What is mathematical thinking and why is it important?

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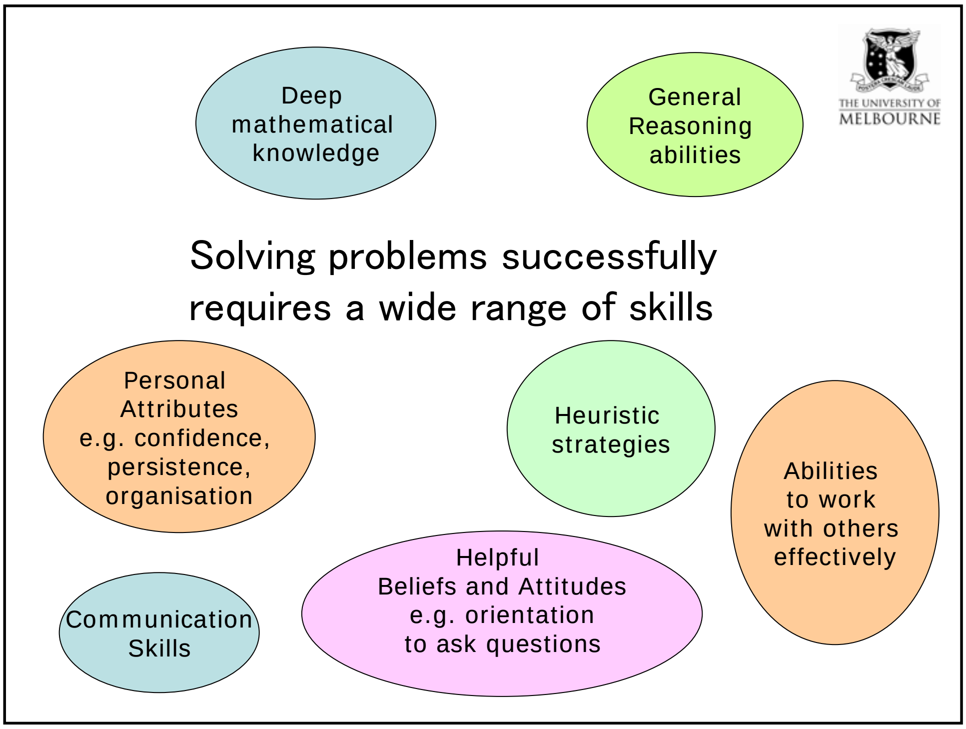
Outline of presentation

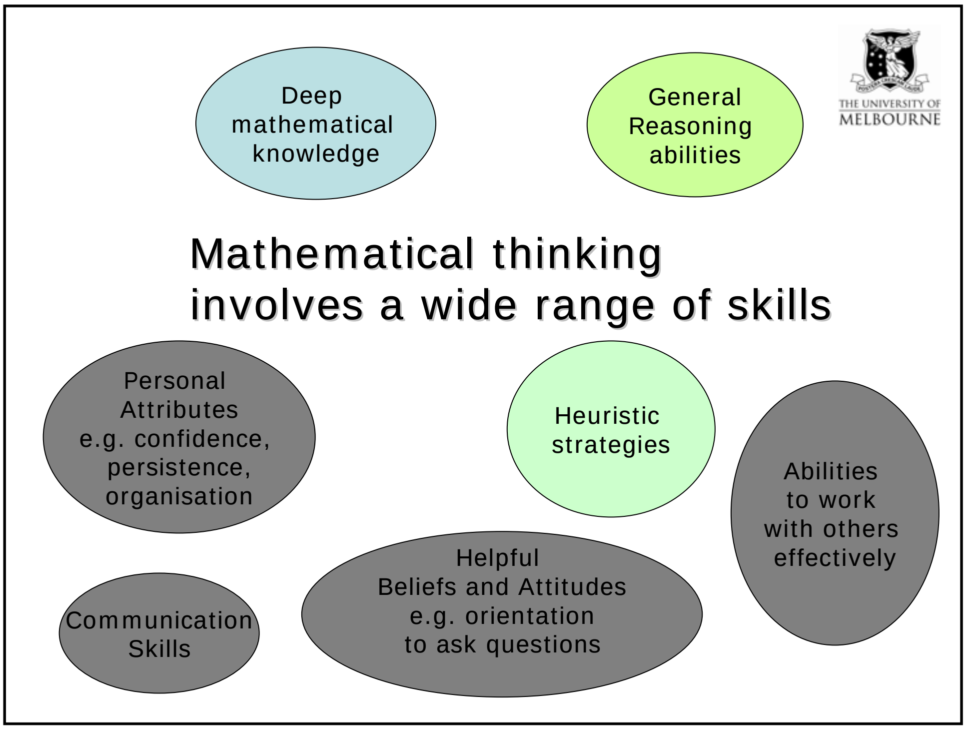
- Mathematical thinking is an important goal of schooling
- Mathematical thinking is important as a way of learning mathematics
- Mathematical thinking is important for teaching mathematics
- Mathematical thinking proceeds by
  - specialising and generalising
  - conjecturing and convincing

Tokyo January 2006

- Jan de Lange – OECD PISA program
- "Mathematical literacy"
  - ability to use mathematics to solve problems for everyday living and for work and for further study
  - involves wide range of key competencies (reasoning, communication, modeling, reproduction , connections)
- In this respect, mathematical thinking is an important goal of schooling, to support science, technology, economic life and development

Solving problems successfully requires a wide range of skills


THE UNIVERSITY OF MELBOURNE
Deep mathematical knowledge
General Reasoning abilities
Solving problems successfully requires a wide range of skills
Personal Attributes e.g. confidence, persistence, organisation
Heuristic strategies
Abilities to work with others effectively
Communication Skills
Helpful Beliefs and Attitudes e.g. orientation to ask questions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Abilities to work with others effectively
Communication Skills
Helpful Beliefs and Attitudes e.g. orientation to ask questions

To become better problem solvers, students need:

- EXPERIENCE - solving non-routine problems in a supportive classroom environment
- REFLECTION - active reflection so that they learn from these experiences
- STRATEGIES – learning about effective heuristic strategies and good problem solving habits, and the processes of mathematical thinking

Thinking Mathematically*

proceeds by alternating between 4 fundamental processes

Specialising ⟷ Generalising

Conjecturing ⟷ Convincing

* Mason, Burton, Stacey *Thinking Mathematically* THAILAND KwanKao'94 & Pearson

Thinking Mathematically*

THE UNIVERSITY OF MELBOURNE

proceeds by alternating between 4 fundamental processes

Specialising ⟷ Generalising

Trying special cases, looking at examples

Conjecturing ⟷ Convincing

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Thinking Mathematically*

THE UNIVERSITY OF MELBOURNE

proceeds by alternating between 4 fundamental processes

Specialising ⟷ Generalising

Looking for patterns and relationships

Conjecturing ⟷ Convincing

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Specialising ⟷ Generalising

Predicting relationships, results

Conjecturing ⟷ Convincing

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Specialising ⟷ Generalising

Finding and communicating reasons why

Conjecturing ⟷ Convincing

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Next:

I will illustrate these four processes of mathematical thinking

in the context of a problem that may be used

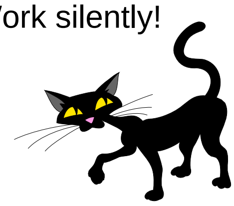
to stimulate mathematical thinking about numbers or

as an introduction to algebra

The Flash Mind Reader

- by Andy Naughton – UK
- http://www.cyberglass.biz/

- First one volunteer will think of a secret number, and I will show you the Flash Mind Reader
- Later everyone can try
- Do not tell anyone your secret number!! Work silently!



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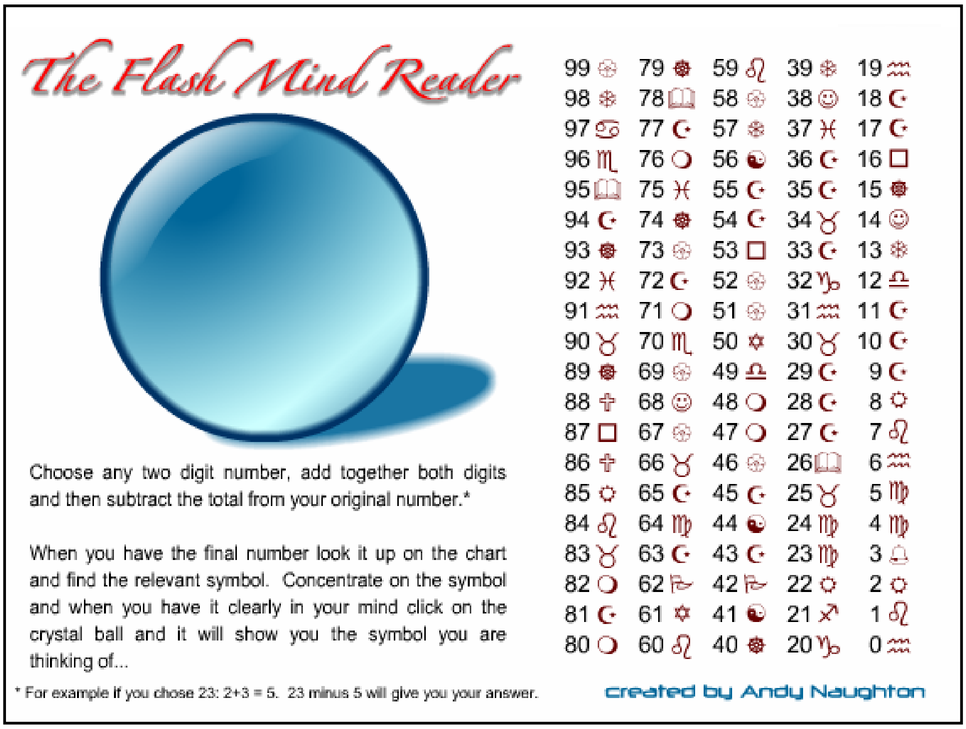

Instructions

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Choose any two digit number, add together both digits and then subtract the total from your original number.*

When you have the final number look it up on the chart and find the relevant symbol. Concentrate on the symbol and when you have it clearly in your mind click on the crystal ball and it will show you the symbol you are thinking of...

For example if you chose 23: 2+3 = 5. 23 minus 5 will give you your answer.

Explanations

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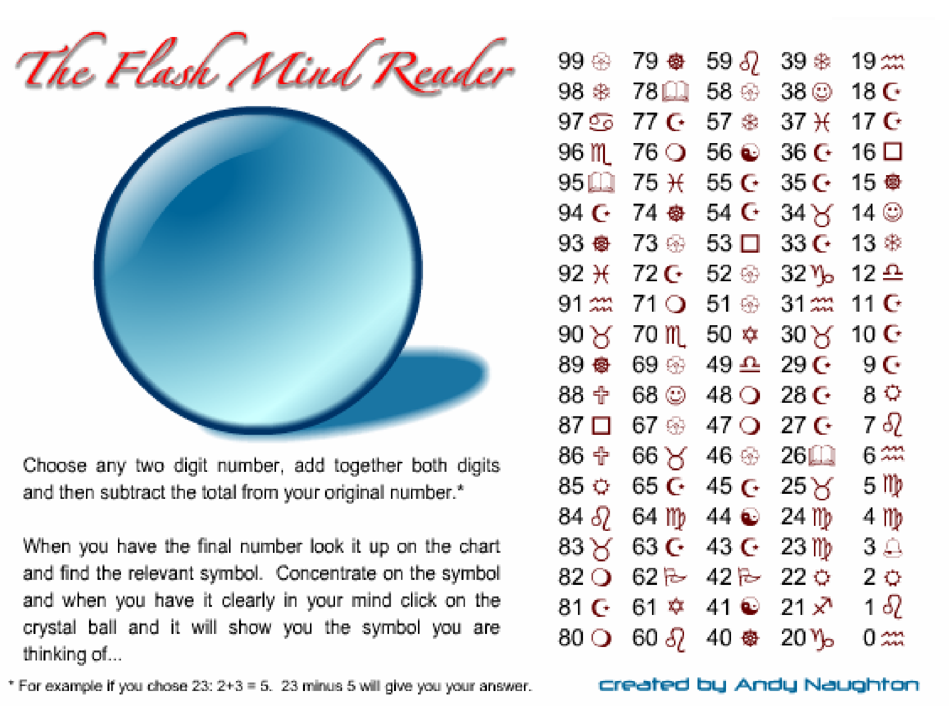
- Maybe it is magic?
- Does it really read your mind?
  - how could we test this?
  - by trying not to concentrate!
- Is there some power over your choice?
  - maybe subconsciously guided in choice
- Is it an optical illusion?
  - maybe it comes from staring at the numbers
- How does it work?



Elements of mathematical thinking

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1. An attitude to look for a logical explanation
2. Gathering information – trying several times, trying different types of numbers
3. Looking carefully at a number

   e.g. 87 – 15 = 72
4. Beginning to work systematically

   e.g. 87 – 15 = 72

   86 – 14 = 72

   85 – 13 = 72

   This begins to reveal how the Mind Reader works

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Specialising

Generalising

And now conjecturing

- The trick works because every number that you choose reduces to a "magic number" such as 72
- And all "magic numbers" have the same symbol, which is the symbol that appears on the screen

And now conjecturing

- The trick works because every number that you choose reduces to a magic number
- And all magic numbers have the same symbol

How can I convince myself that this is true?
And then how can I convince you?

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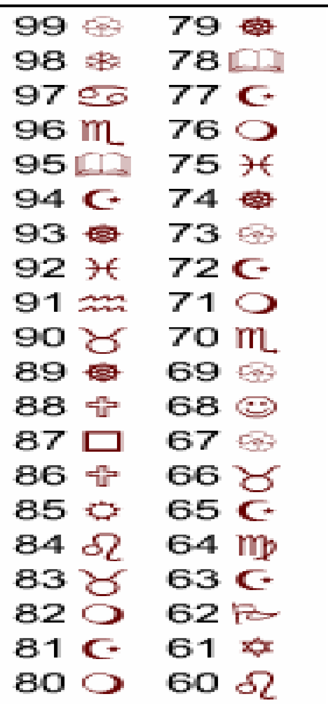

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87 – (8+7) = magic number

87 – (8+7)
=
87 – 7 – 8
= 72
magic number

81	81-1	80	80-8	72
82	82-2	80	80-8	72
83	83-3	80	80-8	72
84	84-4	80	80-8	72
89	89-9	80	80-8	72

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Important heuristics: Working systematically, Making tables

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Seeing the general in a special case

87 – (8+7) = magic number

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87 – (8+7)
=
87 – 7 – 8
= 72
magic number

Better to keep the ‘unclosed expression’ 8+7 than to use 15

81	81-1	80	80-8	72
82	82-2	80	80-8	72
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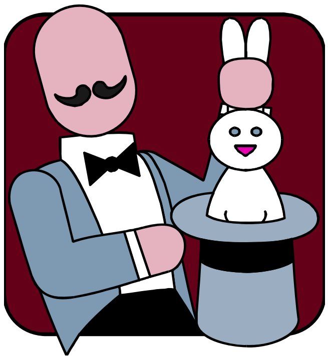
Proofs at various levels all need to show:

- Whatever number is selected, the process leads to the same symbol.
- The symbols for all the “magic numbers” are all the same.
- The “magic numbers” are 81, 72, 63, ….

Proof by Algebra

"ab" = 10 x a + b

"ab" – (a + b)
= (10 x a + b) – (a + b)
= 10 x a - a + b – b
= 9 x a
= multiple of 9

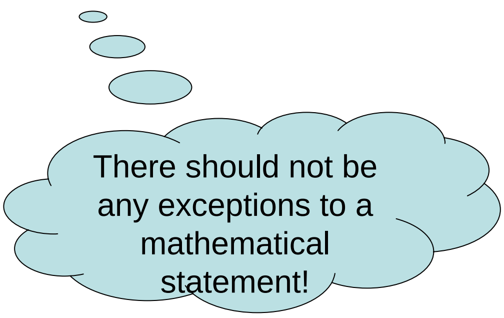


(So the trick works if
all the multiples of 9 the same symbol)

Proof by Algebra

The trick works if all the multiples of 9 have the same symbol.

But they do not!



Refining the proof

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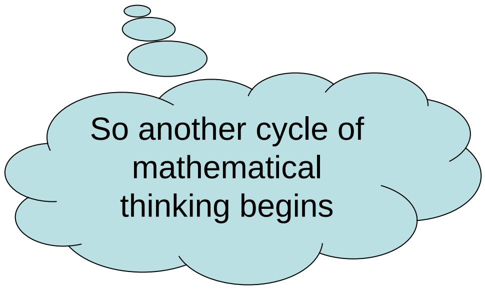

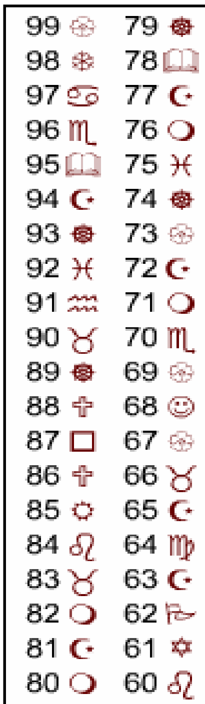

Refining the proof

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Not all the multiples of 9 need the same symbol.

99 and 90 never arise from 2 digit numbers

0 never arises from 2 digit numbers

Clever features of the Flash Mind Reader

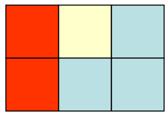
- The graphics
- The special symbol changes each time
- Most of the multiples of 9 must have the special symbol but:
  - some numbers that are not multiples of 9 have the special symbol
  - not all the multiples of 9 have the special symbol (there are some unnecessary multiples)

Teaching potential of the Flash Mind Reader

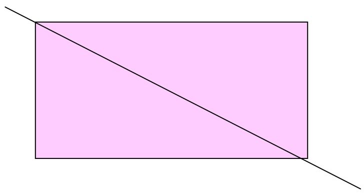
- Pre-algebra – interesting number patterns and relationships
- Elementary algebra – motivates the need to be able to show a result true for a wide range of numbers
- An attitude of looking for and expecting explanations and reasons, rather than magic or just rules.

Logical reasons or *conventions*?

- *360 degrees in a complete rotation*
- a half is more than a third
- a half plus a third is five sixths



- *diagonal joins the opposite corners*
- diagonal divides rectangle into two equal pieces



Mathematical Thinking and the Flash Mind Reader

- Proofs can be at different levels
  - The algebra proof is easiest for us, but the numbers proof is easier for others.
  - The essential principle is to find a convincing argument
- Mathematical thinking aims for generalisations, but they are often reached by looking at special cases (specialising)

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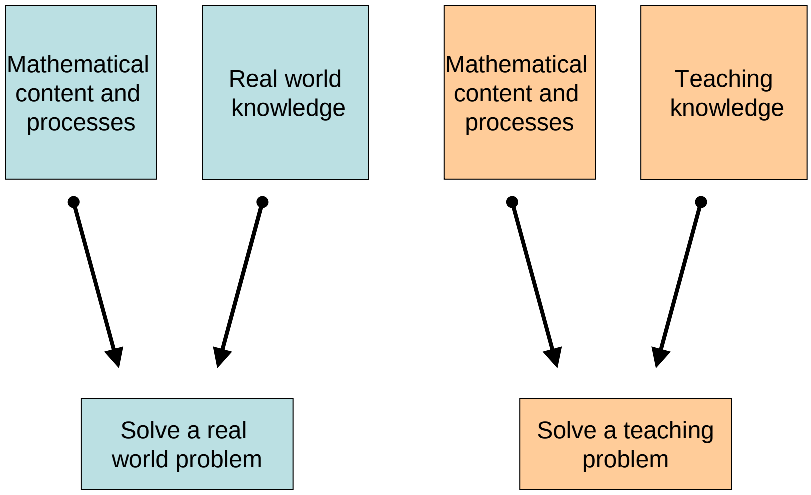
Mathematical thinking is essential for teaching mathematics

Classroom example from the work of Dr Helen Chick

(University of Melbourne)

An analogy



Maths meets Pedagogy

- A complex blend of mathematical knowledge and pedagogical knowledge is necessary for successful maths teaching.
- In the process of delivering a good lesson, teachers engage in extensive mathematical thinking.

Mathematical thinking for teaching is important for:

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- Analysing subject matter
- Planning the lesson to achieve the specified aims
- Understanding the students' thinking, questions and solutions
- Conducting the lesson on minute-by-minute basis

A classroom example

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- Teacher in her fifth year of teaching
- Grade 6 class.
- Pupils aged about 11 years.
- Lesson investigating area *and* perimeter
- First 15 minutes of the class
- This teacher shows greater than average mathematical thinking in her teaching

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T: I want you to draw a rectangle with an area of 20 cm^2, and cut it out.

Good choice of open "reversed" task that encourages investigation.

Knows students have done some area work.

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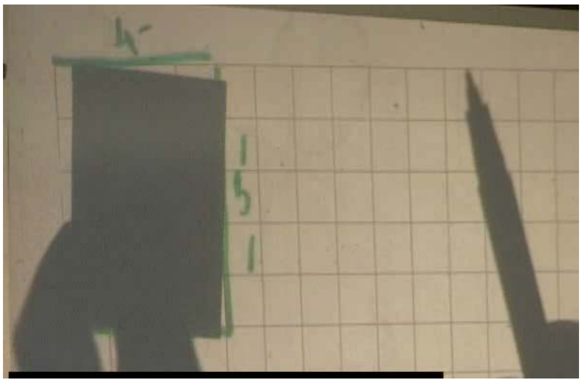
S: Can I do a square?
T: Is a square a rectangle?
T: What's a rectangle?
T: How do you get something to be a rectangle? What's the definition of a rectangle?
S: Two parallel lines
T: Two sets of parallel lines … and …
S: Four right angles.
T: So is that [square] a rectangle?
S: Yes.

Geometry knowledge evident plus knowledge of how definitions are used in maths. She knows that this definition of a rectangle implies that a square is a rectangle.

[Pause as teacher realises that the geometry is okay, but there is a measurement error to address]
T: But has that got an area of 20?
S: [Thinks] Er, no.
T: [Nods and winks]

Recognises student's error, the reason behind it, and that this student can fix it himself.



Teacher gets a student to bring his cut out rectangle, traces it on overhead projector and confirms that its area is 20 cm^2. Class discussion on how multiplying length x width is the same as counting the squares and so gives the area .

Teacher highlights the connections between formula and what it calculates.

Shows that reasoning is a key component of maths (not just rules)

S1: That's how you work out area -- you do the length times the width.
T: When S said that's how you find the area of a shape, is he *completely* correct?
S2: That's what you do with a 2D shape.
T: Yes, for *this* kind of shape. What kind of shape would it *not* actually work for?
S3: Triangles.
S4: A circle.
T: [With further questioning, teases out that LxW only applies to rectangles]

Recognises the limited circumstances in which the formula applies AND knows that students need to be aware of this.

Knows many students overgeneralise rules.

Teacher gets second student to bring her cut out rectangle to the projector, and confirms that its area is 20 cm^2.

Teacher then verifies that all students had done either 10 x 2 or 4 x 5

Reason for choice of original task is now apparent: allows different solutions, and soon will enphasise multiplying.

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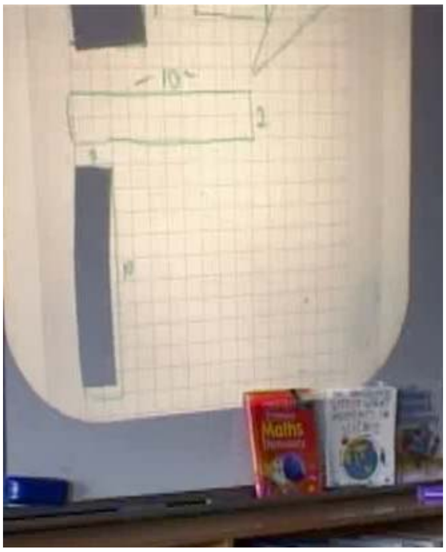

Teacher asks students to see if they can think of other rectangles with area 20.

Students suggest the original ones but oriented differently, and also suggest 1x20

Teacher aware of all the different possibilities; allows students to explore these.

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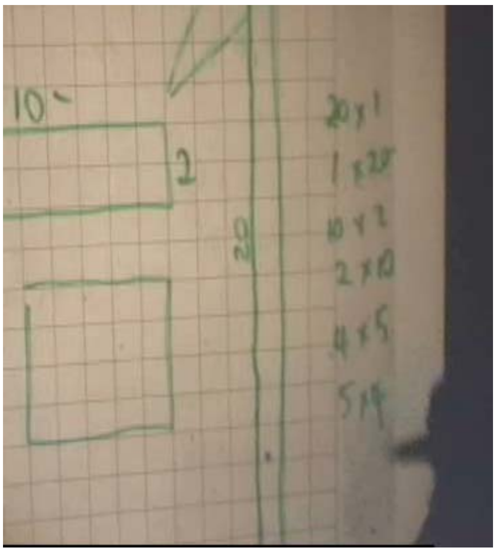


Teacher lists all rectangles found, and asks students to look for a pattern.

Her prompts highlight that all the numbers are factors of 20.

She makes connections to number properties.

T: Are there any other numbers that are going to give an area of 20?
[Pauses, as if uncertain. There is no response from the students at first]
T: No? How do we know that there's not?
S: You could put 40 by 0.5.
T: Ah! You've gone into decimals. If we go into decimals we're going to have *heaps*, aren't we?

First, she was targeting only whole numbers (and factors of 20), but appreciates that other answers exist. Her questions bring this out. Emphases generalising.

What happened next?

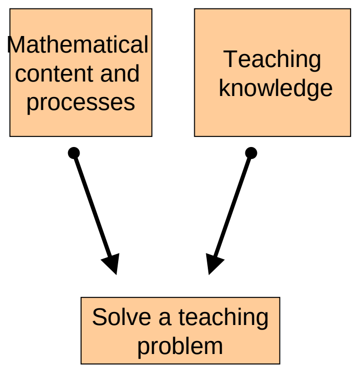
- This was the first 15 minutes of the lesson
- She went on to get the students to:
  - find rectangles with an area of 16 cm^2
  - work systematically when considering all possibilities
  - determine the perimeter of the different 16cm^2 rectangles they had (noting that the perimeters vary while area is always 16cm^2)
  - find many shapes (not just rectangles) with area 12 cm^2 and determine the perimeters

What thinking has teacher used in this segment?

- Maths concepts deeply understood, connections among concepts, and linking concepts and procedures:
  - Area (conceptual meaning, formula, what formula does, when formula applies)
  - Perimeter (meaning, separate from area)
  - Geometry (definitions, sets of shapes, properties)
  - Number (factors, whole numbers, decimals)
- Important general mathematical principles:
  - Working systematically
  - Need for justification, explanation and connections
  - What is a definition in mathematics

Teaching mathematics draws on mathematical thinking for:

- Analysing subject matter
- Planning the lesson to achieve specified aims
- Understanding the students' thinking, questions and solutions
- Conducting the lesson on minute-by-minute basis



Teaching mathematics can be thought of as a form of problem solving, where mathematics is used in combination with pedagogical knowledge to solve a teaching problem.

Summary

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- Mathematical thinking is an important goal of schooling
- Mathematical thinking is important as a way of learning mathematics
- Mathematical thinking is important for teaching mathematics
- Some processes of mathematical thinking

THANK YOU

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