Fraction for Teachers

Knowing What before Planning How to Teach

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Preface

Education is the work to prepare for the future. Developing children who learn mathematics by and for themselves is one of the major issues on mathematics education reforms in the world (See such as Isoda & Katagiri, 2012). After the comparative study of mathematics classroom such as TIMSS video study in 90s, Japanese lesson study is the world-shared methodology as for the tools for professional development because the study indirectly demonstrated the quality of Japanese mathematics teaching and it is established by the lesson study. However, people often misunderstand the lesson study as for the talking about the class rather than studying subject matter. They enjoy the classroom observation likely listening to the music or watching the theatre. However, through listening to the music, and even if we enjoy talking about actors, we cannot prepare the good player ourselves. In Japanese lesson study, most efforts are done for the preparation of the class. The misunderstanding originated due to the limitation of the content guidebook to refer in English. On this reason, I have developed several resources which show the theory for the purpose to improve mathematics education with researches in the world.

For the workshop of SMASE-INSET project under Japan International Cooperation Agency (JICA), Japan and Federal Ministry of Education (FME), Nigeria, this booklet includes the essential theory for enabling teachers to plan the class for developing children who learn mathematics by and for themselves. It focused on the innovation of elementary school mathematics based on the content which is well written in the textbooks in each country and known by teachers. The workshop done in Nigeria was based on the author's experience in Central and South America, South East Asia and Pacific as well as in Japan.

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Japanese Full

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http://www.trios.tsukuba.ac.jp/Profiles/0006/0000997/prof_e.html

Table of Content

Chapter 6. Multiplication and Division of Fraction (1)

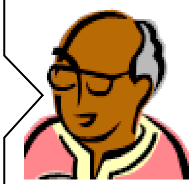
For developing the multiplication and division of fractions, this chapter mainly explains the necessary knowledge needed to produce the idea for it.

Why did Professor recommend us to use the proportional number line?

To explain the meaning of calculation, we must change the representation which explains the process in a visible and understandable way.

If we do not use it, we only teach skill as a rule. Where shall we use it to explain the meaning? We have to teach how to draw.

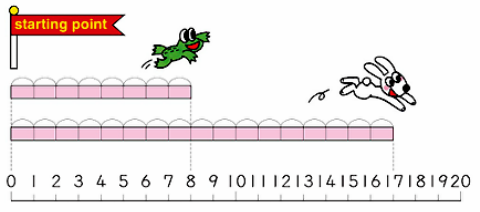
The idea of proportional number line was originated from Rene Descartes (1637). Japanese Math-Educators such as Takeshi Ito invented it to establish the Heuristic Teaching Approach for elementary school mathematics with the extension and integration curriculum sequence in the 1960s. From the 1990s, it became the world famous approach as for the representations to develop the competency for proportional reasoning. Japanese textbooks such as Gakko Tosho established well teaching sequence for developing it.



Number line

In chapter 1 and 2, we recognize that shading activities, the parts for counting, is the root for the misconceptions of fractions for missing the idea of whole as for a unit. For developing children who learn mathematics for and by themselves (see Chapter 1), children are necessary to draw and use appropriate diagrams for explaining their ideas. On the previous chapters, appropriate diagrams need to show original unit with quantity fraction and the unit fraction for measuring its number, like measuring by using the remainder as for operational fraction. The tape and number line with quantities are appropriate diagrams on this condition. For developing children who will draw such a diagram, firstly, we have to develop children who draw the number line by and for themselves.

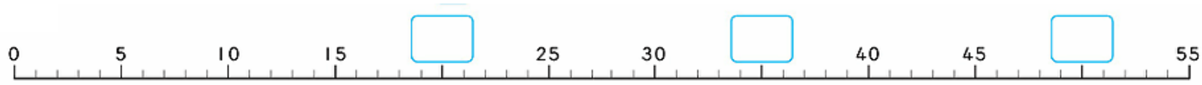
Number line which shows the position of number on the line is introduced by taking the same intervals by the unit of measurement recursively for comparing the size of number. At this moment, it looks like a line of discrete numbers because it is given by **interval** as for the scale on the line and there is only one interval but no number between two numbers. On the process of extension of numbers, when students re-scale it by using smaller units or larger units, it begins to function as number lines which shows the position of various numbers and used for extension of numbers. At the beginning, children learn the '0' is the starting point on the line instead of 'nothing.' On the number line, 0 shows the origin of **position** as for measuring by the interval (unit). The difference of positions shows the distance (the number of intervals: cardinal number).



If teacher does not teach the measuring by the unit from the starting point 0, children may confuse one on counting intervals as scale number 0 on the left instead of the scale number 1 on the right. Children learn the number line as for comparing the size and ordering of numbers, and recognize the number on the base ten system. Taking interval is the preparations for the multiplication and division, too.

5 Write the correct numbers in the ☐.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14		16	17	18	19
20	21		23		25	26		28	
30	31	32		34	35		37	38	39
	41	42		44		46	47		
50	51		53		55		57	58	59



If your children do not know how to draw the number line, let's give them the opportunity to draw it by themselves. It is the activity of measurement by using arbitrary unit.

Multiplication

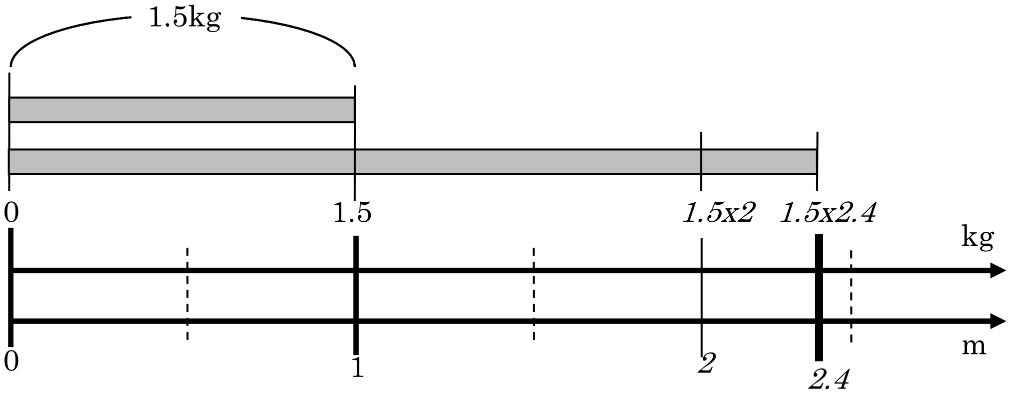
(Gakko Tosho, Grade 2, vol.2 pp2-3, 2005; pp6-7,2011)

In the case for lower elementary school mathematics, children study how to use daily language mathematically on the situation. Definition of arithmetic operations is usually done on the situation in daily context because we have to develop children to use four arithmetic operations on their daily life.

On this issue, multiplication is introduced at the following situations: the number of dishes and the number of objects for each dish.

From the viewpoint of measurement, the situation is used to explain multiplication that the multiplication is measuring the amount of the quantity by the unit of quantity when the unit for the amount is known by the quantity (Ministry of Education, Japan, 1960; Freudenthal, 1983). For example, when the amount is 8 dishes and the unit for the amount is 2 cakes in each dish, the measured amount by the quantity is 8 x 2 =16 (cakes).

This definition works for multiplication of decimal numbers and fractions as well as the situation of repeated addition. For example, when the amount of steel is 2.4 m and the unit weight for the steel is 1.5 kg for one meter, the measured amount by the quantity is 1.5 x 2.4 = 3.6 (kg)[4]:



In relation to how to calculate, multiplication is explained with the repeated addition, however, multiplication of decimal numbers and fractions is not explained by the repeated addition but explained by multiplication table with distribution law on base ten place value system. For explaining the multiplication of decimal numbers and fractions, we can use the proportional number line which represents the meaning of multiplication by measuring.

How to find the expression from the situated problem

At the end of last chapter, you learned how to draw the proportional number line by the task below:

[4] This textbook is using Japanese notation: 1.5 (kg/m) x 2.4 (m) = 3.6 (kg). In English notation, it should be 2.4 (m) x 1.5 (kg/m) when '2.4 x 1.5' is read as '2.4 times 1.5'. In English, '2.4 times' implicates 'multiplied by 2.4'. Thus, as long as you read '1.5 x 2.4' as '1.5 multiplied by 2.4,' Japanese notation of multiplication is understandable. Indeed, 'a x b' can be read as 'a multiplied by b' even in English. English usage has inconsistency.

1 The price of the ribbon is 80 yen per meter. Let's find out how much it would cost for $\square$ m.

The box (in blank) is 2.4 at the end of last chapter. The children who have not yet learned the multiplication of decimal numbers and fractions cannot easily recognize that this task is multiplication. On the other hand, if we put the whole number such as 2 into the box, children who learned the multiplication of whole numbers could easily understand that this task is multiplication because multiplication is introduced in daily situations on the whole numbers.

On the context of extension of numbers and operations, Japanese teachers usually prefer this problem posing strategy like this form and ask children to put any number they want into the box and discuss how. Through putting into a simple number, children recognize this task as multiplication and in the case of whole numbers, they already learned, and in the case of fraction and decimal numbers, they did not yet learn. When the class begins this way, children recognize this task as the task for multiplication and they would like to inquire how to find the answer using what they already learned.

Exercise

Let's draw the proportional number line when the box (blank) is 2 (m), 2.3 (m) or $\frac{3}{2}$ (m) on $\boxed{1}$.

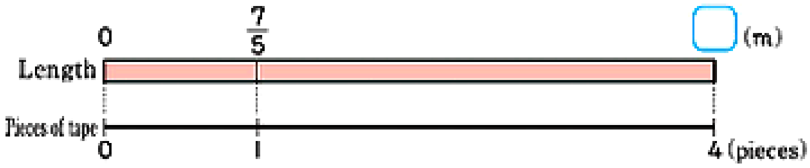
In this exercise, for answering in the case of 2.3 m, we have to change the unit from 1m to 0.1m as well as the case of 2.4m. In the case of fraction, we usually change the unit from 1m to the unit fraction: this case $\frac{1}{2}$ m is the unit for measuring. If 1m is 80 yen, $\frac{1}{2}$ m is $80 \div 2$ yen. If $\frac{1}{2}$ m is $80 \div 2$ yen, $\frac{3}{2}$ m is $80 \div 2 \times 3$ yen. For considering like this, children need to draw a proportional number line and apply multiplication and division on the number line.

Fraction × Whole Number

Please explain the following:

4 We want to make 4 pieces of tape that are each $\frac{7}{5}$ m long.

How much tape do we need?



If 1 piece is $\frac{7}{5}$ m and addition of fractions is known, 4 pieces are $\frac{7}{5} \times 4 = \frac{7}{5} + \frac{7}{5} + \frac{7}{5} + \frac{7}{5} = \frac{28}{5}$. After the explanation, we should ask children as follows: Is it possible to find an easier or faster way? Then, $\frac{7\times4}{5}$ is recognized as a simple way for $\frac{7}{5} \times 4$.

In mathematics, we usually produce shorter and simple ways. Seeking simplicity is a basic value of mathematics.

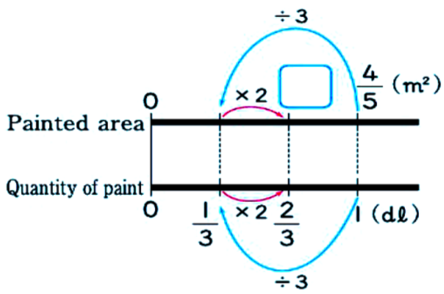
Exercise

In Gakko Tosho textbooks, the proportional number line changes from the tape diagram and number line to two number lines at Grade 5. Let's put the number in the box and answer: "We can cover an area of $\frac{4}{5}$ m² with 1dl paint. How many m² can we cover with

$\square$ dl of the paint?"

Fraction × Fraction

If the box is a whole number, we already learned. If the box is a fraction such as $\frac{2}{3}$ dl, we can draw the proportional number line:



If we develop the way of calculation as well as multiplication of decimal numbers, it can be calculated as follows: $\frac{4}{5} \div 3 \times 2$.

After the explanation, we should ask children as follows: Is it possible to find an easier or faster way? Then, $\frac{4}{5} \div 3 \times 2 = \frac{4\times2}{5\times3}$ is recognized as the simple way for $\frac{4}{5} \times \frac{2}{3}$.

In mathematics, we usually produce shorter and simple ways.

Area Diagram

In school mathematics, area diagram is usually recommended for use in explaining (a+b)(c+d)=ac+ad+bc+bd. On the area diagram, multiplication is a two-dimensional idea and it functions as for the model of commutativity. Some people strongly believe that the area diagram is the best way for explaining multiplication and division because it provides the wall painting/shading metaphor based on two dimensions. The misconception will appear if students do not feel the necessity to draw the same size diagram.

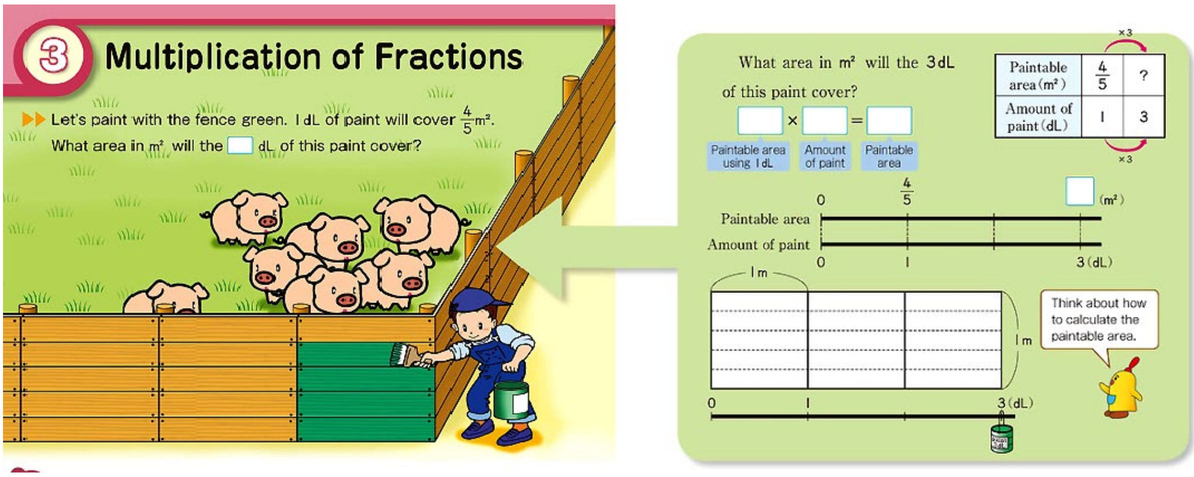
As long as teachers try to explain fraction as dividing fraction it might be true, however, shading activities of the area diagram itself is a major source of the misconceptions if children recognize fraction only by dividing fraction (see Chapter 2).

What is necessary to develop in students by and for themselves is that students are able to draw the area diagram by and themselves as for the tool for reasoning as well as the proportional number lines.

Historically, Euclid produced the theory under the dimension and Descartes overcame the wall between dimensions by the proportional number line which defined multiplication by the measurements.

Exercise

1) Let's solve the following task by three different methods.



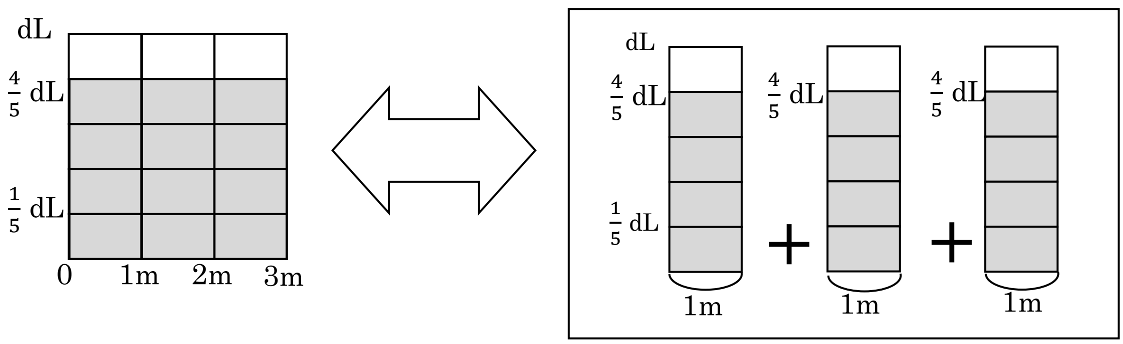
1) Let's compare three methods. Which one do you recommend? Why do we need more?

2) There are a number of students who get the answer using the area diagram such as $\frac{4}{5} \times 3 = \frac{12}{15}$. Why does the area diagram produce such answer?

There are two types of area diagram. The first type is already shown in the textbook. It shows the area of the wall itself for showing the painting. Another type of area diagram represents the following situation:

There is a wall in which we use $\frac{4}{5}$ dL of paint for painting 1 meter of wall. How much liters do we need for painting 3 meters?

For this task, we draw the following two diagrams for the same meaning.



In both diagrams, the denominations of quantities are necessary because children might develop misunderstanding such as $\frac{4}{5} \times 3 = \frac{12}{15}$ if there are no denominations.

Area diagram is fine to explain meaning. However, the children who still keep the dividing fraction, misunderstand the meaning of whole. Indeed, the left hand side of the above diagram can be read as $\frac{4}{5}$ if the square is the whole even though it shows quantity. Here, the key is a unit fraction $\frac{1}{5}$ dL (quantity fraction!) as well as the whole 1 L. If teachers ask students just shading without considering these two units, we are not sure students understand well or not, even though teachers felt success to explain for him/herself.

Major Reference and Further Readings 6

Masami Isoda et al. (2010). Mathematics Education Theories for Lesson Study: Problem Solving Approach and the Curriculum through Extension and Integration, Journal of Japan Society of Mathematical Education, 92(11). 1-158.

Shin Hitotsumatsu et al. (2005). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda et al. (2010). Study with your friends: Mathematics for Elementary School (12 vols). Tokyo: Gakkotosyo.

Masami Isoda (2012). Introductory Chapter: Problem Solving Approach to Develop Mathematical Thinking. *In Isoda, M. & Katagiri, S. edited (2012). Mathematical Thinking: How to Develop it in the Classroom.* Singapore: World Scientific. 1-28.

https://www.criced.tsukuba.ac.jp/math/apec/ICME12/Lesson_Study_set/MATHEMATICAL%20THINKING%20(c)World%20Scientific/MathematicalThinking_chap01(c)WorldScientific.pdf

https://doi.org/10.1142/9789814350853_0001

Masami Isoda, Tenoch Esaú Cedillo Ávalos, et al. (2012). Matemáticas para la educación normal (11vols). Estado de México : Pearson Educación de México.

Tenoch Cedillo, Masami Isoda, et al. (2013) Matemáticas para la educación normal : guía para el aprendizaje y enseñanza de la aritmética, Estado de México : Pearson Educación de México.

Masami Isoda (2015). The Science of Lesson Study in the Problem Solving Approach. *In Inprasitha, M. et.al. edited. Lesson Study: Challenges in Mathematics Education.* Singapore: World Scientific. pp.81-108. https://doi.org/10.1142/9789812835420_0006

Masami Isoda (2015). Dialectic on the Problem Solving Approach: Illustrating Hermeneutics as the Ground Theory for Lesson Study in Mathematics Education. *In Sung, J. edit. Selected Regular Lectures from the 12th International Congress on Mathematical Education*. Cham: Springer. pp.355-381. https://doi.org/10.1007/978-3-319-17187-6_21

Masami Isoda & Aki Murata edited (2011). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.

Masami Isoda, Aki Murata, Aida Istino Yap edited (2015). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.

Masami Isoda & Aki Murata edited (2021). *Study with your friends: Mathematics for Elementary School (12 vols*). Tokyo: Gakkotosyo.
https://support.gakuto.co.jp/mathematics-textbook/

Masami Isoda, Raimundo Olfos. Edited. (2021). *Teaching Multiplication with Lesson Study.* Cham: Springer. DOI: 10.1007/978-3-030-28561-6