Fraction for Teachers

Knowing What before Planning How to Teach

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Preface

Education is the work to prepare for the future. Developing children who learn mathematics by and for themselves is one of the major issues on mathematics education reforms in the world (See such as Isoda & Katagiri, 2012). After the comparative study of mathematics classroom such as TIMSS video study in 90s, Japanese lesson study is the world-shared methodology as for the tools for professional development because the study indirectly demonstrated the quality of Japanese mathematics teaching and it is established by the lesson study. However, people often misunderstand the lesson study as for the talking about the class rather than studying subject matter. They enjoy the classroom observation likely listening to the music or watching the theatre. However, through listening to the music, and even if we enjoy talking about actors, we cannot prepare the good player ourselves. In Japanese lesson study, most efforts are done for the preparation of the class. The misunderstanding originated due to the limitation of the content guidebook to refer in English. On this reason, I have developed several resources which show the theory for the purpose to improve mathematics education with researches in the world.

For the workshop of SMASE-INSET project under Japan International Cooperation Agency (JICA), Japan and Federal Ministry of Education (FME), Nigeria, this booklet includes the essential theory for enabling teachers to plan the class for developing children who learn mathematics by and for themselves. It focused on the innovation of elementary school mathematics based on the content which is well written in the textbooks in each country and known by teachers. The workshop done in Nigeria was based on the author's experience in Central and South America, South East Asia and Pacific as well as in Japan.

May 7, 2013

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Further CV and Publications:

Japanese Full

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English Part
http://www.trios.tsukuba.ac.jp/Profiles/0006/0000997/prof_e.html

Table of Content

Chapter 5. Fraction in relation to ratio and proportion

In relation to the fraction as quotient, fraction is used for representing the ratio ' : ' as (Ratio) = (Comparing Quantity) ÷ (Base Quantity for the Unit of the Ratio). There are several contexts to use ratio. Japanese approach consistently uses Tape diagrams, Number lines, Area diagrams and Tables, as representations for thinking.

Firstly, ratio (value of ratio) is the result of division. There are two contexts in relation to the meaning of division. The following two types of dividing activity: partitive division and quotative division (Chapter 2, Q11).

Partitive Division	Quotative Division
12 candies are divided by 4 children, equally. How many candies one child can receive?	12 candies are distributed by 4 candies for one child. How many children can receive candies?
Ans. 3 candies for one child	Ans. 3 persons

When we see the underlined words in both tasks, the partitive division treats different quantity/denomination such as candies and children, and quotative division treats the same quantity/denomination.

On terminology, rate is the ratio for different quantities. Division by the different quantity (partitive division) is the rate which is unknown quantity represented by the quantity per the different quantity such as (speed (km/h)) = (distance (km)) ÷ (duration (hour)). One person runs 10km/h and another person runs 8km/h. Even if two people run together, they do not run at 18km/h. The quantity such as speed which cannot be added is called connotative quantity. It is usually given specific name such as speed.

Division by the same quantity (quotative division) establishes the ratio of the same quantity which produces the rate to show coefficient/multiple (how many times). If we carefully read the task for quotative division situation, it is not exactly the same quantity/denomination. Because, contextually, it can be represented by (number of children) = (12 (candies)) ÷ (4 (candies for one child)). On the quantity expression, it is $\frac{(Candies)}{(\frac{Candies}{a\ child})} = (Children)$. Both numbers 12 and 4 are number of candies for students. Thus, it is unnecessary to see them as different quantities.

There are countries that use the ratio symbol ' : ' as the for division symbol '÷'. It means that a : b is the same as a÷b and $\frac{a}{b}$. To distinguish the ratio of different quantities (usually with special name as for new quantity) and the ratio of the same quantity (just in case, a<b and it is the narrow usage of WARIAI in Japanese) is reasonable. On the context of ratio a : b, $\frac{a}{b}$ is called value of ratio.

Questions for professional development 5

The following tasks show the various context of ratio. Let's answer the following questions.

Q22.

1) Solve the following task. Provide the answer using percentage.

2) Explain the feature of the task using the following terms: Partitive division, Different Quantities.

3 Some students checked the number of passengers on airlines one day. Which plane is more crowded?

Number of Passengers and Seats

	Small plane	Large plane
Number of passengers	117	442
Number of seats	130	520

A number that is expressed by the quantity being compared when the basic quantity is made 1, like a shooting record or crowding, is called "a ratio."

Ratio = Quantity being compared ÷ Basic quantity

(Gakko Tosho, Grade 5, vol.2, pp57-58, 2005; pp85-86, 2011)

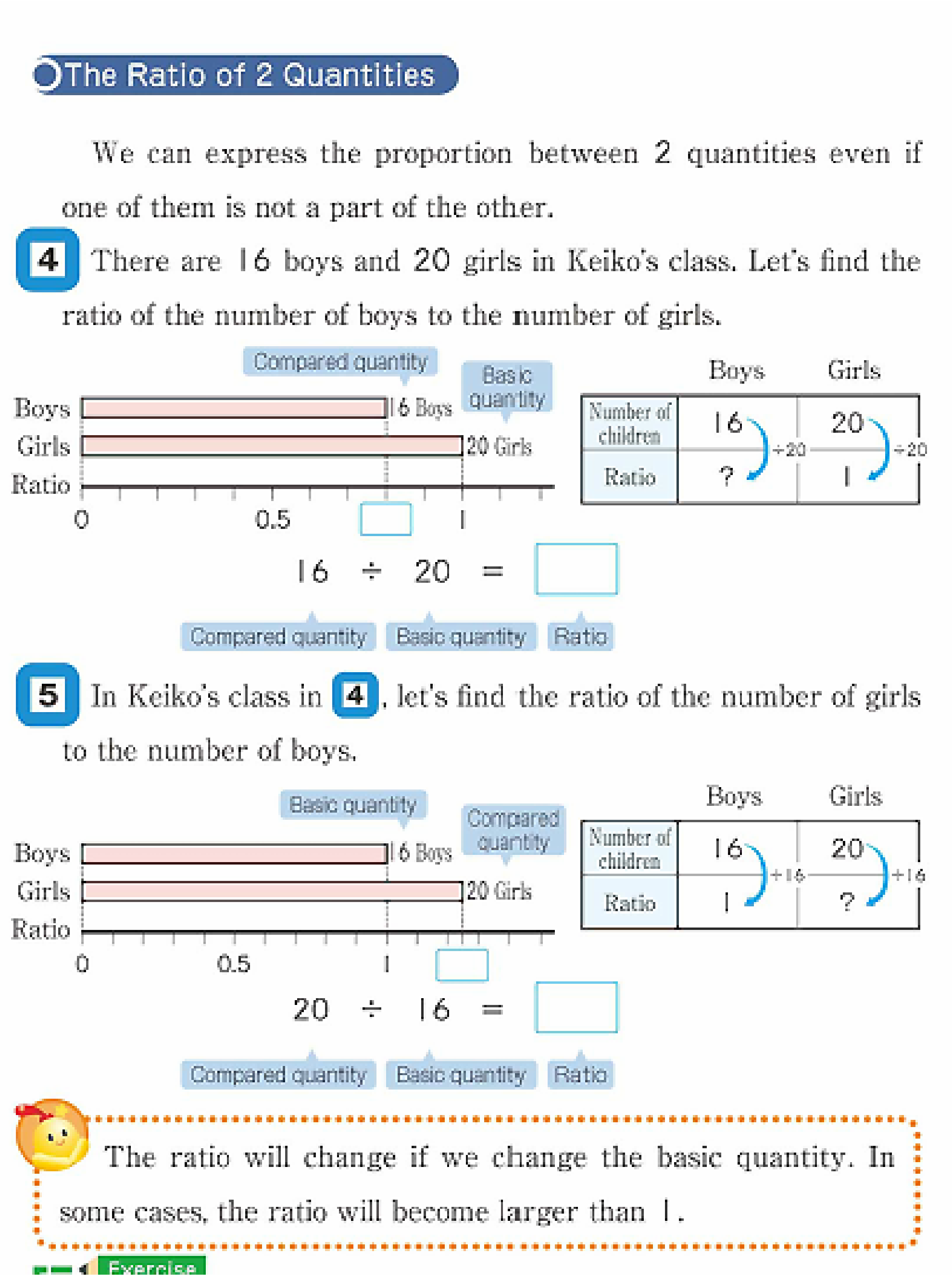
Q23.

1) Solve the following tasks and explain the base quantity for each task.

2) Explain the role of tape and number-line diagram.

3) Explain the situation using the quotative division and the same quantity.

4) $16 \div 20 = \frac{16}{20} = \frac{4}{5}$. Can we explain the task by the part-whole relationship?



The Ratio of 2 Quantities

We can express the proportion between 2 quantities even if one of them is not a part of the other.

4 There are 16 boys and 20 girls in Keiko's class. Let's find the ratio of the number of boys to the number of girls.

	Boys	Girls
Number of children	16	20
Ratio	?	1

16 ÷ 20 = ☐

Compared quantity / Basic quantity / Ratio

5 In Keiko's class in **4**, let's find the ratio of the number of girls to the number of boys.

	Boys	Girls
Number of children	16	20
Ratio	1	?

20 ÷ 16 = ☐

Compared quantity / Basic quantity / Ratio

The ratio will change if we change the basic quantity. In some cases, the ratio will become larger than 1.

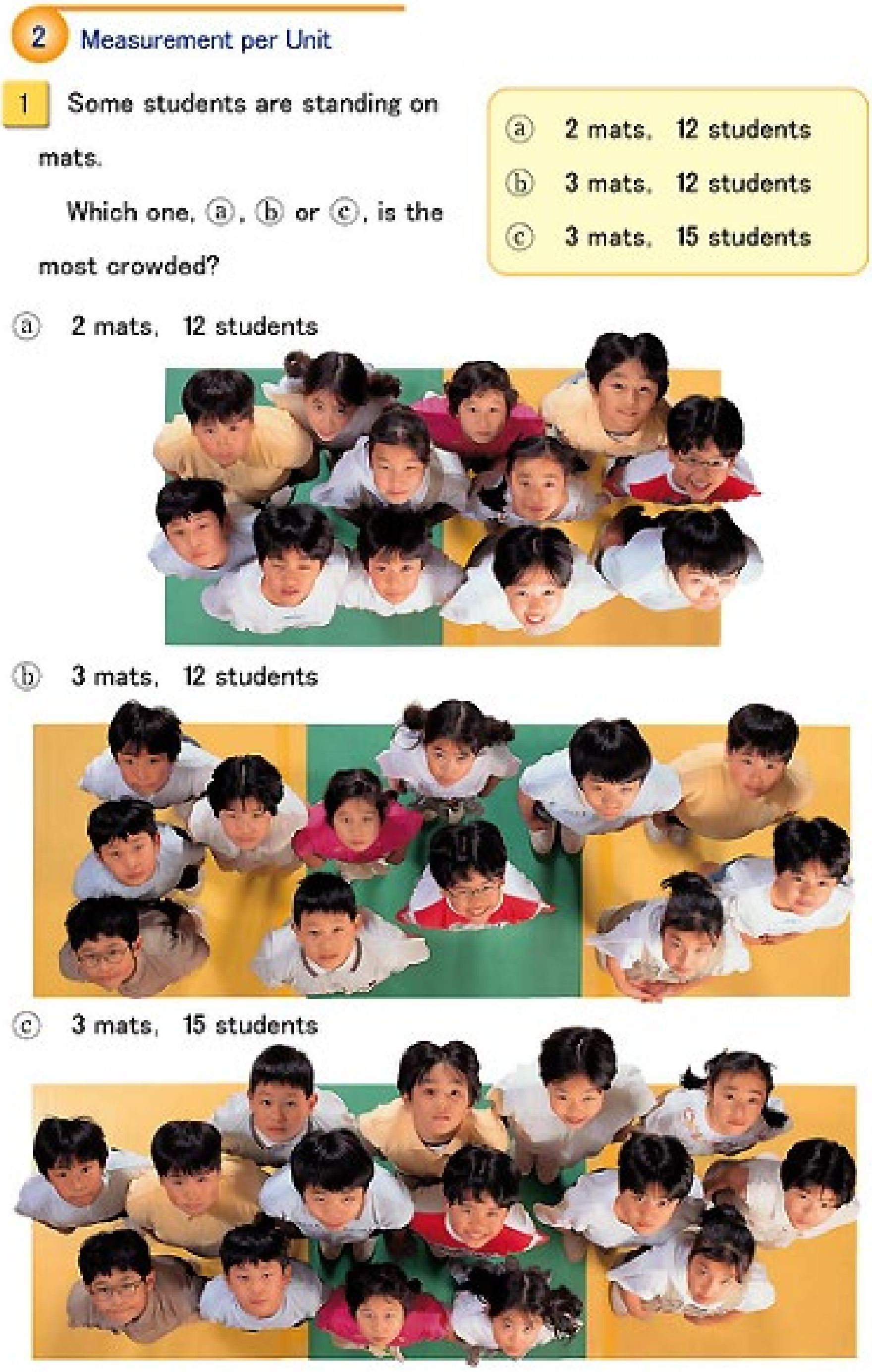
Exercise

(Gakko Tosho, Grade 5, vol.2, pp59, 2005; pp87, 2011)

Q24.

1) Solve the task using ratio.

2) If we consider the case of 6 mats for comparison, we do not need to use the idea of ratio.


2 Measurement per Unit

1 Some students are standing on mats.

Which one, ⓐ, ⓑ or ⓒ, is the most crowded?

ⓐ 2 mats, 12 students
ⓑ 3 mats, 12 students
ⓒ 3 mats, 15 students

ⓐ 2 mats, 12 students

ⓑ 3 mats, 12 students

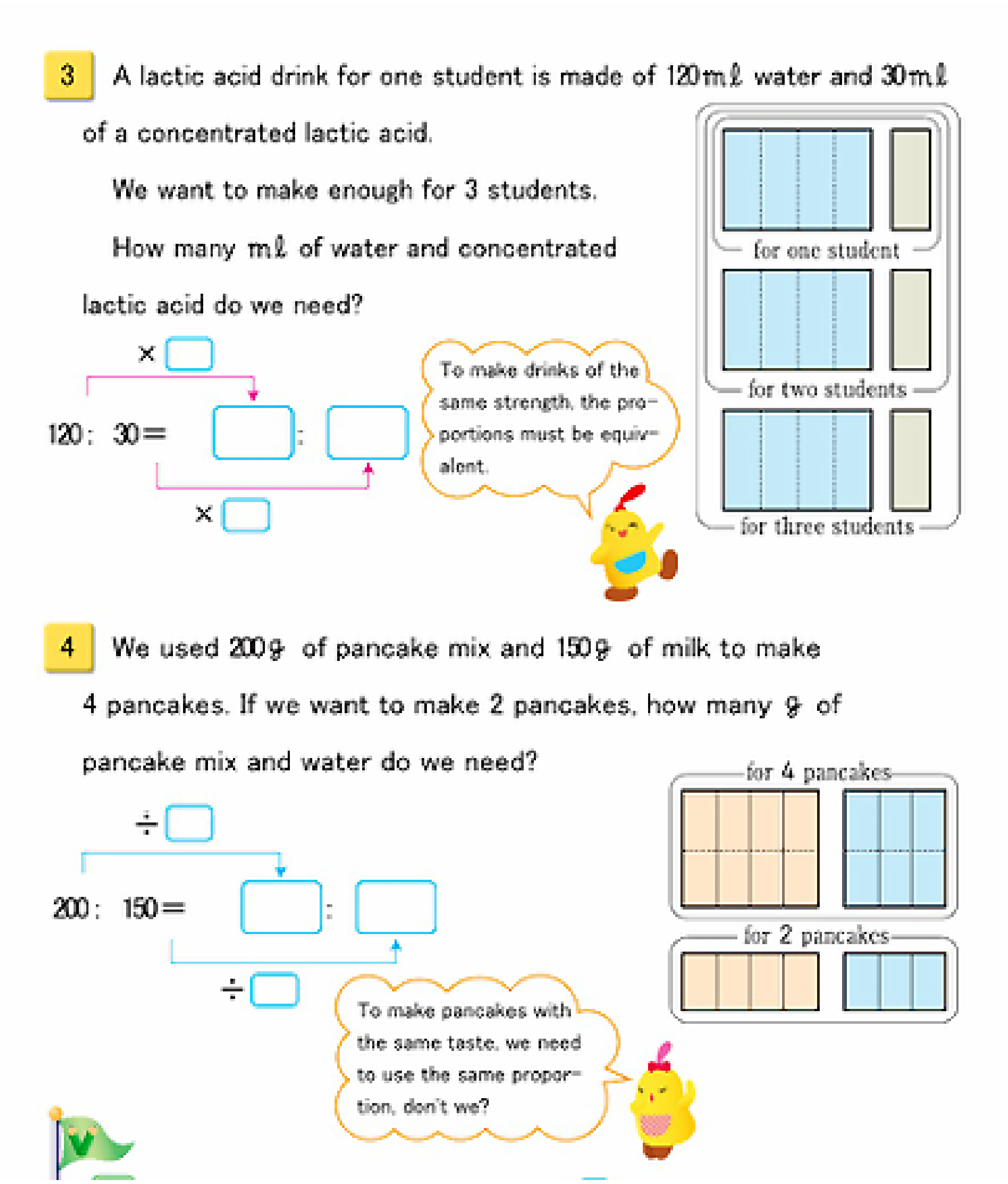
ⓒ 3 mats, 15 students

(Gakko Tosho, Grade 6, vol.1, p74; Grade 5, vol.1, p19)

Q25.

1) Let's solve the following tasks.

2) If you change the ratio to fraction, is it representing the part-whole relationship?



3 A lactic acid drink for one student is made of 120mℓ water and 30mℓ of a concentrated lactic acid.

We want to make enough for 3 students.

How many mℓ of water and concentrated lactic acid do we need?

120 : 30 = ☐ : ☐

4 We used 200g of pancake mix and 150g of milk to make 4 pancakes. If we want to make 2 pancakes, how many g of pancake mix and water do we need?

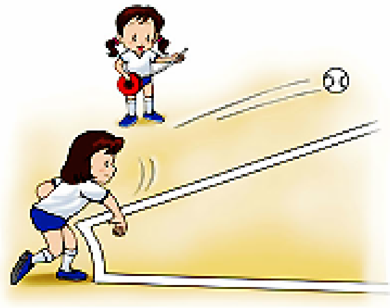
200 : 150 = ☐ : ☐

(Grade 6, vol2,p35,2005; p11, 2011)

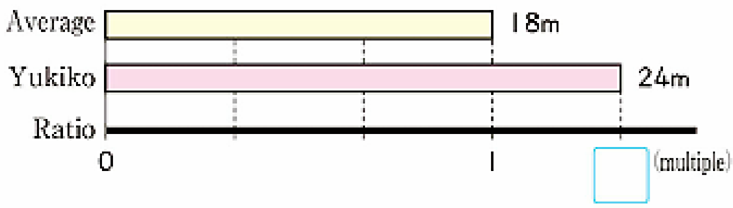
Q26. Compare the following task with Q23. Can we see the fraction by the part-whole relationship?

Ratio Represented by a Fraction

2 Yukiko and her friends played a game by comparing how far they could throw a softball. The average was 18m.



① Yukiko's record is 24m. How many times is her record to the average? Show it by a fraction.



$24 \div 18 = \square$

Compared quantity ÷ Basic quantity = Multiple

Suppose her record is x times the average.

Distance (m)	18	24
Ratio (multiple)	1	x

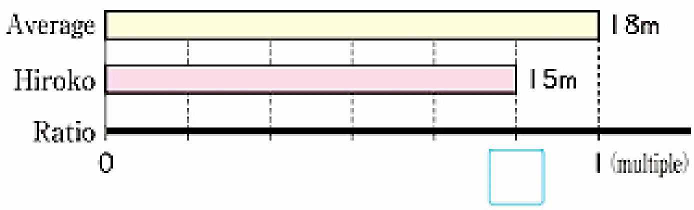
$\times x$

$18 \times x = 24$

$x = 24 \div 18$

Ratio is sometimes expressed as fractions.

② Hiroko's record was 15m. Her record is how many times the average?



$15 \div 18 = \square$

Compared quantity ÷ Basic quantity = Multiple

Suppose her record is x times the average.

Distance (m)	18	15
Ratio (multiple)	1	x

$\times x$

$18 \times x = 15$

$x = 15 \div 18$

Exercise

Let's fill the $\square$ with fractions.

① 15m is $\square$ times of 9 m. ② 35kg is $\square$ times of 42kg.

(Gakko Tosyo,Grade6, vol2,p27,2005; Grade 6, vol.1,p53,2011)

Fraction as Ratio

On the context of ratio a : b, $\frac{a}{b}$ is called value of ratio. However, in some countries, fraction is usually explained on the context of the part-whole relationship. Contextually, it is correct, however, in the previous chapter, the author already explained that this narrow understanding of fraction has developed huge misconceptions of fraction itself. Fraction $\frac{n}{m}$ ($m \neq 0$) in mathematics is a representation of rational numbers. It is the value of ratio n:m which cannot be defined by the part-whole relationship. For understanding fraction, we should know the various meanings and must use suitable representations for every meaning.

Indeed, on the context of usage on the ratio, we cannot generally explain the fraction by the part-whole relationship. For example, speed is the quantity such as km per hour. Distance is not the part of time duration. The ratio of boy and girl is 2:3. In fraction, it is $\frac{2}{3}$. It means that if there are three girls, the boys must be two: $\frac{2}{3} \times 3 = 2$. Here, the boy is not the part of the girl. The boy is the part of the whole human, not the girl. The ratio of boy and human 2:5 can be seen as a part-whole relationship.

Thus, through the learning sequence, children relearn all meanings or contexts of using fraction, and at the end of it. They can be re-presented/integrated by the idea of ratio.

Fraction as ratio is usually used in the context of multiplication. The ratio of boys and girls is 2:3. The value of ratio is $\frac{2}{3}$. $\frac{2}{3}$ is used for knowing the number of boys when the number of the girls is given: (the number of boys)= $\frac{2}{3}$ x (the number of girls). This is the proportion. On the equation to represent the proportion, the ratio is used for the coefficient. Until students well understand the proportionality[3], we cannot say that the dividing fraction is the fraction as ratio.

Let's think about the various representation of proportionality. Firstly, proportionality appeared from each row of the multiplication table. However, on the multiplication table, children do not clearly think co-variable. It is difficult to distinguish the co-variable with the term of multiples from the co-variable with the term of difference/increase at the introduction of rows because they are developing each row by adding multiplier based on accumulation.

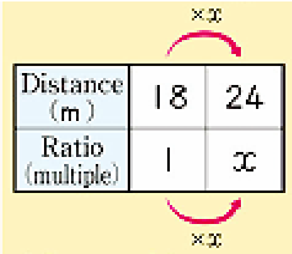
Exercise

If there is proportionality between the length of ribbon and its price, please fill in the ?.

Price (Cost)	80	?
Length of ribbon (m)	1	2.4

[3] Here, proportionality means properties to correspond proportion such as if x value become 2x, 3x, and so on, corresponding y value become 2y, 2y and so on.

Rule of Three: two by two matrix table



In the situation of the proportionality, if three numbers are given on the table, we can get the one in the remaining part of the table. Historically, this rule is called 'Rule of Three' which was the ancient's method for ratio and proportional reasoning. The idea itself is well known at the age of Egyptians (BC17C). It has existed before the invention of expressions and equations for algebra. Historically, the Rule of Three in 17th century is represented the arrangement of table with only four numbers.

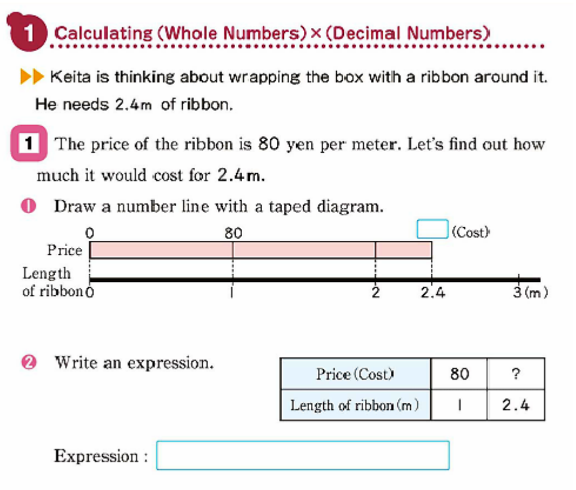
In school mathematics, the rule of three was taught just as a rule in relation to ratio and proportion. Later, the rule of three is represented by the three different expressions or formula, however we should know that the original rule of three itself was used without current expressions. It was used to be the number-arrangement on the paper to consider the ratio and proportion. Currently, we distinguish it by different formula of multiplication and division depending on the arithmetic situations. However, rule of three is based on the ratio and we do not need to care to distinguish the multiplication or division expressions depending on the situations. If we array numbers and consider the relationship with idea of ratio and proportion by using multiplication and division, it is enough. On this meaning, the person who explains the rule of three as the three different types of formulas is not using the rule of three but thinking based on current arithmetic representation oriented to algebra which should be represented by the different expressions. Likely using rule of three in ancient era, table and arrow representation for representing ratio must be easier for everyone and closer to the ancients approach because rule of three does not need to distinguish the situations by the three different types of expressions. On the algebraic representation of situation, if given quantities on ratio x to y, the value of x/y is ratio and y is the base. The formulas, x/y=a, x=ya and y=x/a, correspond to three types of situations. On the rule of three, if the situations are written on the two by two matrix table as one unknown on three known, we do need to distinguish the place of given numbers and multiples,un-necessary to memorize three formulas. If we use the rule of three meaningfully on ratio and proportion, we do not need to distinguish these three types of formulas.

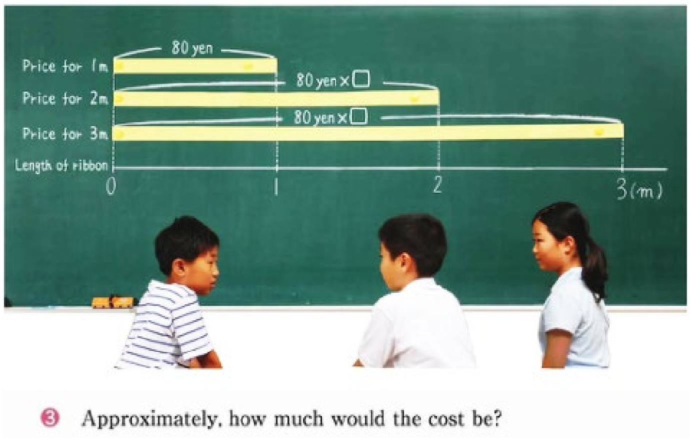
Tape and Number-Line diagram: Proportional number line

Proportionality or Rule of Three are represented on the tape and number-line diagram as well as table. Japanese has been using these representations since 1960s.

If the rule of three on the table is just used for the rule, relatively, the tape and number-line diagram are better to represent the meaning of proportionality because it works to represent co-variable with the images of the size of number on the line. Even if children do not know the term of proportion it supports to establish the way to develop expression and its calculation. We call it proportional number line. If we use the rule with tape diagram instead of table, it shows meaningful magnitude.

For explanation of meaning, we usually use various representations. Expressions can be derived from them. Proportional number line is also useful for studying fractions. The diagram of operational fraction with the remainder in Chapter 2, Figure 3, is also seen as a kind of proportional number line. Children draw table and diagram if their teacher clearly taught how to draw and use them. Teachers have to learn how to draw the proportional number lines.



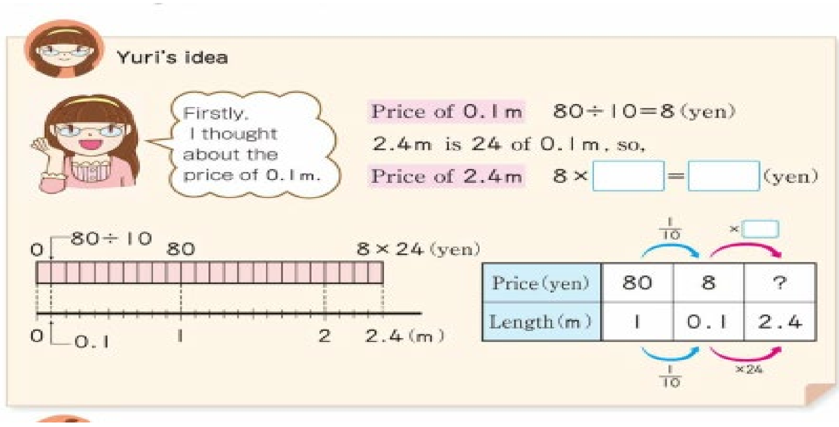


(Gakko Tosho Grade 5, Vol.1, pp29-30,2005;pp30-31,2011)

Exercise

Let's get the price for 2.4m by the proportional number line.

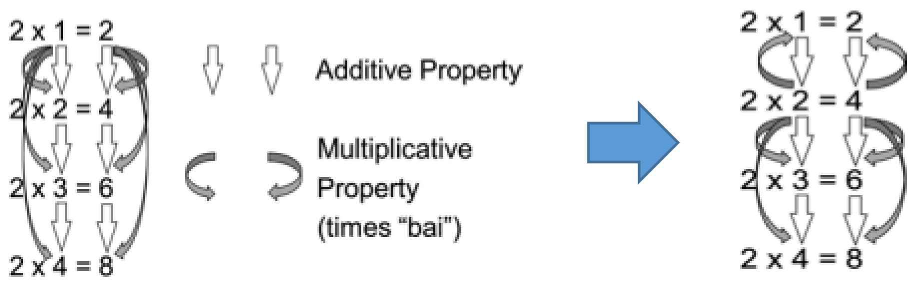
For knowing how and what, please explain the way to use the proportional number line by yourself.



(Gakko Tosho Grade 5, Vol.1, pp30, 2005;pp32,2011)

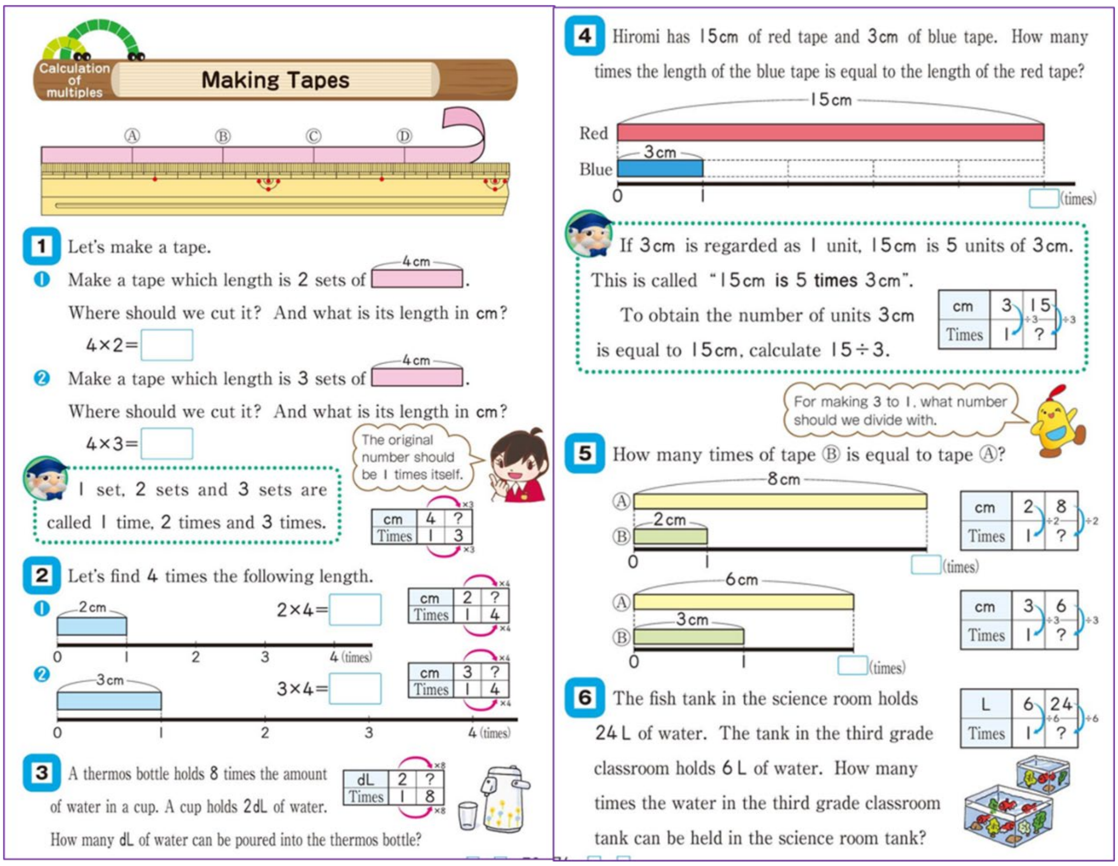
Multiplication as for the Origin of Proportionality

Origin of proportionality is multiplication table. At the introduction of multiplication table, every row can be seen by additive property and later, we re-see it for multiplicative property. It is the origin of proportionality.



(Isoda&Olfos, 2021. p.96)

For developing the proportionality, Gakko Tosho textbook (2021, Grade 3) introduces proportional number line as follows after students learned multiplication table.


Calculation of multiples

Making Tapes

1 Let's make a tape.

❶ Make a tape which length is 2 sets of 4 cm.

Where should we cut it? And what is its length in cm?

4×2=

❷ Make a tape which length is 3 sets of 4 cm.

Where should we cut it? And what is its length in cm?

4×3=

The original number should be 1 times itself.

1 set, 2 sets and 3 sets are called 1 time, 2 times and 3 times.

2 Let's find 4 times the following length.

❶ 2 cm 2×4=

❷ 3 cm 3×4=

3 A thermos bottle holds 8 times the amount of water in a cup. A cup holds 2 dL of water. How many dL of water can be poured into the thermos bottle?

4 Hiromi has 15 cm of red tape and 3 cm of blue tape. How many times the length of the blue tape is equal to the length of the red tape?

If 3 cm is regarded as 1 unit, 15 cm is 5 units of 3 cm. This is called "15 cm is 5 times 3 cm".

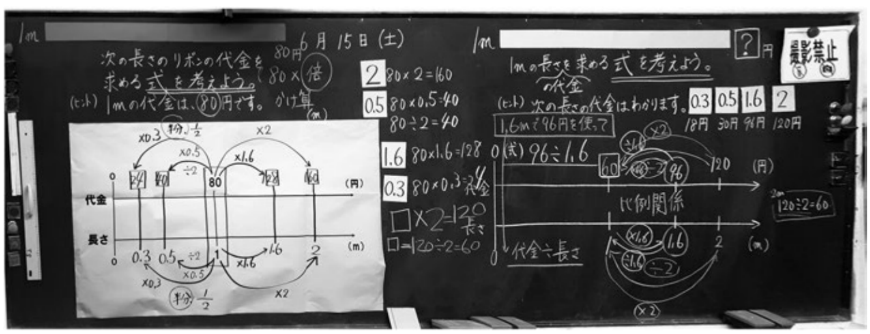
To obtain the number of units 3 cm is equal to 15 cm, calculate 15÷3.

For making 3 to 1, what number should we divide with.

5 How many times of tape Ⓑ is equal to tape Ⓐ?

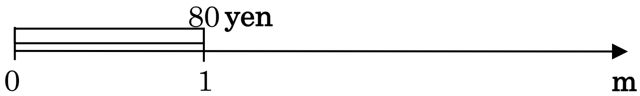
6 The fish tank in the science room holds 24 L of water. The tank in the third grade classroom holds 6 L of water. How many times the water in the third grade classroom tank can be held in the science room tank?

After students learned proportion in grade 5, it is drawn as follows (Isoda, Olfos 2021).



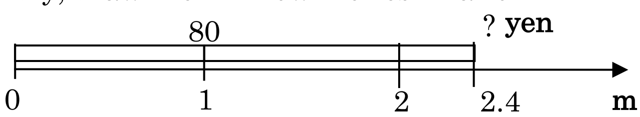
How to draw the proportional number line

Firstly, draw the amount for unit which is known:

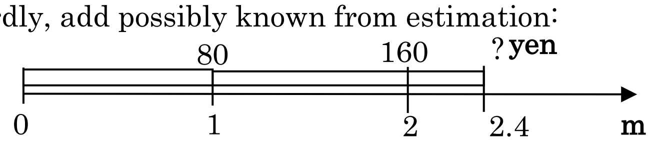


One of the objectives to draw the proportional number line is searching the way of calculation. Its procedure for drawing is like this. Arrows show the calculation based on the proportionality.

Secondly, draw the unknown for estimation:

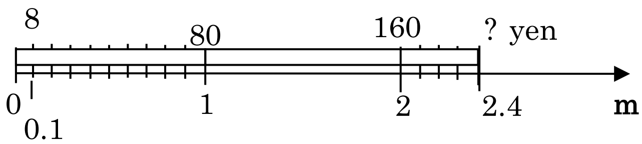


Thirdly, add possibly known from estimation:

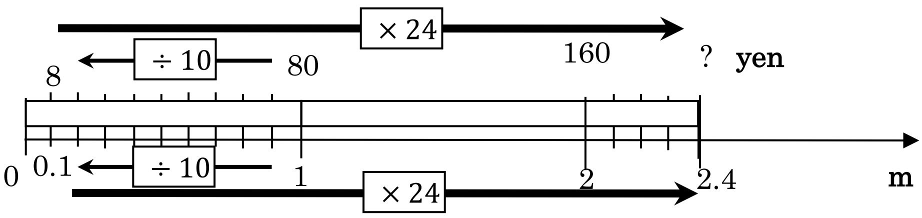


The price of 2.4m is not sure. If I **change the unit** 'm' to 'cm,' it is 80 yen for 100cm. How much in 240cm. Aha, 8 yen for 10cm: it means 0.1m!

If not clear, let's **change the unit for measuring the unknown**:



Finally, search the way of calculation under the proportionality:



80 × 2.4 =? The length corresponds the price. **The re-measuring process is represented by the arrows**. The arrow sequence represents 80 ÷ 10 × 24. Aha!!

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