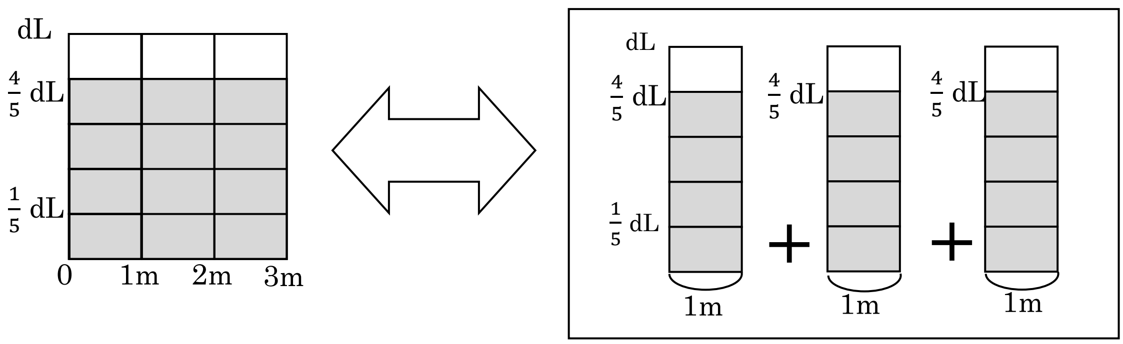
1) Let's compare three methods. Which one do you recommend? Why do we need more?

2) There are a number of students who get the answer using the area diagram such as $\frac{4}{5} \times 3 = \frac{12}{15}$. Why does the area diagram produce such answer?

There are two types of area diagram. The first type is already shown in the textbook. It shows the area of the wall itself for showing the painting. Another type of area diagram represents the following situation:

There is a wall in which we use $\frac{4}{5}$ dL of paint for painting 1 meter of wall. How much liters do we need for painting 3 meters?

For this task, we draw the following two diagrams for the same meaning.



In both diagrams, the denominations of quantities are necessary because children might develop misunderstanding such as $\frac{4}{5} \times 3 = \frac{12}{15}$ if there are no denominations.

Area diagram is fine to explain meaning. However, the children who still keep the dividing fraction, misunderstand the meaning of whole. Indeed, the left hand side of the above diagram can be read as $\frac{4}{5}$ if the square is the whole even though it shows quantity. Here, the key is a unit fraction $\frac{1}{5}$ dL (quantity fraction!) as well as the whole 1 L. If teachers ask students just shading without considering these two units, we are not sure students understand well or not, even though teachers felt success to explain for him/herself.

Major Reference and Further Readings 6

Masami Isoda et al. (2010). Mathematics Education Theories for Lesson Study: Problem Solving Approach and the Curriculum through Extension and Integration, Journal of Japan Society of Mathematical Education, 92(11). 1-158.

Shin Hitotsumatsu et al. (2005). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda et al. (2010). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda (2012). Introductory Chapter: Problem Solving Approach to Develop Mathematical Thinking. *In Isoda, M. & Katagiri, S. edited (2012). Mathematical Thinking: How to Develop it in the Classroom.* Singapore: World Scientific. 1-28.

https://www.criced.tsukuba.ac.jp/math/apec/ICME12/Lesson_Study_set/MATHEMATICAL%20THINKING%20(c)World%20Scientific/MathematicalThinking_chap01(c)WorldScientific.pdf

https://doi.org/10.1142/9789814350853_0001

Masami Isoda, Tenoch Esaú Cedillo Ávalos, et al. (2012). Matemáticas para la educación normal (11vols). Estado de México : Pearson Educación de México.

Tenoch Cedillo, Masami Isoda, et al. (2013) Matemáticas para la educación normal : guía para el aprendizaje y enseñanza de la aritmética, Estado de México : Pearson Educación de México.

Masami Isoda (2015). The Science of Lesson Study in the Problem Solving Approach. *In Inprasitha, M. et.al. edited. Lesson Study: Challenges in Mathematics Education.* Singapore: World Scientific. pp.81-108. https://doi.org/10.1142/9789812835420_0006

Masami Isoda (2015). Dialectic on the Problem Solving Approach: Illustrating Hermeneutics as the Ground Theory for Lesson Study in Mathematics Education. *In Sung, J. edit. Selected Regular Lectures from the 12th International Congress on Mathematical Education*. Cham: Springer. pp.355-381. https://doi.org/10.1007/978-3-319-17187-6_21

Masami Isoda & Aki Murata edited (2011). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.

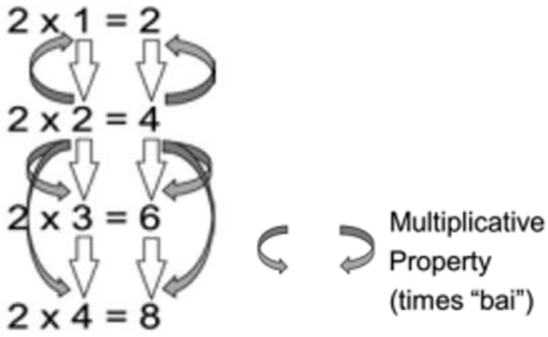
Masami Isoda, Aki Murata, Aida Istino Yap edited (2015). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.

Masami Isoda & Aki Murata edited (2021). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.
https://support.gakuto.co.jp/mathematics-textbook/

Masami Isoda, Raimundo Olfos. Edited. (2021). *Teaching Multiplication with Lesson Study.* Cham: Springer. DOI: 10.1007/978-3-030-28561-6

Chapter 7. Multiplication and Division of Fraction (2)

The reasoning for ratio and proportion is already begun from the multiplication at the second grade. Proportional number lines and Rules of three on the table are the representation of the proportionality: Children should study them before learning ratio and proportion for multiplication and division of fraction by and for themselves by using what they already learned.



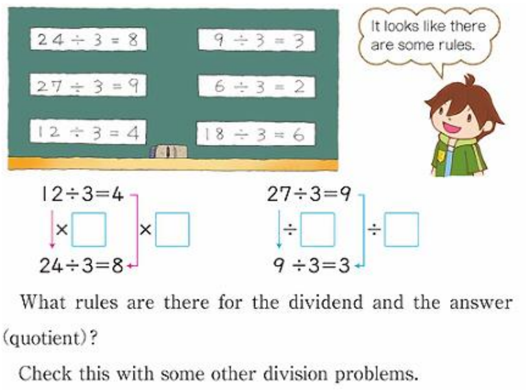
Rules of Multiplication and Division

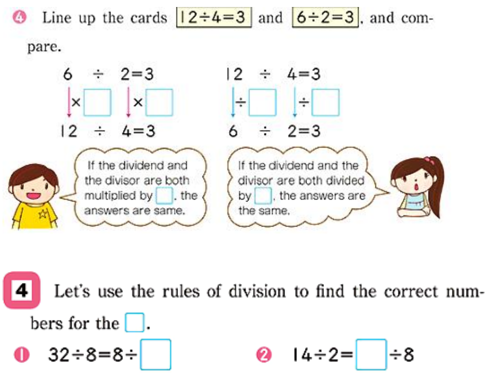
Rules of Multiplication and Rules of Division which appears the comparison of expressions for multiplication and division can be seen as the representation of the proportionality. Both rules are also useful for the extension of multiplication and division into decimals and fractions from the whole number.

Exercise:

The followings are the samples of the rules of multiplication and division:

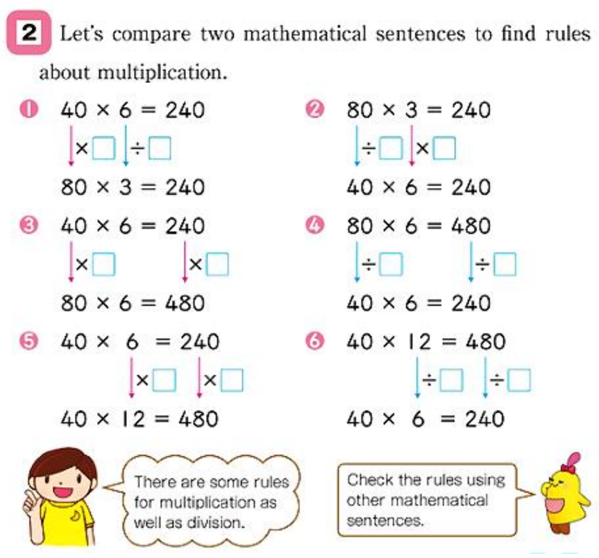
1) Let's find the rules of division.





Gakkotosho G4, vol.1, pp21-24 (2011)

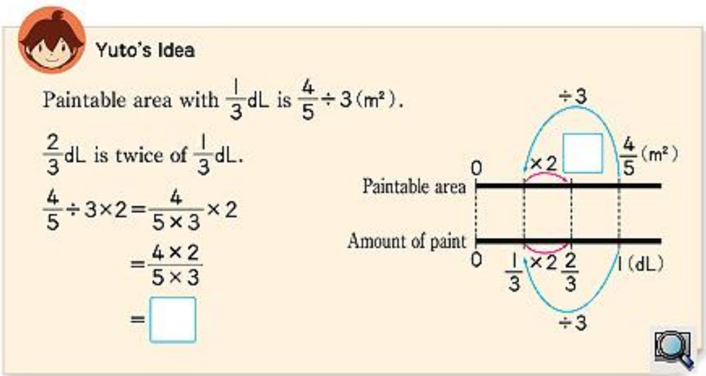
2) Let's find the rules of multiplication:

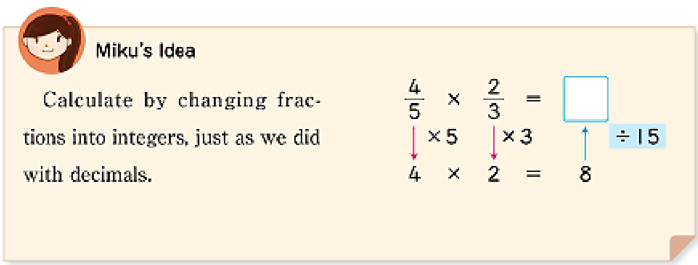


Gakkotosho G4, vol.1, p89 (2011)

3) Let's explain those rules by using four number table for the Rule of three.

4) Rules of multiplication are easier ways to find the answer of multiplication of fractions. Let's compare the following two approaches and explain how to use the rule of multiplication below (see page 49).

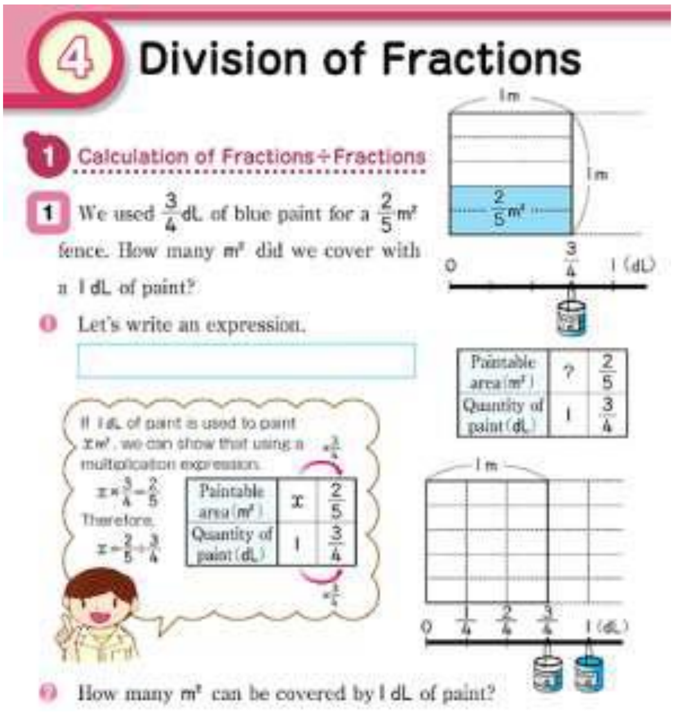




Gakkotosho, G6, vol.1, p36 (2011)

Exercise: Fraction divided by Fractions

1) Let's find the answer of the following task by the proportional number line, rule of three, rule of division, and area diagram.



4 Division of Fractions

1 Calculation of Fractions ÷ Fractions

1 We used $\frac{3}{4}$ dL of blue paint for a $\frac{2}{5}$ m² fence. How many m² did we cover with a 1 dL of paint?

1. Let's write an expression.

If 1 dL of paint is used to paint x m², we can show that using a multiplication expression.

$x \times \frac{3}{4} = \frac{2}{5}$

Therefore,

$x = \frac{2}{5} \div \frac{3}{4}$

Paintable area (m²)	x	$\frac{2}{5}$
Quantity of paint (dL)	1	$\frac{3}{4}$

Paintable area (m²)	?	$\frac{2}{5}$
Quantity of paint (dL)	1	$\frac{3}{4}$

2. How many m² can be covered by 1 dL of paint?

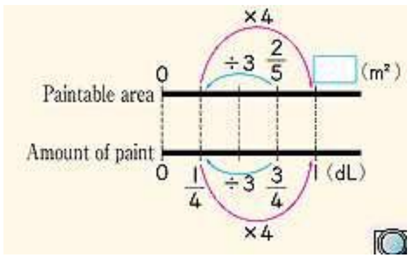
Gakkotosho G6. vol.1, pp44-45 (2011).

2) Explain the following approaches to obtain the answers.

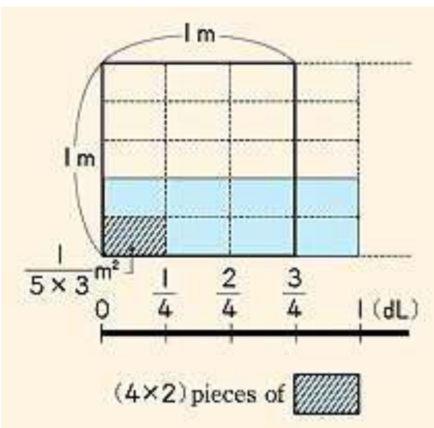
a) Rule of Division

$$\frac{2}{5} \div \frac{3}{4} = \left(\frac{2}{5} \times \frac{4}{3}\right) \div \left(\frac{3}{4} \times \frac{4}{3}\right)$$

b) Proportional Number Line



c) Area Diagram



Gakkotosho G6. vol.1, pp46-47 (2011).

3) Let's compare the three approaches. Which approach uses the unit fraction for changing the measurement scale? Which approach directly shows the way for the multiplication of inverse fraction? Which approach has the possibility to produce the wrong answer $\frac{8}{20}$?

Major Reference and Further Readings 7

Masami Isoda et al. (2010). Mathematics Education Theories for Lesson Study: Problem Solving Approach and the Curriculum through Extension and Integration, Journal of Japan Society of Mathematical Education, 92(11). 1-158.

Shin Hitotsumatsu et al. (2005). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda et al. (2010). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda (2012). Introductory Chapter: Problem Solving Approach to Develop Mathematical Thinking. *In Isoda, M. & Katagiri, S. edited (2012). Mathematical Thinking: How to Develop it in the Classroom.* Singapore: World Scientific. 1-28.

https://www.criced.tsukuba.ac.jp/math/apec/ICME12/Lesson_Study_set/MATHEMATICAL%20THINKING%20(c)World%20Scientific/MathematicalThinking_chap01(c)WorldScientific.pdf

https://doi.org/10.1142/9789814350853_0001

Masami Isoda, Tenoch Esaú Cedillo Ávalos, et al. (2012). Matemáticas para la educación normal (11vols). Estado de México : Pearson Educación de México.

Tenoch Cedillo, Masami Isoda, et al. (2013) Matemáticas para la educación normal : guía para el aprendizaje y enseñanza de la aritmética, Estado de México : Pearson Educación de México.

Masami Isoda (2015). The Science of Lesson Study in the Problem Solving Approach. *In Inprasitha, M. et.al. edited. Lesson Study: Challenges in Mathematics Education.* Singapore: World Scientific. pp.81-108. https://doi.org/10.1142/9789812835420_0006

Masami Isoda (2015). Dialectic on the Problem Solving Approach: Illustrating Hermeneutics as the Ground Theory for Lesson Study in Mathematics Education. *In Sung, J. edit. Selected Regular Lectures from the 12th International Congress on Mathematical Education*. Cham: Springer. pp.355-381. https://doi.org/10.1007/978-3-319-17187-6_21

Masami Isoda & Aki Murata edited (2011). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.

Masami Isoda, Aki Murata, Aida Istino Yap edited (2015). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.

Masami Isoda & Aki Murata edited (2021). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.
https://support.gakuto.co.jp/mathematics-textbook/

Masami Isoda, Raimundo Olfos. Edited. (2021). *Teaching Multiplication with Lesson Study*. Cham: Springer. DOI: 10.1007/978-3-030-28561-6

Summary

As discussed at the exercises of Chapter 1, this textbook is written for developing children who learn mathematics by and for themselves. For developing children, we should set the appropriate task sequence from what they already learned. Difficulty of fraction is related with various meanings of fraction and originated from many teachers who believe that the meaning of fraction is only dividing fraction. This textbook explained that it is the source of misconceptions. It demonstrated various meanings of fractions with terminology and the necessity of teachers considering them for overcoming difficulties.

In Japan, the terminology to distinguish the conceptual difference of fraction and so on were already known in the 1960s, at least, as a result of lesson study since 1873 with surveys of various countries' mathematics education. Terminology has been used to establish curriculum and task sequence on the textbooks. The author learned it from Prof. Tatsuro Miwa on his lectures for undergraduate students in 1981.

Due to difference of curricula sequence and textbooks, teachers who read this textbook may feel uncomfortable to use it. It might be normal as long as keeping the mindset to teach procedure by exercise with minimum explanation by teachers. However, if teachers try to develop children who explain their mathematical ideas by using what they already learned, the terminology might be significant. Indeed, Japanese teachers who show excellent practice use it for clarifying what children already learned and how they use it for thinking, and knowing necessity as for preparation of future learning.

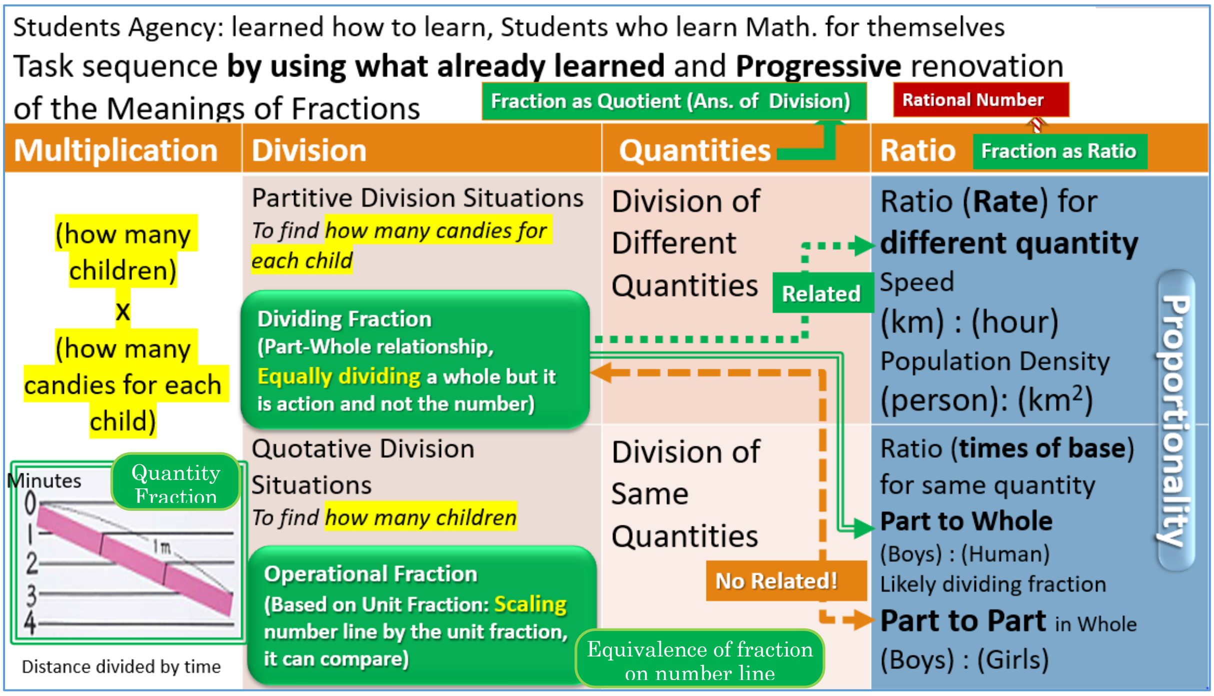


Figure 7. Various meanings of fraction and their development

This textbook used Gakko Tosho textbooks, see QR codes in the reference, which have been written by teachers in Elementary School of the University of Tsukuba which has been leading lesson study since 1873.

For using the terminology of this textbook, the task sequence to develop children who learn mathematics by and for themselves will be considered by Figure 7. Please note, it illustrates the necessary process for conceptual change under the extension and integration up to the fraction as ratio as for the base of rational number.

As long as teachers believe fraction means dividing fraction, dividing fraction may include operational fraction in a broad sense, however, in a narrow sense, the action for dividing into equally corresponds to the action for partitive division and the action for scaling by the unit fraction corresponds to the action for quotative division. On this difference, a narrow meaning of the dividing fraction should be distinguished with operational fraction. To change dividing fraction as comparable fraction, denomination of quantities is necessary for fixing and showing the size of the whole. To recognize fraction as number, equivalence of fraction on the number line and to see fraction as the answer of division are necessary. To consider the multiplication and division of fraction, it is necessary to treat the fraction under proportionality. In relation to proportionality, fraction is ratio. In relation to the meaning of division, there are three types on ratio. At the introductory stage, dividing fraction is a pair of numbers and not a number itself. For extension of number, we should discuss existence, comparison, and operations. After completion of these three discussions, fraction becomes the representation of rational number.

This book focused on fraction. Teaching of decimals can be considered analogically.

Further Readings

Tenoch Cedillo, Masami Isoda, et al. (2013) Matemáticas para la educación normal : guía para el aprendizaje y enseñanza de la aritmética, Estado de México : Pearson Educación de México.

Masami Isoda y Raimundo Olfos, coodinadores ; [contribuyentes] Ubiratan D'Ambrósio [et al.] (2011). Enseñanza de la multiplicación : desde el estudio de clases japonés a las propuestas iberoamericanas, Valparaíso [Chile] : Ediciones Universitarias de Valparaíso, Pontificia Universidad Católica de Valparaíso

Masami Isoda, Raimundo Olfos (2009).El enfoque de resolución de problemas : en la enseñanza de la matemática : a partir del estudio de clases. Valparaíso : Ediciones Universitarias de Valparaíso.

Masami Isoda, Max Stephens, Yutaka Ohara, Takeshi Miyakawa (2007). Japanese Lesson Study in Mathematics: Its Impact, Diversity and Potential for Educational Improvement. NJ: World Scientific.

Masami Isoda, Shigeo Katagiri (2012). Mathematical Thinking: How to Develop it in the Classroom. NJ: World Scientific.

http://math-info.criced.tsukuba.ac.jp/museum/dbook_site/

http://www.criced.tsukuba.ac.jp/en/seameo_index.html

http://www.criced.tsukuba.ac.jp/math/apec/

http://www.criced.tsukuba.ac.jp/renkei/msa/

https://www.criced.tsukuba.ac.jp/online-contents.html