Fraction for Teachers

Knowing What before Planning How to Teach

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Preface

Education is the work to prepare for the future. Developing children who learn mathematics by and for themselves is one of the major issues on mathematics education reforms in the world (See such as Isoda & Katagiri, 2012). After the comparative study of mathematics classroom such as TIMSS video study in 90s, Japanese lesson study is the world-shared methodology as for the tools for professional development because the study indirectly demonstrated the quality of Japanese mathematics teaching and it is established by the lesson study. However, people often misunderstand the lesson study as for the talking about the class rather than studying subject matter. They enjoy the classroom observation likely listening to the music or watching the theatre. However, through listening to the music, and even if we enjoy talking about actors, we cannot prepare the good player ourselves. In Japanese lesson study, most efforts are done for the preparation of the class. The misunderstanding originated due to the limitation of the content guidebook to refer in English. On this reason, I have developed several resources which show the theory for the purpose to improve mathematics education with researches in the world.

For the workshop of SMASE-INSET project under Japan International Cooperation Agency (JICA), Japan and Federal Ministry of Education (FME), Nigeria, this booklet includes the essential theory for enabling teachers to plan the class for developing children who learn mathematics by and for themselves. It focused on the innovation of elementary school mathematics based on the content which is well written in the textbooks in each country and known by teachers. The workshop done in Nigeria was based on the author's experience in Central and South America, South East Asia and Pacific as well as in Japan.

May 7, 2013

Masami Isoda, PhD

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Pictures of the English Edition of Japanese-Mathematics Textbook are extracted from **'Study with Your Friends MATHEMATICS for Elementary School** (Gakko-Tosho; 2005)'. When user extracts the pictures from the booklet, he/she needs the permission from Gakko-Tosho: Katsuaki Serizawa (e-mail: katsuaki.serizawa@gakuto.co.jp), GAKKO TOSHO CO., LTD. 3-10-36 Higashi-jujo Kita, Tokyo, 114-0001, Japan. **https://support.gakuto.co.jp/mathematics-textbook/**

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Further CV and Publications:

Japanese Full

http://www.trios.tsukuba.ac.jp/Profiles/0006/0000997/profile.html

English Part http://www.trios.tsukuba.ac.jp/Profiles/0006/0000997/prof_e.html

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Chapter 2: Dividing Fraction and Quantity Fraction

On the Fourth APEC-Tsukuba Lesson Study Conference, Gould. P.(2010), Inspector of NSW, Australia, lectured the roots of misconception of fraction in the case of Australian students with Australian textbook. He mentioned that the standard fraction notation itself encourages a 'count' interpretation of the regional 'parts of a whole' model (Figure 1). He quoted Hackenberg (2007) who asked the students to draw seven-fifths of a candy bar if the drawing of the rectangle on their paper represented a candy bar. Although both girls correctly created seven-fifths of the rectangle, when asked about the size of the pieces in the bars they had drawn, the girls maintained that the pieces were sevenths. Depending on his report, the major root of misconception is just counting of the parts without considering the whole. On his lecture, he also mentioned Japanese better approach.

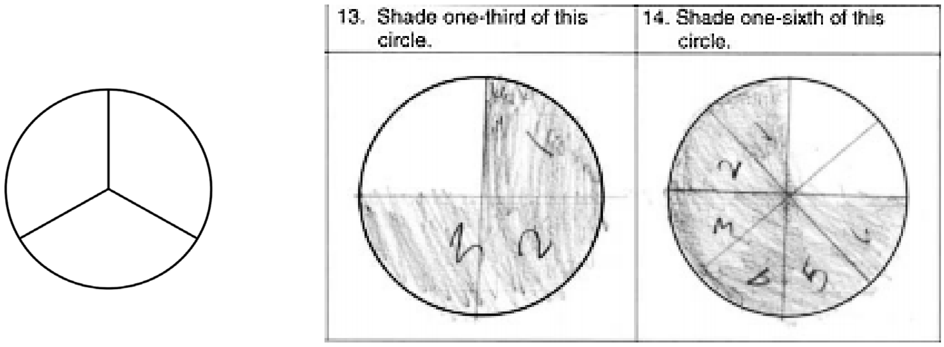


Figure 1 Circle Model and Answer of Grade 5 student (Gould, P. 2010)

On the Fifth APEC-Tsukuba Lesson Study Conference, Lewis, C. (2011) lectured about her research for professional development which compared three groups: Lesson Study with Japanese Resource for fractions, Lesson Study without it, and Ordinal professional development programs. In her lecture, she mentioned two points: Firstly, Japanese approach for introducing fractions is better approach for understanding the meaning of fractions. Her recommended Japanese Approach is shown on the Figure 3a &3b.

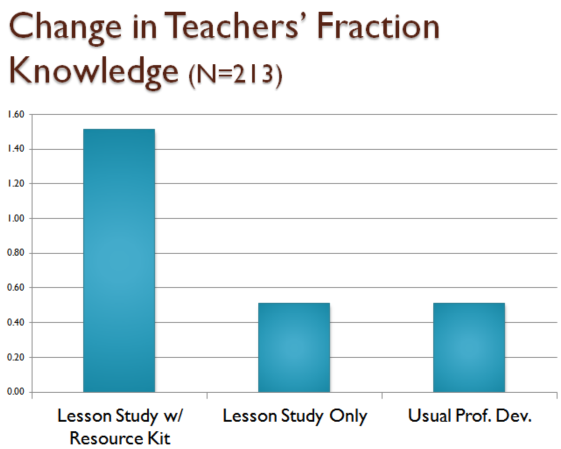


Figure 2 The effect of Lesson Study with Japanese Resource (Lewis, 2011)

14 Fractions

We cut a tape with a length that is equal to the length of the blackboard and measured it with a 1 m stick.

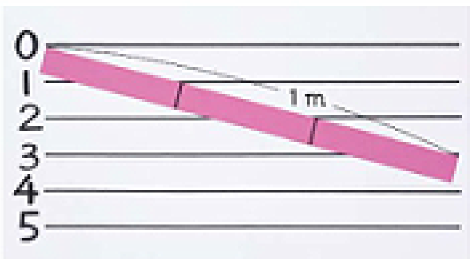
The length is 1 m and a remaining part. How many meters is the remaining part?

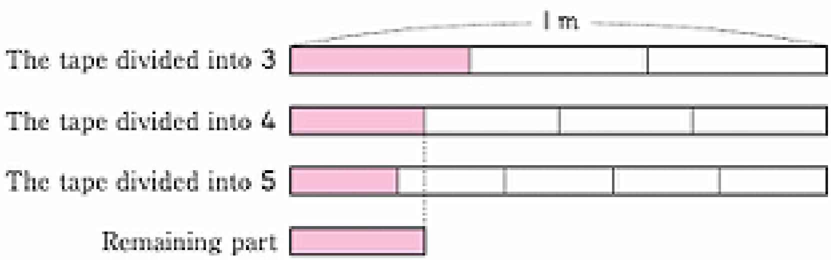
1 Fractions

1 Divide a 1 m tape into 3, 4 and 5 equal parts, respectively.

Compare each part with the remaining part.

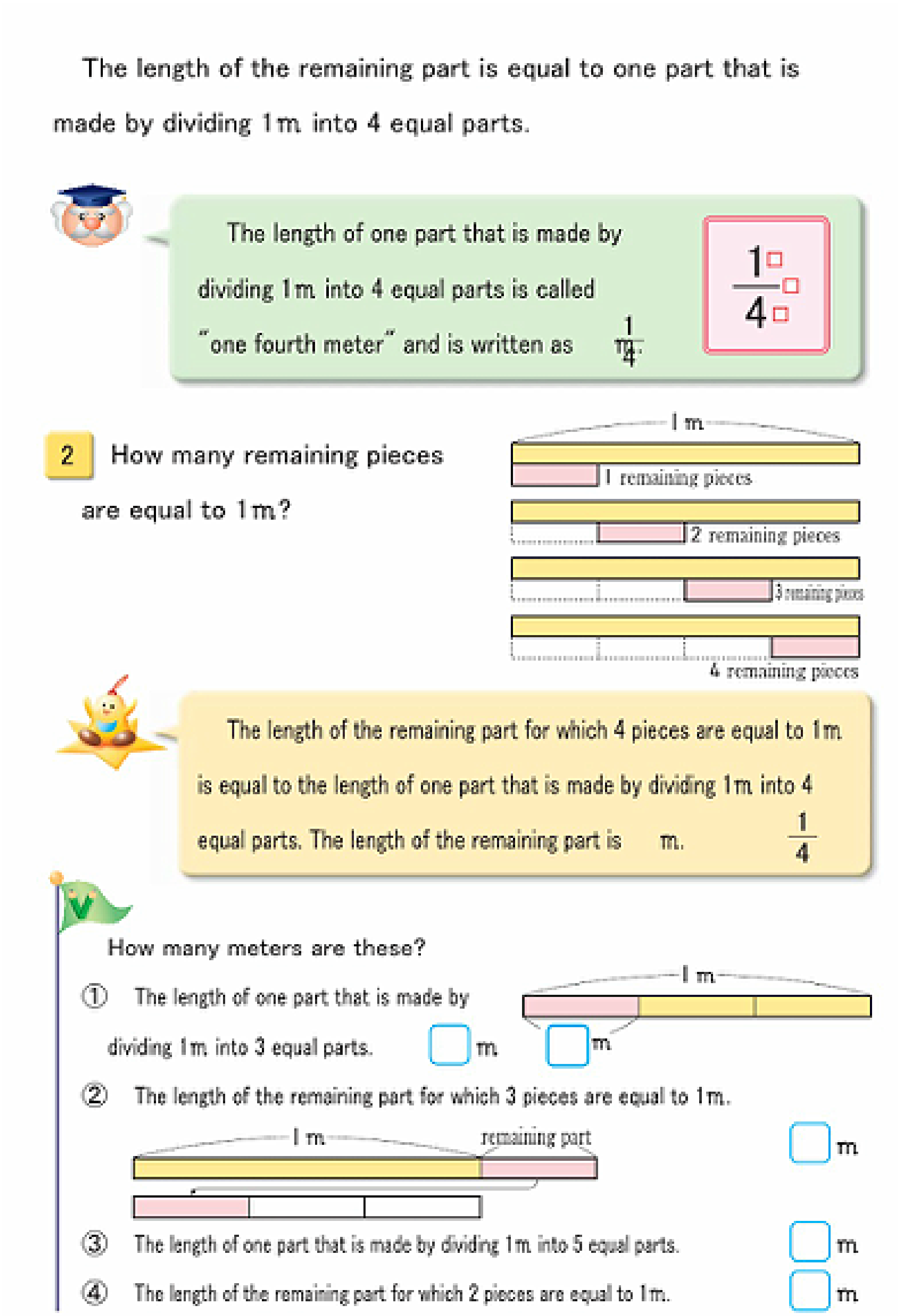
How to Divide a 1 m Tape into 3 Equal Parts





Learn new ways to express a length that is shorter than 1 m.

Figure 3a. Study with your friends: Mathematics for elementary school, Gakkotosho (2005, Grade 4, vol.2, pp.65-66; 2011, Grade 3, vol.2, pp88-89)



The length of the remaining part is equal to one part that is made by dividing 1m into 4 equal parts.

The length of one part that is made by dividing 1m into 4 equal parts is called "one fourth meter" and is written as $\frac{1}{4}$m.

2 How many remaining pieces are equal to 1m?

The length of the remaining part for which 4 pieces are equal to 1m is equal to the length of one part that is made by dividing 1m into 4 equal parts. The length of the remaining part is $\frac{1}{4}$ m.

How many meters are these?

① The length of one part that is made by dividing 1m into 3 equal parts. ☐ m

② The length of the remaining part for which 3 pieces are equal to 1m. ☐ m

③ The length of one part that is made by dividing 1m into 5 equal parts. ☐ m

④ The length of the remaining part for which 2 pieces are equal to 1m. ☐ m

Figure 3b. Study with your friends: Mathematics for elementary school, Gakkotosho (2005, Grade 4, vol.2, pp.65-66; 2011, Grade 3, vol.2, pp88-89)

Secondly, only the case using Japanese Resource for fractions has improved teachers' fraction knowledge, significantly.

Both lectures recommended Japanese approach. One of their recommended approaches is shown in Figure 3a and 3b which uses the remainder for measuring the unit length 1m.

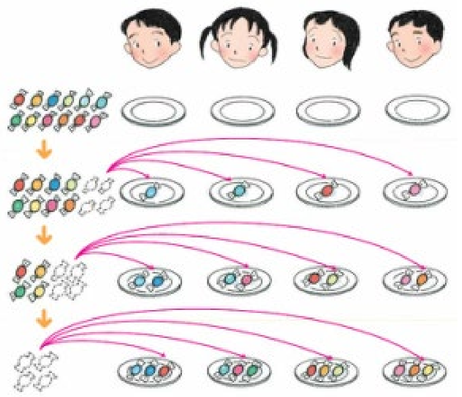
Questions for professional development 2

Q10. Let's analyze the ideas of Som and Ano in Chapter 1 from the view point of Gould, P. (2009).

Q11. Let's view Figure 3a and 3b. What is the common idea between Figure 3 and the suggestion of Professor in Chapter 1?

Q12. In division, there are two meanings as follows. Let's analyze the dividing activity in Figure 3a and 3b from the viewpoint of two different meanings.

Partitive division	Quotative Division
12 candies are divided by 4 children, equally. How many candies one child can receive?	12 candies are distributed by 4 candies for one child. How many children can receive candies?



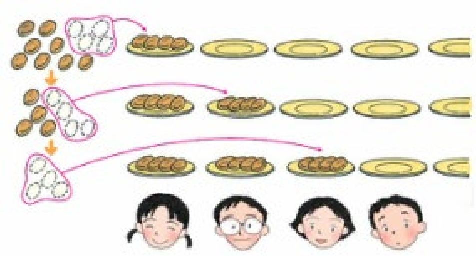


Figure 4. Gakko Tosho Grade 3 (Vol2. p4,p8, 2005; vol1, p60, p64, 2011)

Both activities for dividing are different for children however both activities include the repeated subtraction of the same amount from the total amount (Only see the left-hand-part of every picture, in Figure 4 left and right). Partitive division establishes the equally partition at first. Quotative division quotes the same amount recursively until we cannot quote, not sure the number of partitions at the beginning.

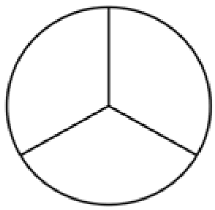
Q13. Please do the activity in the Figure 3 by yourself and explain the difference with your traditional approaches for teaching fraction based on Pizza model such as on the figure 1.

In mathematics, fraction n/m as for rational number is defined when n and m are integers (or rational number) and m is not equal 0.[1] If n or m are irrational or complex numbers, the fraction is not rational number. In mathematics, the concept is usually explained by the definition and the equivalent properties with exemplars. However, in school mathematics, especially for the elementary school, the meaning is explained using the different representations, model or situation with action (See Chapter 1, Q7,1). If we only teach number and calculation, children usually explain the ways of calculation as a rule (See Chapter 1, Q7, 3) when we ask them to explain 'Why?'. If readers are hoping children to explain the meaning, you have to assist children to use the further representations and models with situation. To meet these demands, Q12 on this chapter has asked you to engage in the same activity by yourself. When you use the 1m tape and you recognize the length of blackboard is 125cm, you can well prepare the activity on Figure 3.

Fraction is used in various contexts/situations with different meanings even if it is clearly defined mathematically. First part of this booklet explains how Japanese distinguish those meanings.[2] Second part explains how Japanese teach four operations of fraction.

Dividing Fraction

Many people believe that fraction is Dividing Fraction. It is the part-whole relationship which means that fraction n/m is dividing the whole into m equal parts and selecting n parts from them. It is deeply related with the activity of partitive division. 2/3 means the part-whole relationship which shows the whole is equally divided into three parts and selecting two parts. If the pizza is divided into three parts and take two parts, it means 2/3 of the pizza.



Paper folding, Origami, of a square paper includes the possible action for dividing the object equally, such as half and quarter. On this action, children usually think that dividing fraction cannot be larger than the one whole. However, as shown in Chapter 1 where people lost the denomination, children usually lose the process dividing the whole into equal parts. As Gould mentioned, the misconception for dividing fraction is originated from giving children worksheets to shade the object already equally divided through counting. On these worksheets, children are not needed to recognize the whole and to think how to divide it at first, because they just count the number of parts on an equally divided object. At the same time, on the ill-solution in Figure 1, it is a relationship to see the confusion of the fraction as ratio which allows to think the part-part relationship instead of the part-whole relationship described in Chapter 5. The task

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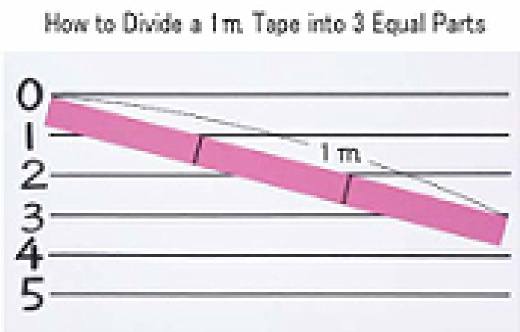
[1] Fraction, rational number, as a number system will be discussed at later chapters.

[2] Internationally, there are various technical terms to explain the meaning of fraction on situations such as Behr, M., et al. (1983) and Charalamous, C. Y & Pitta-Pantazi, D. (2007). However, it is un-usual to distinguish dividing fraction and operational fraction. Japanese technical terms are more precise to establish fraction in relation to what students learned on multiplication and division (see Isoda&Olfos, 2021, p.91).

itself expected to think part-whole relationship, thus, the major problem is related with their losing of whole for equally-dividing the whole on the context. For developing better understanding, we have to begin the activity to set a whole as one unit by children and ask them to divide it into the number of equal parts. After dividing into equal parts, children chose the necessary number of parts from the whole.

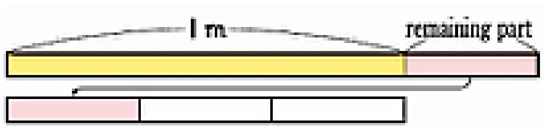
Any length of tape can be divided equally if we use the parallel lines with same interval. Necessity of this diagram is related with dividing the whole to the equal parts.



Exercise:

Draw 1/3 of Pizza on your notebook and compare the size of it with other's drawing.

Operational (measuring) Fraction



Operational Fraction (measuring fraction) can be seen as a kind of dividing fraction however it tries to measure the whole as a unit by the remaining part. From the viewpoint of division, it can be seen the activity of quotative division. In the following, three times of the remaining part is 1m. It means that 1/3 of 1m is the unit for counting however the whole, that should be divided, is 1m. Additionally, the original length is longer than the whole 1m. Mixed fractions and improper fractions have already existed even though children do not study them.

It is strange! We cannot always measure the 1m using the remaining part. What shall we do when we get another remaining when we measure the whole by the part? The operation does not work!

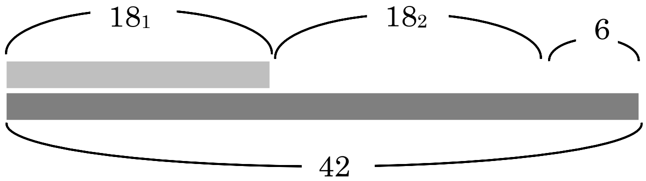
Yes, you usually give me the good question. Euclid (BC 3C) approved the way to find the unit (Greatest Common Divisor) for the measurement. It works with whole numbers (Integers) and called Euclidian Algorithm.

Euclidian Algorithm

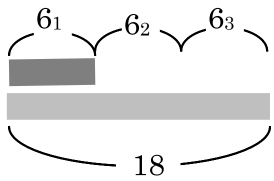
For example, Greatest Common Divisor of 18 and 42 is: 42=6x7, 18=3x6, 3 and 7 are prime numbers, then 6 is GCD of 18 and 42. It will be found using the remainders as follows.

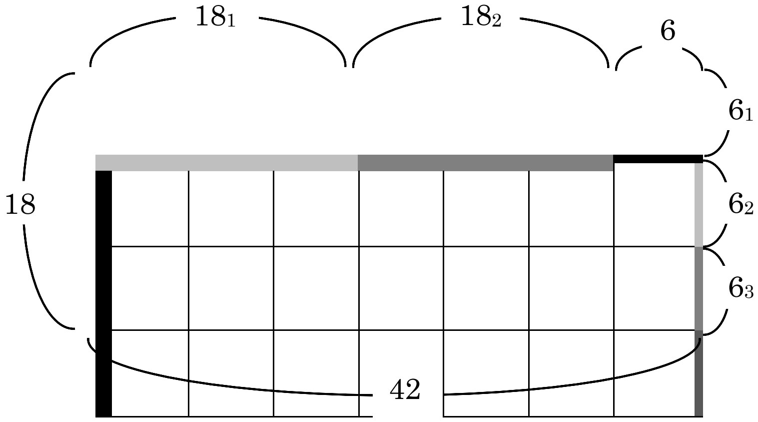
42=2x18+6 → 18=3x6

42 divided by 18, the remainder is 6.



18 divided by the remainder 6.





The rectangular diagram is the better way to demonstrate Euclidian Algorithm.

Exercise:

Let's find the GCD of (41, 18) by the rectangular diagram.

Quantity Fraction

Quantity fraction is the fraction with denomination. 1/3 of 1m is 1/3m in quantity fraction. The dialectic in Chapter 1 is done between quantity fraction (2/3m) and dividing fraction (2/3 of 2m). If we lose the denomination, we cannot distinguish quantity and dividing fraction like the communication in Chapter 1. It means that we cannot arrange the position of fraction on the number line if we do not have the denomination of the quantity. Quantity fraction allows us to compare the size of fraction in relation to the unit quantity. Dividing fractions are not easy to compare on the number line because the size of the whole unit is not clear and it looks always less than one.

In Figure 3a and 3b, all fractions are denominated with meter. All of them are quantity fraction based on the unit meter. In Japanese textbook, fraction is introduced by the quantity fraction with operational fraction and dividing fraction. It is better than the traditional approach introduced by dividing fraction for recognizing what is a unit (whole). Because it allows us to extend the fraction larger than the whole unit, and enables us to compare the size of fraction on the number line.

Unit Fraction for measuring unit

Unit fraction is the fraction in which numerator is one such as 1/3. Unit fraction is the unit for measuring up to improper fraction. Any fraction is represented as follows: (the specific number) x (the unit fraction). It is the necessary base to extend proper fraction to improper fraction. Thus, in fractions, there are two types of unit. Firstly, the whole is the unit. Secondly, unit fraction. The unit fraction is the unit for counting the numerator.

The idea of unit fraction on base ten place value system is related with decimals. For example, 1 mm as unit quantity for length is usually introduced as dividing 1 cm by 10, equally. The relationship between 1cm and 1 mm is a base to introduce decimals such as 10 mm is 1 cm and 1 mm is 0.1 cm. 1 cm and 1 mm scales are given on the ruler and tape measure, it is the base for number line which begins from 0 to +∞. On this context, operational fraction and unit fraction are the bases for extension of numbers.

In world known Japanese approach, fraction is didactically explained using various technical terms and introduced based on quantity and operational fraction, the remainder as measurement, for preparing the unit fraction (Figure 3b). If children understand the quantity fraction and the unit fraction, they can easily represent the fraction on the number line such as in Chapter 1, and easier to think four operations of fractions.

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