Mathematics Education to Develop Students Agency
Part III: Summary
Mathematical Thinking and Activities

ISODA Masami, Prof/PhD
Director of CRICED
University of Tsukuba, Japan

Instead of just saying Higher Order Thinking Skills in General and Authentic Mathematics, we preferred to use Terminology to design mathematics' lessons and task sequence. It is terms to explain mathematical thinking (invariant), at first and conceptual differences (variant), at second.

Can you explain what is higher order thinking in mathematics?
Higher order thinking is explained by Mathematical Thinking

For example:

- Instead of just saying Critical Thinking, we discussed dialectic approaches as a part of problem-solving approaches which explains how to design arguments in classroom by using students' misconception on the manner 'if your saying is true, what will happen?'
  See Part II, Lesson 1 & 3, and Part III, Lesson 4
- Discuss possibility of generality by using Counter example, by changing Representations, by posing the questions with other cases.

2

Instead of just saying Higher Order Thinking Skills in General and Authentic Mathematics, we preferred to use Terminology to design mathematics' lessons and task sequence. It is terms to explain mathematical thinking (invariant), at first and conceptual differences (variant), at second.

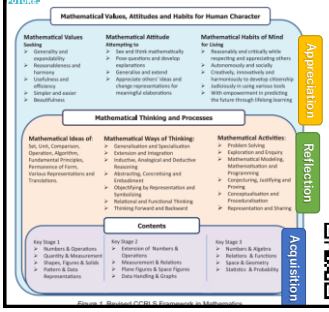
Can you explain what is Authentic Mathematics in mathematics?
Authentic Mathematics is explained by Problem Solving Approaches on the appropriate task sequence which uses what students already learned.

Example 2:

- Instead of just saying authentic mathematics,
  - We discussed the nature of mathematics curriculum, not the nature of pure mathematics, which includes huge numbers of inconsistency/contradictions.
    See Part III, Lesson 1 & 4
  - We discussed the problematic appeared from children on Problem Solving Approach on the task sequence which are explained by second terminology.

3

On SEA-BES: CCRLS in Mathematics



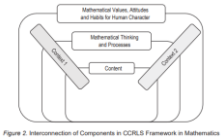


Figure 2. Interconnection of Components in CCRLS Framework in Mathematics

Context 1: Real World Problem Solving
Context 2: Problem Solving on the appropriate task sequence

4

What is Authentic Math on SEA-BES: SSRLS?

Nature of Mathematics

Mathematics has been recognized as a necessary literacy for citizenship and not only living economically but also to establish a society with fruitful arguments and creations for better living. It has been taught as a basic language for all academic subjects using visual and logical-symbolic representations. In this information society, mathematics has increased its role to establish 21st century skills through reviewing mathematics as the science of patterns for future prediction and designing with big data which produces innovation not only for technology advancement but also for business model.

- Mathematics is unique common language in the world

Mathematics is an essential subject to establish common reasoning for sustainable development of society through viable argument in understanding each other and develop critical reasoning as the habits of mind. Mathematics should be learned as basis for all subjects. For clarifying the framework in CCRLS on mathematics and by knowing the role of mathematics education, the humanistic and philosophical natures of mathematics are confirmed as follows:

- Mathematics is unique subject for understanding others through Re-presenting.

The Humanistic nature of mathematics is explained by the attitudes of competitiveness and understanding of others by challenging mathematicians such as Blaise Pascal, Rene Descartes, Isaac Newton and Gottfried Wilhelm Leibniz. For example, if you read the letter from Pascal to Pierre de Fermat, you recognise the competitive attitude of Blaise Pascal to Fermat's intelligence and seek the way to be understood on the excellence of his finding on Pascal's Triangles. If we read Pascal's **Pensées,** you recognise how Pascal denied Descartes geometry using algebra from the aspect of ancient Greek geometry. On the other hand, Descartes tried to overcome the difficulties of ancient geometry by algebra. If you read the letter from Descartes to Elisabeth, you recognise how Descartes appreciated and felt happy the Royal Highness Elisabeth used his ideas of algebra in geometry. Despite being a princess, Elisabeth had been continuously learning mathematics in her life.

- Mathematical activity is based on human activity with competitiveness and understanding others

5

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The Philosophical nature of mathematics can be explained from ontological and epistemological perspectives. From the ontological perspective, mathematics can be seen as a subject for universal understanding and common scientific language. Plato and Aristotle are usually compared from this perspective. Plato believes that the existence of the world of "idea" and mathematics existed in the world of "idea" on Platonism. In this context, mathematical creation is usually explained by the word "discover" which means taking out the cover from which it has already existed. At the moment of discovery, reasonable, harmony and beautifulness of mathematical system is usually felt. Aristotle tried to explain about reaching an idea from the "material" to the "form". This explains that abstract mathematics can be understood with concrete materials using terms such as "modelling", "instruments", "embodiment", "metaphor" and "change representation". On this context, "invention" is also used for development of mathematical instruments, representation. and so on. From both ontological perspectives, mathematics can be understood and acquired by anyone and if acquired, it serves as a common scientific language that is used to express in any subject. Once representing the ideas using the shared common language, the world can be perceived in the same view autonomously.

From the epistemological perspective, mathematics can be developed through processes that are necessary to acquire mathematical values and ways of thinking. From this perspective, idealism and materialism are compared. In the context of Hegel, a member of German idealism, Imre Lakatos explained the development of mathematics through proof and refutation by using counter example. In another words, beyond the contradiction is the nature of mathematical activity and it provides the opportunity to think mathematically for overcoming. On this context, mathematics is not fixed but an expandable system that can be restructured through a process of dialectic in constructing viable arguments. Plato also used dialectic for reaching ideas with examples of mathematics. The origin of dialectic is known as the origin of indirect proof. In education today, dialectic is a part of critical thinking for creation. Parallel perspectives for mathematical developments are given by George Polya and Hans Freudenthal. For the discovery of mathematics, Polya explained mathematical problem-solving processes with mathematical ideas and mathematical ways of thinking in general. Freudenthal enhanced the activity to reorganize mathematics by the term mathematisation on the principle of reinvention.

Ontology

Idealism

Plato: idea

Discover

Reasonable, harmony & beautifulness on the system.

Materialism

Aristotle

Invention

Modeling, instruments and representation.

Epistemology

Idealism:

Hegel's Dialectic

Lakatos's Proof and Refutation

Plato's Dialectic

Polya's problem solving

Freudental's mathematization

6

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Genetic epistemologist Jean Piaget established his theory for operations based on various theories, including the discussion of Freudenthal and explained mathematical development of operations by the term reflective abstraction. Reflection is also a necessary activity for mathematisation by Freudenthal. On materialism, under the Vygotskyian perspective, intermediate tools such as language become the basis for reasoning in the mind. Under his theory, high-quality mathematical thinking can be developed depending on the high-quality communication in mathematics classrooms. Dialectical-critical discussion should be enhanced in the mathematics class. From both the epistemological perspectives, mathematics can be developed through the processes of communication, problem solving and mathematisation which include reorganization of mathematics. Those processes are necessary to acquire mathematical values and ways of thinking through reflection.

Constructivism

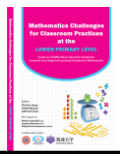
Piaget, Dewey & Freudenthal

Learning through Reflection

Social Constructivism

Vigotsky

Thinking is developed through internalization of communication



7

Summary: Basic Principles

1. Principle for Teaching Approach

Teaching is the activity by using what students learned before and preparation for Future

- It is based on the appropriate task sequence.
- Task is given by teacher and problematic emergent from students.
- Problem solving approach or teaching through problem solving are not aimed to find answers for given task but aimed to realize students' arguments to solve problematic for students.
- On problem solving approach, the objective of lesson is not to solve given task but to solve problematic for students by themselves.
- Problematic is not easily to solve by using what students already learned however if they utilize, extend and arrange various representations of learned ideas, they are able to solve. It is new creation by students and points of arguments in the lesson.
- Misconceptions are the essential product of appropriate learning through mathematics curriculum and a part of task sequence. We should not deny them based on what teachers know because misconceptions are indicators which students produce their own ideas based on what they learned. It is the opportunity to develop critical discussions.
- In the classroom communication, understanding others means re-present others' idea by him/herself and producing another example by him/herself. If we understand others and produce another example, it will be possible to say if your saying is true.
- Teaching approach should be preferred depending on the objectives.

8

Summary: Basic Principles

2. Principle to design Curriculum

Terminology to explain conceptual difference is aimed to explain the task sequence and objectives for each lesson.

- In mathematics, we internationally share the technical terms. Each term's concept itself explains mathematical invariant but it does not well explain mathematics curriculum itself because the meaning of the same technical term usually change and extend in the curriculum sequence. On mathematics, concepts usually well defined and never change. Thus, in case, we use the word 'concept' we should explain it exactly like mathematicians. Such an explanation is necessary to make clear what is essential however it is NOT the terminology to explain curriculum and task sequence.
- Instead of explaining mathematical technical terms, on this lecture we more focused on the terminology which explain conceptual differences to explain how the meaning of mathematical terms change on our curriculum. It explains conceptual differences on the task sequence.
- We also discussed levels and phases as terminology to explain conceptual differences and task sequence such as Part III, Topic 2 Figure, Topic 3 Measurement and Topic 4 Pattern and Data.

We learned various meaning of mathematics curriculum.
In the case of Philippines, curriculum includes up to the lesson plan.
In many countries, curriculum standards does not include terminology beyond technical terms in mathematics.

9

Summary: Basic Principles

3. Principle to develop Mathematical Thinking in Curriculum

Terminology to explain mathematical thinking which aimed to explain invariant processes of thinking which should be taught through the curriculum.

- Repetition of the same process is the opportunity to learn mathematical thinking however for 'learning how to' students need the opportunity to reflect on the process and appreciate it.
- Even though terminology of invariant mathematical thinking never explains each student thinking itself, it explain how it will be possible to repeat on the task sequence on the curriculum.
- Wide meaning of mathematical thinking include values, attitudes and habits even through narrow meaning of mathematical thinking is mathematical ideas and ways of thinking.
- Variant Terminology to explain conceptual differences are also include unique idea and ways of thinking
- Various Mathematical activity such as problem solving, modeling and inquiry always characterized by specific process such as PPDAC cycle.

10

These are final reviews and summary.

11