Mathematical Thinking and How to Teach It

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Table of Contents

Chapter 1

The Aim of Education and Mathematical Thinking

1. The Aim of Education: Scholastic Ability From the Perspective of "Cultivating Independent Persons"

School-based education must be provided to achieve educational goals. "Scholastic ability" becomes clear when one views the aim of school-based education.

The Aim of School Education

The aim of school education is described as follows in a report by the Curriculum Council:
"To cultivate qualifications and competencies among each individual school child, including the ability to find issues by oneself, to learn by oneself, to think by oneself, to make judgments independently and to act, so that each child or student can solve problems more skillfully, regardless of how society might change in the future."

This guideline is a straightforward expression of the preferred aim of education.
The most important ability that children need to gain at present and in the future, as society, science, and technology advance dramatically, is not the ability to correctly and quickly execute predetermined tasks and commands, but rather the ability to determine for themselves what they should do, or what they should charge themselves with doing.
Of course, the ability to correctly and quickly execute necessary tasks is also necessary, but from now on, rather than adeptly imitating the skilled methods or knowledge of others, the ability to come up with one's own ideas, no matter how small, and to execute one's own independent, preferable actions (ability full of creative ingenuity) will be most important. This is why the aim of education from now on is to instill the ability (scholastic ability) to take these kinds of actions. Furthermore, this is something that must be instilled in every individual child or student. From now, it will be of particular importance for each individual school child to be able to act independently (rather than the entire class acting independently as a unit). Of course, not every child will be able to act independently at the same level, but each school child must be able to act independently according to his or her own capabilities. To this end, teaching methods that focus on the individual are important.

2. The Scholastic Ability to Think and Make Judgments Independently Is Mathematical Thinking

– Looking at Examples –

The most important ability that arithmetic and mathematics courses need to cultivate in order to instill in students this ability to think and make judgments independently is mathematical thinking. This is why cultivation of this "mathematical thinking" has been an objective of arithmetic and mathematics courses in Japan since the year 1950. Unfortunately, however, the teaching of mathematical thinking has been far from adequate in reality.
One sign of this is the assertion by some that "if students can do calculations, that is enough."

The following example illustrates just how wrong this assertion is:
Example: "Bus fare for a trip is 4,500 yen per person. If a bus that can seat 60 people is rented out, however, this fare is reduced by 20% per person. How many people would need to ride for it to be a better deal to rent out an entire bus?"
This problem is solved in the following manner:

Solving Method When a bus is rented –
One person's fee: 4,500×0.8=3,600
In the case of 60 people: 3,600×60=216,000
With individual tickets, the number of people that can ride is

216,000÷4,500=48 (people).
Therefore, it would be cheaper to rent the bus if more than 48 people ride.

Sixth-graders must be able to solve a problem of this level. Is it sufficient, however, to solve this problem just by being able to do formal calculation (calculation on paper or mental calculation, or the use of an abacus or calculator)? Regardless of how skilled a student is at calculation on paper, and regardless of whether or not a student is allowed to use a calculator at will, these skills alone are not enough to solve the problem. The reason is that before one calculates on paper or with a calculator, one must be able to make the judgment "what numbers need to be calculated, what are the operations that need to be performed on those numbers, and in what order should these operations be performed?" If a student is not able to make these judgments, then there's not much point in calculating on paper or with a calculator. Formal calculation is a skill that is only useful for carrying out commands such as "calculate this and this" (a formula for calculation) once these commands are actually specified. Carrying out these commands is known as "deciding the operation." Therefore, "deciding the operation" for oneself in order to determine which command is necessary to "calculate this and this" is an important skill that is indispensable for solving problems.
Deciding the operation clearly determines the meaning of each computation, and decides what must be done based on that meaning. This is why "the ability to clarify the meanings of addition, subtraction, multiplication, and division and determine operations based on these meanings" is the most important ability required for computation.
Actually, there is something more important – in order to correctly decide which operations to use in this way, one must be able to think in the following manner "I would like to determine the correct operations, and to do so, I need to recall the meanings of each operation, and think based on these meanings." This thought process is one kind of mathematical thinking.

Even if a student solves the group discount problem as described above, this might not be sufficient to conclude that he truly understood the problem. This is why it is important to "change the conditions of the problem a little" and "consider whether or not it is still possible solve the problem in the same way." These types of thinking are neither knowledge nor a skill. They are "functional thinking" and "analogical thinking."
For instance, let's try changing one of the conditions by "changing the bus fare from 4,500 yen to 4,000 yen." Calculating again as described above results in an answer of 48 people (actually, a better way of thinking is to replace 4,500 above with 4,000 – this is analogical thinking). In this way, one should gain confidence in one's method of solving the problem, as one realizes that the result is the same: 48 people.
The above formulas are expressed in a way that is insufficient for students in fourth grade or higher. It is necessary to express problems using a single formula whenever possible.
When these formulas are converted into a single formula based on this thinking, this is the result:

$4{,}500\times(1-0.2)\times 60\div 4{,}500$

When viewed in this form, it becomes apparent that the formula is simply $60\times(1-0.2)$
What is important here is the idea of "reading the meaning of this formula." This is important "mathematical thinking regarding formulas." Reading the meaning of this formula gives us:

full capacity × ratio

For this reason, even if the bus fare changes to 4,000 yen, the formula $60\times 0.8=48$ is not affected. Furthermore, if the full capacity is 50 persons and the group discount is 30%, then regardless of what the bus fare may be, the problem can always be solved as "$50\times 0.7=35$; the

group rate (bus rental) is a better deal with 35 or more people." This greatly simplifies the result, and is an indication of the appreciation of mathematical thinking, namely "conserving cogitative energy" and "seeking a more beautiful solution."
Students should have the ability to reach the type of solution shown above independently. This is a desirable scholastic ability that includes the following aims:

- **Clearly grasp the meaning of operations, and decide which operations to use based on this understanding**
- **Functional thinking**
- **Analogical thinking**
- **Expressing the problem with a better formula**
- **Reading the meaning of a formula**
- **Economizing thought and effort (seeking a better solution)**

Although this is only a single example, this type of thinking is generally applicable. In other words, in order to be able to independently solve problems and expand upon problems and solving methods, the ability to use **"mathematical thinking"** is even more important than knowledge and skill, because it enables to drive the necessary knowledge and skill.
Mathematical thinking is the "scholastic ability" we must work hardest to cultivate in arithmetic and mathematics courses.

3. The Hierarchy of Scholastic Abilities and Mathematical Thinking
As the previous discussion makes clear, there is a hierarchy of scholastic abilities. When related to the above discussion, and limited to the area of computation (this is the same as in other areas, and can be generalized), these scholastic abilities enable the following (from lower to higher levels):

1. The ability to memorize methods of formal calculation and to carry out these calculation
2. The ability to understand the rules of calculation and how to carry out formal calculation
3. The ability to understand the meaning of each operation, to decide which operations to use based on this understanding, and to solve simple problems
4. The ability to form problems by changing conditions or abstracting situations
5. The ability to creatively make problems and solve them

The higher the level, the more important it is to cultivate independent thinking in individuals. To this end, mathematical thinking is becoming even more and more necessary.

Chapter 2

The Importance of Teaching to Cultivate Mathematical Thinking

2.1 The Importance of Teaching Mathematical Thinking

As we found in the previous chapter, the method of thinking is the center of scholastic ability. In arithmetic and mathematics courses, mathematical thinking is the center of scholastic ability. However, in Japan, in spite of the fact that the improvement of mathematical thinking was established as a goal more than 45 years ago, the teaching of mathematical thinking is by no means sufficient.

One of the reasons that teaching to cultivate mathematical thinking does not tend to happen is, teachers are of the opinion that students can still learn enough arithmetic even if they don't teach in a way to cultivate the students' mathematical thinking. In other words, teachers do not understand the importance of mathematical thinking.

The second reason is that, in spite of the fact that mathematical thinking was established as a goal, teachers do not understand what it really is. It goes without saying that teachers cannot teach what they themselves do not understand.

Therefore, we shall start out by explaining how important the teaching of mathematical thinking is.

A simple summary follows.

Mathematical thinking allows for: (1) An understanding of the necessity of using knowledge and skills (2) Learning how to learn by oneself, and the attainment of the abilities required for independent learning

(1) The Driving Forces to Pursue Knowledge and Skills

Mathematics involves the teaching of many different areas of knowledge, and of many skills. If children are simply taught to "use some knowledge or skill" to solve problems, they will use that knowledge or skill. In this case, however, children will not realize why they are being told to use such a knowledge or skill. Also, when new knowledge or skills are required for problem solving and students are taught what skill to use, they will be able to use that skill to solve the problem, but they will not know why this skill must be used. Students will therefore fail to understand why the new skill is good.

What is important is "how to realize" which previously learned knowledge and skills should be used. It is also important to "sense the necessity of" and "perceive the need or desirability of using" new knowledge and skills.

Therefore, it is necessary for something to act as a drive towards the required knowledge and skills. Children first understand the benefits of using knowledge and skills when they possess and utilize such a drive. This leads them to fully acquire the knowledge and skills they have used.

Mathematical thinking acts as this drive.

(2) Achieving Independent Thinking and the Ability to Learn Independently

Possession of this driving force gives students an understanding of how to learn by themselves.

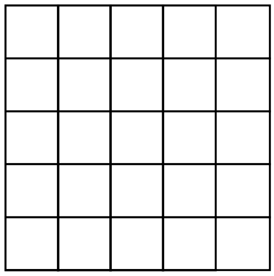
Cultivating the power to think independently will be the most important goal in education from now on, and in the case of arithmetic and mathematics courses, mathematical thinking will be the most central ability required for independent thinking. By mastering this skill even further, students will attain the ability to learn independently.

The following specific example serves to clarify this point further.

2.2 Example: How Many Squares are There?

This instructional material is appropriate for fourth-grade students.

How many squares are there in the following figure?



1. The Usual Lesson Process

This is usually taught in the following way (T refers to the teacher, and C the children):

T: There are both big and small squares here. Let's count how many squares there are in total.

T: (When the children start counting) First, how many small squares are there?

C: 25.

T: Which squares are the second smallest?

C: (Indicate the squares using two by two segments)

T: Count the number of those squares.

T: Which squares are the next biggest size, and how many are there?

The questions continue in this manner in order of size. In each case, the teacher asks one child the number, and then asks another child if this number is correct. Alternatively, the teacher might recognize the correctness of the number, and comment "yes, that's the right number."

The teacher has the children count squares in order of size, and then has the children add the numbers together to derive the grand total.

2. Problems with This Method

a) When the teacher instructs children to count squares based on size, the children do not realize for themselves that they should sort the squares into groups. As a result, the children do not understand the need to sort, or the thinking behind sorting.

b) The number of squares of each size is determined either by the majority of the children's answers, or based on the teacher's approval. These methods are not the right way of determining the correct answer. Correctness must be determined based on solid rationales.

c) Also, if instruction regarding this problem ends this way, children will only know the answer to this particular problem. The important things they must grasp, however, are what to focus upon in general, and how to think about problems of this nature.

Teachers should, therefore, follow the following teaching method:

3. Preferred Method

(1) Clarification of the Problem – 1

The teacher gives the children the previous diagram.

T: How many squares are there in this diagram?

C: 25 (many children will probably answer this easily).
Some children will probably respond with a larger number.
The children come up with the answer 25 after counting just the smallest squares. Those who think the number is higher are also considering squares with more than one segment per side. This is the source of the issue, which is not about the correct answer, but the vagueness of the mathematical problem.
The teacher should then have the children discuss "which squares they are counting when they arrive at the number 25," and inform them that "this problem is vague and does not clearly state which squares need to be counted." The teacher concludes by clarifying the meaning of the problem, saying "let's count all the squares, of every different size."

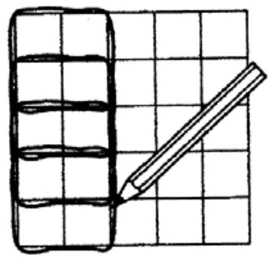
(2) Clarification of the Problem – 2
First, the teacher lets all the children count the squares independently. Various answers will be given when the teacher asks for totals, or the children may become confused while counting. The children will realize that most of them (or all of them) have failed to count correctly. It is then time to **think of a way of counting that is a little better and easier** (this becomes a problem for the children to solve).

(3) Realizing the Benefit of Sorting
The children will realize that the squares should be sorted and counted based on size. The teacher has the children count the squares again, this time sorting according to size.

(4) Knowing the Benefit of Encoding
Once the children are finished counting, the teacher asks them to give their results. At this point, when the teacher asks "how many squares are there of this size, and how many squares are there of that size..." he/she will run into the problem of not being able to clearly indicate "which size."
At this point, naming (encoding) each square size should be considered. It is important to make sure that the children realize that calling the squares "large, medium, and small" is not preferable because this naming system is limited. However, the children learn that naming the squares in the following way is a good system, as they state each number.

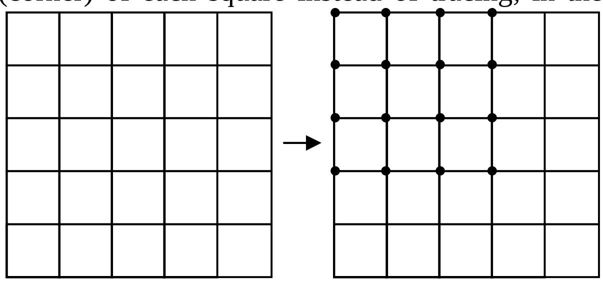
Squares with 1 segment	25
Squares with 2 segment	16
Squares with 3 segment	9
Squares with 4 segment	4
Squares with 5 segment	1
Total	55

(5) Judging the Correctness of Results More Clearly, Based on Solid Rationale
The correctness or incorrectness of these numbers must be elucidated, so have one child count the squares again in front of the entire class. The student will probably count the squares while tracing each one, as shown to the right. This will result in a messy diagram, and make it hard to tell which squares are being counted. Tracing each square is inconvenient, and will make the students feel their counting has become sloppy.



(6) Coming up with a More Accurate and Convenient Counting Method

There is a counting method that does not involve tracing squares. Have the students discover that they can count the upper left vertex (corner) of each square instead of tracing, in the following manner: place the pencil on the upper left vertex and start to trace each square in one's head, without moving the pencil from the vertex.



By using this system, it is possible to count two-segment squares as shown in the diagram to the right, by simply counting the upper left vertices of each square. This counting method is easier and clearer.

This method takes advantage of the fact that "squares and upper left vertices are in a one-to-one relationship." In other words, in the case of two-segment squares, once a square is selected, only one vertex will correspond to that square's upper left corner. The flip side of this principle is that once a point is selected, if that point corresponds to the upper left corner of a square, then it will only correspond to a single square of that size. Therefore, while sorting based on size, instead of counting squares, one can also count the upper left corners.

Instead of counting squares, this method "uses a <u>functional thinking</u> by counting the easy-to-count upper left vertices, which are functionally equivalent to the squares (in a one-to-one relationship)."

(7) Expressing the Number of Squares as a Formula

When viewed in this fashion, the two-segment squares shown in the diagram have the same number as a matrix of four rows by four columns of dots. When one realizes that this is the same as 4×4, it becomes apparent that the total number of squares is as follows:

$5\times5+4\times4+3\times3+2\times2+1\times1$ (A)

Students will understand that it's a good idea <u>to think of ways to devise different expression methods</u>, and to <u>express problems as formulas</u>.

(8) Generalizing

This makes <u>generalization</u> simple. In other words, consider what happens when "the segment length of the original diagram is increased by 1 to a total of 6." All one needs to do is to add 6×6 to formula (A) above. Thus, the <u>thought process of trying to generalize</u>, and the <u>attempt to read formulas</u> is important.

(9) <u>Further Generalization</u>

For instance (for students in fifth grade or higher), when this system is applied to other diagrams, such as a diagram constructed entirely of rhombus, how will this change the formula? (Answer: It will not change the above formula at all.)

By generalizing to see the case of parallelograms (as long as the counting involves only parallelograms that are similar to the smallest parallelogram, the diagram can be seen in the same way), the true nature of the problem becomes clear.

4. <u>Mathematical Thinking</u> is the Key Ability Here

What kind of ability is required to think in the manner described above? First, what knowledge and skills are required? The requirements are actually extremely simple:

Understanding the meaning of "square," "vertex," "segment," and so on

The ability to count to around 100

The ability to write the problem as a formula, using multiplication and addition

Possession of this understanding and skills, however, is not enough to solve the problem. An additional, more powerful ability is necessary. This ability is represented by the underlined parts above, from (1) to (9):

Clarification of the Meaning of the Problem
Coming up with an Convenient Counting Method
Sorting and Counting
Coming up with a Method for Simply and Clearly Expressing How the Objects Are Sorted
Encoding
Replacing to Easy-to-Count Things in a Relationship of Functional Equivalence
Expressing the Counting Method as a Formula
Reading the Formula
Generalizing

This is mathematical thinking, which differs from simple knowledge or skills.

It is evident that mathematical thinking serves an important purpose in providing the ability to solve problems on one’s own as described above, and that this is not limited to this specific problem. Therefore, the cultivation of a number of these types of mathematical thinking must be the aim of this class.

Chapter 3

The Meaning of Mathematical Thinking and How to Teach It

Characteristics of Mathematical Thinking

Although we have examined a specific example of the importance of teaching that cultivates mathematical thinking during each hour of instruction, for a teacher to be able to teach in this way, he must first have a solid grasp of "what kinds of mathematical thinking there are."
After all, there is no way a person could teach in such a way as to cultivate mathematical thinking without first understanding the kinds of mathematical thinking that exist.
Let us consider the characteristics of mathematical thinking.

1. Focus on Sets
Mathematical thinking is like an attitude, as in it can be expressed as a state of "attempting to do" or "working to do" something. It is not limited to results represented by actions, as in "the ability to do," or "could do" or "couldn't do" something.
For instance, the states of "working to establish a perspective" and "attempting to analogize, and working to create an analogy" are ways of thinking. If, on the other hand, one has no intention whatsoever of creating an analogy, and is told to "create an analogy," he/she might succeed in doing so due to having the ability to do so, but this does not mean that he/she consciously thought in an analogical manner.
In other words, mathematical thinking means that when one encounters a problem, one decides which set, or psychological set, to use to solve that problem.

2. Thinking Depends on Three Variables
In this case, the type of thinking to use is not determined by the problem or situation. Rather, the type of thinking to use is determined by the problem (situation), the person, and the approach (strategy) used. In other words, the way of thinking depends on three variables: the problem (situation), the person involved, and the strategy.
Two of these involve the connotative understanding of mathematical thinking. There is also denotative understanding of the same.

3. Denotative Understanding
Concepts are made up of both connotative and denotative components. One method which clarifies the "mathematical thinking" concept is a method of clearly expressing connotative "meaning." Even if the concept of mathematical thinking is expressed with words, as in "mathematical thinking is this kind of thing," this will be almost useless when it comes to teaching, because even if one understands the sentences that express this meaning, this does not mean that they will be able to think mathematically.
Instead of describing mathematical thinking this way, it should be shown with concrete examples. At a minimum, doing this allows for the teaching of the type of thinking shown.
In other words, mathematical thinking should be captured denotatively.

4. Mathematical Thinking is the Driving Force Behind Knowledge and Skills
Mathematical thinking acts as a guiding force that elicits knowledge and skills, by helping one realize the necessary knowledge or skills for solving each problem. It should also be seen as the driving force behind such knowledge and skills.
There is another type of mathematical thinking that acts as a driving force for eliciting other types of even more necessary mathematical thinking. This is referred to as the "mathematical attitude."

3.2 Substance of Mathematical Thinking

It is important to achieve a concrete (denotative) grasp of mathematical thinking, based on the fundamental thinking described in section 3.1. Let us list the various types of mathematical thinking.

First of all, mathematical thinking can be divided into the following three categories:

II. Mathematical Thinking Related to Mathematical Methods

III. Mathematical Thinking Related to Mathematical Contents

Furthermore, the following acts as a driving force behind the above categories:

I. Mathematical Attitudes

Although the necessity of category I was mentioned above, further consideration as described below reveals the fact that it is appropriate to divide mathematical thinking into II and III.

Mathematical thinking is used during mathematical activities, and is therefore intimately related to the contents and methods of arithmetic and mathematics. Put more precisely, a variety of different methods is applied when arithmetic or mathematics is used to perform mathematical activities, along with various types of mathematical contents. It would be accurate to say that all of these methods and types of contents are types of mathematical thinking. It is because of the ways of thinking that the existence of these methods and types of contents has meaning. Let us focus upon these types of contents and methods as we examine mathematical thinking from these two angles.

For this reason, three logical categories can be derived.

Specific details are provided below.

List of Types of Mathematical Thinking

I. Mathematical Attitudes

1. Attempting to grasp one’s own problems or objectives or substance clearly, by oneself
   (1) Attempting to have questions
   (2) Attempting to maintain a problem consciousness
   (3) Attempting to discover mathematical problems in phenomena
2. Attempting to take logical actions
   (1) Attempting to take actions that match the objectives
   (2) Attempting to establish a perspective
   (3) Attempting to think based on the data that can be used, previously learned items, and assumptions
3. Attempting to express matters clearly and succinctly
   (1) Attempting to record and communicate problems and results clearly and succinctly
   (2) Attempting to sort and organize objects when expressing them
4. Attempting to seek better things
   (1) Attempting to raise thinking from the concrete level to the abstract level
   (2) Attempting to evaluate thinking both objectively and subjectively, and to refine thinking
   (3) Attempting to economize thought and effort

II. Mathematical Thinking Related to Mathematical Methods

1. Inductive thinking
2. Analogical thinking
3. Deductive thinking
4. Integrative thinking (including expansive thinking)
5. Developmental thinking
6. Abstract thinking (thinking that abstracts, concretizes, idealizes, and thinking that clarifies conditions)
7. Thinking that simplifies

8. Thinking that generalizes
9. Thinking that specializes
10. Thinking that symbolize
11. Thinking that express with numbers, quantifies, and figures

III. Mathematical Thinking Related to Mathematical Contents

1. Clarifying sets of objects for consideration and objects excluded from sets, and clarifying conditions for inclusion (Idea of sets)
2. Focusing on constituent elements (units) and their sizes and relationships (Idea of units)
3. Attempting to think based on the fundamental principles of expressions (Idea of expression)
4. Clarifying and extending the meaning of things and operations, and attempting to think based on this (Idea of operation)
5. Attempting to formalize operation methods (Idea of algorithm)
6. Attempting to grasp the big picture of objects and operations, and using the result of this understanding (Idea of approximation)
7. Focusing on basic rules and properties (Idea of fundamental properties)
8. Attempting to focus on what is determined by one's decisions, finding rules of relationships between variables, and to use the same (Functional Thinking)
9. Attempting to express propositions and relationships as formulas, and to read their meaning (Idea of formulas)

Chapter 4

Detailed Discussion of Mathematical Thinking Related to Mathematical Methods

The previous chapter listed types of mathematical thinking as pertains to methods, but what does this mean in concrete terms? This chapter examines the meaning of each type.

1) Inductive Thinking

① Meaning

Inductive thinking is a method of thinking that proceeds as shown below.

What is **Inductive Thinking**
(1) Attempting to gather a certain amount of data
(2) Working to discover rules or properties in common between these data
(3) Inferring that the set that includes that data (the entire domain of variables) is comprised of the discovered rules and properties
(4) Confirm the correctness of the inferred generality with new data

② Examples

Example 1: Creating a multiplication times table
The meaning of multiplication is "an operation used to add the same number multiple times." Using this meaning, create the times table as shown below. For instance, the 4s row would have the following:

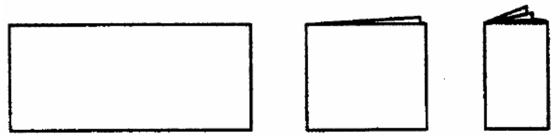
4×2=4+4=8
4×3=4+4+4=12
4×4=4+4+4+4=16
4×5=4+4+4+4+4=20

It is possible to seek a number of results in this way, but when one experiences the hassle of doing the same kind of addition over and over, one considers "whether or not it is possible to do this more simply."
Reexamination of the above results discover that "every time the number to be multiplied increases by 1, the answer increases by 4." Using this, one can deftly complete the rest of the times table, as shown below:

4×6=20+4=24
4×7=24+4=28

This is an example of gathering data, then reexamining the data to induce a rule.

Example 2: "Fold a single piece of paper perfectly in half, from left to right. How many creases will there be after the 10th fold, when you continue folding so that all the rectangles are folded into two halves each time?"
If one actually attempts to perform this experiment, it will become apparent that folding 10 times is impossible (this experience is important).



First Fold   Second Fold

Folding from the start, however, when the number of folds is still small and folding is still easy (thinking that simplifies), allows one to attempt to discover the rule that describes the relationship between the number of folds and the number of creases.

Number of Folds	Number of Creases
1	1
2	3

The results of two folds are shown in the chart above, which indicates that the number of creases is 1 and then 3. This might lead one to induce that the number of creases will then increase to 5, 7, and so on, starting with 1 and going up in odd numbers.

3	7
4	15

To verify this, try folding one more time. This gives the following result, which reveals the error of the previous induction.
Furthermore, this data shows that "the number of creases goes up by 2, 4, and 8," resulting in the induction that the number of creases goes up in the pattern 2, 4, 8, 16, and so on, or by doubling the previous number with each iteration.
Verify the induced rule with new data (the 5^{th} fold).
This type of thinking is inductive thinking. This example shows how to attempt to find rules while gathering data.

③ Points to Remember Regarding Teaching

It is important that inductive thinking is used in situations where it is valid. In other words, it is necessary to teach students the benefits of inductive thinking. One of these is the experience of problems that deductive thinking cannot solve well.
Also, since inductive rules are not always correct, students must learn the necessity of verifying rules with new data.
It is also a good idea to teach students that induction includes the following:

- Cases where one collects a certain amount of data and reexamines the data to discover rules
- Cases where one discovers rules while gathering data in an attempt to find generalities, and
- Cases where one gathers data while predicting rules, and verifies the same

2) Analogical Thinking

① Meaning

Analogical thinking is an extremely important method of thinking for establishing perspectives and discovering solutions.

What is **Analogical Thinking**?
Given Proposition A, one wants to know its properties, rules or solution methods.
However, when one does not know these things, one can recall an already known Proposition

A', which resembles A (assuming that regarding A', one already knows the properties, rules, solution method, and so on, which are referred to as P'). One then works to consider what can be said about P' of A', and with respect to A as well.

② Examples

Example 1: In the previous example for inductive thinking, we created a times table for the 4s row. Let us continue by creating the times table for the 6s row. This is created in order from 6×1, and resembles the 4s row. The thinking at this point is "if I can find a rule the same as that applied to create the 4s row, then I can easily complete the entire row." Furthermore, a rule has already been found during the creation of the 4s row. One considers "perhaps if there is a similar rule for the 6s row, and if I find it in the same way, then this should be possible." Next, proceed the same way as for the 4s row. This is analogical thinking.
Proceed as follows:
As in the case of the 4s row, start by writing the following while remembering that "every time the number multiplied increases by 1, the answer must also increase by a certain fixed amount."

6×1=6
6×2=6+6=12
6×3=6+6+6=18

By examining the situation based on this thinking, one discovers that "every time the multiplied number increases by 1, the answer increases by 6." Discovering this rule makes it easy to complete the 6s row.
Furthermore, other rows can also be easily created in the same manner. This is the benefit of analogical thinking.

Example 2: The comparison and measurement of width and weight are similar to the comparison and measurement of length. After one has learned how to compare and measure length, one can then learn how to compare and measure weight.
Although length and weight are not the same, they are similar in that both involve a comparison of magnitude. For this reason, one recalls how one worked with lengths. When lengths are compared, one compares them directly, and if this is not possible, either one or both lengths are copied to something easy to compare to, such as a string, and then they are compared directly.
Furthermore, in order to clearly state the differences in compared lengths, the appropriate unit is selected, and used to indicate the numerical measurements. In order to give the measurements universality, legal units are used.
When it is time to discuss weight, one first considers that it can probably be dealt with in the same way as length, and therefore thinks of how to compare weights directly. Next, one considers methods of indirect comparison as well, moreover considering to measure with the weight of say, a one-yen coin as the unit. Finally, one considers that there must be legal units that can be used in order to take measurements with universality.
The comparison and measurement of weight will be learned independently in this fashion, while appreciating the benefits of each stage. The focus here is on analogical thinking, which is used to analogize from the comparison and measurement of length.
Even in the case of the comparison and measurement of width, it can be shown by an analogy of the above that analogical thinking performs an important and effective function.

In this way, analogical thinking is an effective method of thinking to establish a perspective and to discover solutions.

③ Points to Remember Regarding Teaching

When considering perspectives on solution methods and results, the point is to have the students think “have I already learned something similar?” or “can I treat this in the same way?” or “can the same be said about this problem?” Analogical thinking, however, relies on similarities, and considers whether or not the same thing can be stated. Therefore, it does not always provide correct results.
For instance, with regard to the addition of the decimal fractions 2.75+43.8, a student has already learned how to add 237+45 or 13.6+5.8. Attempt to create an analogy based on this previous knowledge.
With previous additions, the student would write the addition problems down as shown below, and add using the numbers aligned on the right side.

```
  237     13.6
+  45   +  5.8
-----   ------
```

If the student analogizes this form for the new problem and attempts to write the problem down with numbers aligned on the right, it will look like this:

```
 2.75
+43.8
-----
```

Of course, this is a mistake. Instead, the student now analogizes by aligning place value in the ones column when writing the problem before adding:

```
  2.75
+43.8
------
```

The action of then clarifying whether an analogy is correct or not is important.

3) Deductive Thinking

① Meaning

What is **Deductive Thinking**?
This method of thinking uses what is already known as a basis and attempts to explain the correctness of a proposition in order to assert that something can always be stated.

② Examples

Example 1:
Consider how multiples of 4 or 8 are arranged in the following number table (arrangements of every 4 numbers or every 8 numbers go without saying – look for other characteristic arrangements).
First, write down the multiples of 4 and the multiples of 8 in the number table. The bold and gothic numbers below are multiples of 4, and every other gothic number is a multiple of 8.

0	1	2	3	**4**	5	6	7	**8**	9
10	11	**12**	13	14	15	**16**	17	18	19
20	21	22	23	**24**	25	26	27	**28**	29
30	31	**32**	33	34	35	**36**	37	38	39
40	41	42	43	**44**	45	46	47	**48**	49
50	51	**52**	53	54	55	**56**	57	58	59
60	61	62	63	**64**	65	66	67	**68**	69
70	71	**72**	73	74	75	**76**	77	78	79
80	81	82	83	**84**	85	86	87	**88**	89
90	91	**92**	93	94	95	**96**	97	98	99

Once the student has written part of the number table, he/she can induce that “it is possible to move from one multiple of 8 to another by going down one row, and then left two columns.” When stated the same way for multiples of 4, it is also possible to induce that “it is possible to move from one multiple of 4 to another by going down one row, and then left two columns.”
Then considering “why it is possible to make this simple statement” and “whether or not it is still possible to state this for numbers over 99, and why this is the case” is deductive thinking. Next, consider what to base an explanation of this on. One will realize at this point that it is possible to base this on how the number table is created. This is also deductive thinking, and is based upon the following.
Since this number table has 10 numbers in each row, “going one position to the right increases the number by one, and going one position down increases the number by ten.”
Based upon this, it is evident that going down one position always adds 10, and going left two positions always subtracts 2. Combining both of these moves always results in an increase of 8 (10-2=8). Therefore, if one adds 8 to a multiple of 4 (or a multiple of 8), the result will always be a multiple of 4 (8). This explains what is happening.

By achieving results with one’s own abilities in this way, it is possible to gain confidence in the correctness of one’s conclusion, and to powerfully assert this conclusion. Always try to explain the truth of what you have induced, and you will feel this way. Also, think about general explanations based on clear evidence (the creation of the number table). This is deductive thinking.

Example 2: Deductive thinking is not just used in upper grades, but is used in lower grades as well.
Assume that at the start of single-digit multiplication in 3rd grade, the problem “how many sheets of paper would you need to hand out 16 sheets each to 8 children” is presented. When the children respond with “16×8,” the teacher could run with this response and say, “all right, let’s consider how to find the answer to this.”
This is not adequate, however. The students must be made to thoroughly understand the fundamental reasoning behind the solution. It is important that students independently consider “why this is the way the problem is solved.”
The child will probably explain the problem by saying that “in this problem, eight 16s are added: 16+16+16+16+16+16+16+16.” This is based on the meaning of multiplication

(repeated addition of the same number), and is a deduction that generally explains why multiplication is the way to solve the problem.
Furthermore, the response to “let’s think about how to perform this calculation,” will probably be “the answer when you add eight 16s is 128.” When asked the reason, the answer will probably be “this multiplication is the addition of eight 16s.”
Deductive thinking is used to explain this calculation and the foundation behind it.

③ Points to Remember Regarding Teaching

Establishing this need to attempt to think deductively is more important than anything. To do this, one must be able to use one’s own abilities to discover solutions through analogy or induction. Through this, students will gain the desire to assert what they have discovered, and especially to think deductively and appreciate the benefits of thinking deductively.
When one thinks deductively, an attitude of attempting to grasp the foundational properties one already possesses, and of clarifying what the conditions are, is important. For this reason, encourage students to consider “what kinds of things they understand” and “what kinds of things they can use.”
Also, when one thinks deductively, one uses both **synthetic thinking**, whereby one considers conclusions based on presumptions concerning “what can be said” based on what is known, as well as **analytical thinking**, whereby one considers presumptions based on conclusions concerning “what needs to be valid for that to be said.” Students should have experience using both methods of thinking.

4) Integrative Thinking

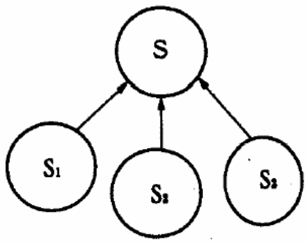
① Meaning

What is **Integrative Thinking**?
Rather than leaving a large number of propositions disconnected and separate, this thinking method abstracts their essential commonality from a wider viewpoint, thereby summarizing the propositions as the same thing.

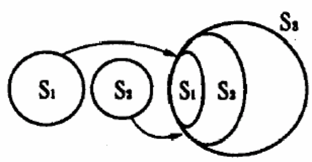
Integrative thinking does not always take the same form, but can be divided into three categories.

Type I Integration (High-Level Integration)

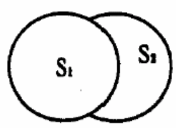
When there is a number of propositions (these can be concepts, principles, rules, theories, methods of thinking, and so on), this method of thinking views the propositions from a wider and higher perspective, and discovers their shared essence in order to summarize a more general proposition (S).



Type II Integration (Comprehensive Integration)
By reexamining a number of propositions S_1, S_2 and S_3, this type of thinking integrates S_1 and S_2 into S_3.



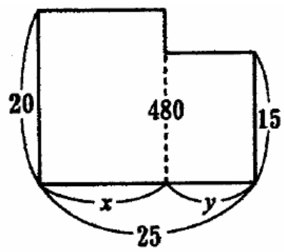
Type III Integration (Expansive Thinking)
In order to expand a certain known proposition to a larger scale that includes the original proposition, this type of thinking changes the conditions a little in order to make the proposition more comprehensive. In other words, this thinking incorporates and merges one new thing after another.



This is expansive thinking, which also includes developmental aspects.

② Examples

Example 1:



1. Someone purchased one type of stamps for 20 yen each, and another type of stamps for 15 yen each, paying a total of 480 yen. How many stamps of each type were purchased?
2. Boys took 20 sheets of paper each, and girls took 15 sheets of paper each. The total number of boys and girls was 25, and the total number of sheets of paper was 480. How many boys, and how many girls were there?
3. Something moved at a speed of 20 m/second at first, and then 15 m/second, for a total distance of 480 m in a total of 25 seconds. How many seconds did it move at each speed?

When one solves these problems individually, since they involve extremely different situations, they appear to be completely different problems. Once one draws an area figure in order to solve each problem, however, it becomes apparent that each problem can be summarized as the same problem. This is because it becomes evident that each problem has the same area figure as shown in the figure to the right, which corresponds to the integration of Type I Integration (S).

Example 2: Students are learning a lot about the multiplication and division of whole numbers, decimal fractions, and fractions. Learning each different method well and working with each type of number in different ways is somewhat cumbersome. Consider whether or not it is possible to summarize each different method of computation and understand them as a whole. This is integrative thinking. Thinking in this way makes it possible to express whole numbers and decimal fractions as fractions too, which enables the conversion of fraction division into fraction multiplication:

$3/4 \div 0.7 \div 8$ can be converted to $3/4 \div 7/10 \div 8/1 = 3/4 \times 10/7 \times 1/8$

In this fashion, the multiplication and division of whole numbers, decimal fractions, and fractions are all summarized as the multiplication of fractions. This is an example of Type II Integration, whereby other types of multiplication and division are integrated into the same level of fraction multiplication.

Example 3: Students are taught that the multiplication $a \times b$ involves bringing together sizes a in the number b and calculating the resulting total size. In this case, of course, neither a, which represents the size of one element, or b, which represents the number of elements, is 0. Therefore, although it is possible to represent, for instance, the number of points scored if "6 balls hit the 4-point target" with the multiplication as $4 \times 6 = 24$, it is not possible to use a formula to express "how many points are scored if no balls hit the 5-point target." The cases where either a or b is 0 are not included in multiplication. In order to eliminate this exception, since the context is the same for both the situation where one number is 0 as in "the score when 0 balls hit the 5-point target," and the situation where neither number is 0 as in "the score when 6 balls hit the 4-point target," by using the same multiplication operation, it is possible to extend the meaning of multiplication to include $5 \times 0 = 0$.
In this way, it is possible to express problems whether a number is 0 or not. In other words, the exception disappears. This is an example of Type III Integration.

③ Points to Remember Regarding Teaching

If multiple instances of the same thing are left as they are, then there is a cumbersome necessity to know about each different instance. Is there some way economize thinking and effort? Also, when there are exceptions, one must always think of them as something different, which is not very satisfying. Having students experience this is the first priority because it strengthens their desire to think in an integrative manner. The second priority is to ensure that students look at multiple things, and consider what is common to them all and how to see them as the same thing.

5) Developmental Thinking

① Meaning

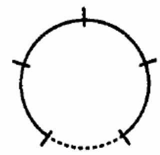
What is **Developmental Thinking?**
Developmental thinking is when one achieves one thing, and then seeks an even better method, or attempts to discover a more general or newer thing based on the first thing. There are two types of developmental thinking.
Type I Developmental Thinking: Changing the conditions of the problem in a broad sense.
By "changing the conditions of the problem," this means:
(1) Change some conditions to something else, or try loosening the conditions.
(2) Change the situation of the problem.
Type II Developmental Thinking: Changing the perspective of thinking

② Examples

Example 1: "20 trees are planted 4m apart along a straight road. How long is the road in meters? Note, however, that the trees are planted on both ends of the road." Assume that the student discovers that this tree-planting problem use the following relationship:
(space between trees) = (number of trees) – 1 ... (1)
By considering whether or not this relationship is only valid when the space is 4m, or when there are 20 trees, or when the road does not curve, it becomes apparent that it is indeed valid regardless of the space or number of trees, or even whether the road curves (see the following diagram). This is an example of Type I Developmental Thinking (1) as described above.

In both of the two cases above, there are 5 trees spaced 4m apart, and the relationship "(space between trees) = (number of trees) – 1" applies.
Furthermore, by using developmental thinking to consider what happens when the method of planting trees is changed, it is possible to develop the relationships when (2) trees do not need to be planted on one end, (3) trees do not need to be planted on either end, or (4) the road is circular:
(space between trees) = (number of trees) ... (2)
(space between trees) = (number of trees) + 1 ... (3)
(space between trees) = (number of trees) ... (4)

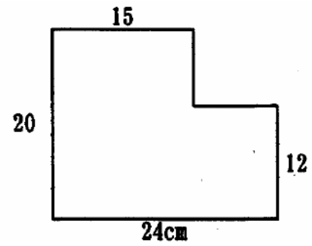
(4)



From relationship (1) above, we can see that:
In (2), the number of spaces increases by 1 from (1), so spaces = trees – 1 + 1 = trees
In (3), the number of spaces increases by 1 each for both sides,
so spaces = trees – 1 + 2 = trees + 1
In (4), the number of spaces between trees at both ends in (1) increases,
so spaces = trees – 1 + 1 = trees

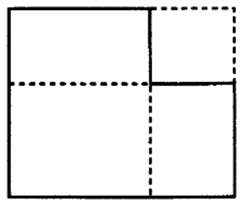
Situations (2), (3), and (4) can be developed from the original (1) in this way, after which the relationship between the different formulas can be summarized. This is an example of Type II Developmental Thinking as described above.

Example 2: The following equation can be used to find the area of the figure to the right:



20×15+12×(24-15)=408

Rather than being satisfied with this single method, however, continue by considering whether there is a different or better way. Also, by changing how you look at this shape and thinking in a developmental manner, it is possible to discover the following type of solution as well.



- (20-12)×15+12×24=408
- (20-12)×15+12×15+12×(24-15)=408
- 20×24-(20-12)×(24-15)=408
- 20×15+12×24-12×15=408

By changing one's perspective in this way, and reexamining the shape to consider different solving methods, it is sometimes possible to discover various different methods. This is an example of Type II Developmental Thinking.

③ Points to Remember Regarding Teaching

The basic philosophy behind teaching this type of thinking is to inspire students to seek better and new methods, and discover or create new problems.

Types I and II of Developmental Thinking involve "clarifying the conditions," "changing the conditions," and "strengthening or weakening part of the conditions," or "changing the situation" and "changing the domain." Also, if one can write a formula for a story problem or other such problem, or consider how to write a story problem for a formula, then one can take advantage of developmental thinking. Functional thinking and the "what if not?" technique (thinking about the case where something is not true) are effective here.

Also, have the students think back and clarify "what perspective has been used during consideration" then "reexamine based on this from a different perspective." An effective way to make students change their perspective is to change the perspective of the problem. For instance, try changing the constituent elements or operations of the figure in question to different constituent elements or operations. Also, even if one method or solution works, rather than being satisfied at this point, have the students try another method or attempt to discover a better solution. The basic method is to give students a certain proposition, and have them consider that proposition's opposite, or contraposition, or reverse.

Also note that not all of the above problems can be completely explored by integrative or developmental thinking. Attaining some kind of perspective of this type is essential to teach these thinking methods, however. When a solution is evolved, if the result is still based on the same type of thinking, and if it has the same structure, then it can be integrated. Integrating clarifies the essential conditions, and enables developmental speculation that can be used to discover further new problems and solutions.

Integrative thinking and developmental thinking stimulate each other in this way, and can be utilized in mutually complimentary ways.

6) Abstract Thinking

i. Thinking that abstracts
ii. Thinking that concretizes
iii. Thinking that idealizes
iv. Thinking that clarifies conditions

① Meaning

What is **Abstract Thinking**?
Abstract thinking is a method of thinking that, first of all, attempts to elicit the common properties of a number of different things.
Also, **thinking that concretizes** is also used in the end for abstracting propositions, so it is treated as the second type of abstract thinking and is included in our discussion.
Considering the ideal state where a variety of different conditions are constant, or ideal cases where conditions or properties satisfy mathematical definitions, principles, or rules, can often clarify the situation. Thinking about ideal states in this manner is referred to as **thinking that idealizes**, and is the third type of abstract thinking.
The fourth type of abstract thinking is an attempt to clarify conditions, which is necessary for abstraction.

② Examples

Example 1: Showing students a round top and telling them “this is called a circle” is not enough when it comes to teaching the concept of the circle. Since the top will have properties such as material, size, a painted pattern, and a method of use, the students will not yet ignore these aspects, and may think of, for instance, a round wooden top as a circle. The other properties must be ignored. Instead, show students tops of various different sizes, and have them also consider various other circular objects including cups, to elicit commonalities such as “all of these shapes have the same length from one point (the central point) to the edge.” Abstract thinking is used to clarify shared properties here.
These abstracted properties are referred to as the concept’s connotation.
Next, consider the concept from the opposite direction, and think of the objects that have these properties. For instance, have the students recognize the fact that large, round toys are also circles, as well as objects that only consist of the perimeter of a circle, such as rings. Have the students consider egg-shaped items and balls, and rings with one break, or other items that are not quite circular, so that they may determine whether or not these things are circles. Clarify what a circle is, and what is similar but not quite a circle. This process will make the concept of a circle clear.
The thinking method of concretization is used at this time to gather many different concrete examples, and to clarify the denotation, or extension of the concept. Continue finding various different properties of circles. This is also abstraction, and enriches the connotation of the concept.
In general, concepts include both connotations and denotations (extensions), the abundant clarification of which forms a concept. Thinking that involves abstraction (and elimination) is used to do this.

Example 2: If the meaning of the statement “when two numbers are added together, even if the order of the numbers is reversed, the sum remains the same” is not clear, try a concrete example, such as 3 and 5. The statement is now “when 3 and 5 are added together, even if the order of 3 and 5 is reversed, the sum remains the same,” so the meaning is easy to understand. By converting an abstract and general statement to a concrete statement, the meaning can be made obvious. This type of concretizing is important, and since the goal is actually abstraction, it can be included as a type of abstract thinking.

Example 3: 4th grade students can be taught the computational laws of multiplication for whole numbers in the following way:

○×□=□×○
(○×□) ×△=○×(□×△)
○× (□+△)=○×□+○×△

For instance, consider concrete examples using these laws for explaining the distributive law. It is possible to explain that "if a certain number of flowers is planted in a rectangular shape, if ○ is the number of columns, □ is the number of rows of red flowers, and △ is the number of rows of white flowers, then both sides express the total number of red and white flowers, and this equation is valid." This is also an example of concretization.
When 5th grade students learn the multiplication of decimal fractions, consider whether or not the above rules apply to decimal fractions as well. With this equation, it is not possible to consider whether or not the rules apply, so try concretization.
In other words, try replacing ○, □, and △ with specific numbers such as 2.5, 3.7, and 1.8, and examine whether or not the relationship holds true. This way of thinking is an example of concretization.

Example 4: To compare the dimensions of two cups, fill one cup with water, and then pour the water into the other cup. Although the first cup might not be completely filled, or there might still be some water left in the first cup after pouring, it is necessary to imagine that the first cup has been completely filled, and that all the water was poured into the other cup. Idealization is used to do this.
This is also included in abstraction because some conditions are eliminated, and other condition is abstracted.

Example 5: Even with the extremely simple problem "of two people, A and B, whose house is closer to the school?," if conditions such as A) compare not by straight distance, but by distance along the roads, B) assume that the person who walks to school in the shortest time is the closest, C) both people walk at the same speed, or D) the walking speed is around 60 meters/minute, it is possible to make comparisons based on the actual time walked by each person, as well as which house is closer in meters.
This type of thinking is important when it comes to the clarification of conditions because it is used to abstract and clarify conditions from many different conditions, or to clarify conditions in order to make them harder to forget.

③ Points to Remember Regarding Teaching

- When there are a number of different things, the first priority is to clarify the perspective of consideration, or "what are we examining?" Furthermore, have the students consider "what is the same, and what is shared?" as they abstract points in common. At the same time, students must be made to consider "what is different?," thereby clarifying points that are different, and don't need to be considered at this time. These points are then ignored. In other words, this type of thinking clarifies what can be ignored.
  Furthermore, this type of thinking is not limited to just abstraction, but also involves the concretization of "finding other new things that have in common." This clarifies what has been abstracted even further.
- When a problem is solved, the first thing to do is understand the meaning of the problem. In other words, the right attitude of grasping the problem clearly is important. To take this kind of attitude, it becomes necessary to think how to clarify "what the conditions of the problem are," "whether or not the conditions are

sufficient, insufficient, or too numerous to solve the problem," as well as "what is sought." This type of thinking attempts to clarify the conditions referred to here.

7) Thinking that Simplifies

① Meaning

Thinking that Simplifies 1: Although there are several conditions, and although one knows what these conditions are, when it is necessary to consider all of the conditions at once, sometimes it is difficult to do this from the start. In cases like this, it is sometimes beneficial to temporarily ignore some of the conditions, and to reconsider the problem from a simpler, more basic level. This type of thinking is referred to as "thinking that simplifies."
Thinking that Simplifies 2: Thinking that replaces some of the conditions with simpler conditions, is also a type of thinking that simplifies.

Keep in mind, however, that general applicability must not be forgotten during the process of simplification. Although the problem is simplified, there is no point in simplifying to the extent that the essential conditions of the original problem or generality are lost. This applies to idealization as well.

② Examples

Example 1: If the following problem is difficult, try considering each condition, one at a time: "if 4 pencils are purchased at 30 yen each, along with 6 sacks at 20 yen each, what is the total cost?"

- The cost of 4 pencils at 30 yen each
- The cost of 6 sacks at 20 yen each
- The cost of both pencils and sacks

By simplifying the problem in this way, one can think of the equations as follows:

$30\times4=120$
$20\times6=180$
$120+180$

This makes it easy to realize that the solution is equal to $30\times4+20\times6$

Example 2: If the computations necessary to solve the following problem are not clear, one can replace 36.6 or 1.2 with simple whole numbers to simplify the problem: "If A weighs 36.6 kg, which is 1.2 times as much as B, how much does B weight?" For instance, try converting this to "A weighs 36 kg, which is 2 times as much as B." Obviously, this would be $36\div2$. Once one understands this simple version of the problem, it is possible to extrapolate that the original problem can be solved by calculating $36.6\div1.2$.

③ Points to Remember Regarding Teaching

When considering story problems with large numbers and decimal fractions or fractions and so on, or story problems with many conditions, if the numerical relationships are obscured by the size of numbers or the large number of conditions, have the students think about "why the problem is difficult" and "what can be done to make it understandable" so that they realize where the difficulty is (for instance, complicated numbers or conditions). Have them try

“replacing numbers with simple whole numbers” or “thinking about the conditions one at a time.”
The goal of incorporating this type of thinking into the process of teaching a class is to teach children how to proceed on their own, so that they think of simplification on their own initiative.
The previous section on idealization is very similar in this respect.

8) Thinking that Generalizes

① Meaning

What is **Thinking that Generalizes?**
This type of thinking attempts to extend the denotation (the applicable scope of meaning) of a concept. This type of thinking also seeks to discover general properties during problem solving, as well as the generality of a problem’s solution (the solving method) for an entire set of problems that includes this problem.

② Examples

Example 1: Create a times table for say, the 3s:

$3\times2=3+3=6$
$3\times3=3+3+3=9$
$3\times4=3+3+3+3=12$

To avoid the hassle of repeatedly the repeated additions, however, try to find a simpler method. One discovers that the previously mentioned method can be reexamined revealing that “the answer goes up by 3 with each number” (inductive thinking). And this can be generalized and applied as follows:

$3\times5=12+3=15$
$3\times6=15+3=18$

Next, consider whether or not the same kind of rule can be applied for the 4s, or the 6s (analogical thinking). Verify this assumption and then use it. This is an example of generalizing this rule to “when the multiplier increases by 1, the answer increases by that number.”
Generalization uses inductive and analogical thinking in this way.

③ Points to Remember Regarding Teaching

When teaching how the addition problem 5+3 can be used, for instance, try having the children create a problem that uses this equation. This gradually leads to the generalization of the meaning of addition. As the meaning of one concept is understood, making new problems is important in that it teaches about the kinds of situations to which these concepts can be applied, and encourages students to seek other situations to which they can apply. Concepts are thus gradually generalized in this manner.
Properties, rules and other factors are often generalized through the use of multiple concrete cases in this way. In this type of situation, examining numerous individual cases is cumbersome. It is necessary to make the students wonder if there’s a better way. The basic way to do this is to actually have them solve various different concrete problems. This will lead the students to wonder if “there’s a simpler way” or “if a helpful rule can be found.”

Generalization includes cases where one example is generalized, as well as cases where generalization is considered first, and then applying it to a special case. Teach this repeatedly until the children "think what can always be said" and "think of rules that always apply."

9) Thinking that Specializes

① Meaning

Thinking that specializes is a method that is related to thinking that generalizes, and is the reverse of generalization.

What is **Thinking that Specializes?** In order to consider a set of phenomena, this thinking method considers a smaller subset included in that set, or a single phenomenon in that set.

The meaning of specialization is clarified by thinking about when it is used and how it is considered.

Thinking that specializes is used in the following cases: 1. By changing a variable or other factor of a problem to a special amount without losing the generality of the problem, one can sometimes understand the problem, and make the solution easier to find. 2. By considering an extreme case, one can sometimes attain a clue as to the problem's solution. The result of this clue or method can then be used to assist in finding the general solution. 3. Extreme cases or special values can be used to check whether or not a possible solution is correct.

② Examples

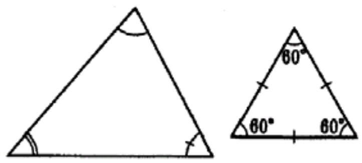
Specialization is often used to assist with generalization. Therefore, the examples provided here also take advantage of thinking that generalize in many places.

Example 1: The question of whether or not the following applies to fractions as well is given as a 6^{th} grader's problem: ○×(□+△)=○×□+○×△
If students have trouble understanding the meaning of this problem, then the first priority is naturally to make them understand the meaning of this problem. To do this, try replacing ○, □, and △ with special numbers. For instance, try 1/2, 1/3, and 3/4. This allows the students to check whether or not the following equation is true:

1/2×(1/3+3/4)=1/2×1/3+1/2×3/4

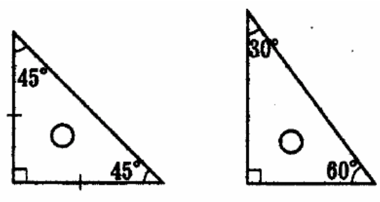
At this point, the students will understand that the problem is to figure out whether or not this rule always applies, no matter what fractions are used.
By trying out special numbers, one can understand the meaning of a problem.

Example 2: It is important for 5^{th} graders to gain the perspective that "if you try collecting three angles, it looks like it will work out" when looking for the total of all three inner angles in a triangle. Students gather angles based on this perspective.
Specialization is useful for achieving this perspective. Try considering special cases for triangles, as describe below.
Each angle in an equilateral triangle is 60°, so the sum of

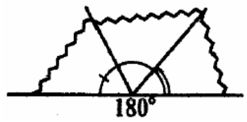


the three angles is 180°.

In the case of two types of triangle rulers, the angles are 45°, 45°, and 90°, or 30°, 60°, and 90°. Both of these also add up to 180°.



Since the totals were 180° for all of the special cases, one can infer that the total is also 180° generally for all triangles. Also, since 180° is the angle of a straight line, if one collects all three angles and brings them together, one would predict that a straight line will result. This leads to the idea of trying to collect all three angles and place them on a single point.



③ Points to Remember Regarding Teaching

In the above example, teachers often start out by telling students to "cut the angles off the triangles and bring them together at a single point." The problem with this method is students do not understand why they are collecting the three angles. If the total was 170° or 200°, rather than 180°, however, then the teacher would not tell the students to do this.
By gaining the perspective that the three angles of a triangle might add up to 180°, one understands why collecting the angles in this way is a good idea. This is why thinking about "known (simple) special cases" is important for developing a perspective regarding "how it looks" or "what can be done."
The attempt to <u>find a perspective</u> on general properties or rules can lead to an attitude of thinking about specialization in certain cases.

10) Thinking that Symbolize

① Meaning

What is **Thinking that Symbolize?**
Thinking that symbolize attempts to express problems with symbols and to refer to symbolized objects. This type of thinking also includes the use and reading of mathematical terms to express problems briefly and clearly. This type of thinking proceeds one's thought based on the formal expression of problems.

② Examples

Example 1: The formula for the area of a triangle is "area = base × height ÷ 2," but always writing this out in full is a hassle. One can therefore write this simply as "a=b×h÷2." This is thinking that symbolizes.

Example 2: If, while teaching addition, with the example of "3 and 5 make 8," students only write the answer 8, he or she will not know what the original amounts were, or what operation was used to result in 8. In order to clearly express this, it is necessary to use 5, 3, and 8, as well as a symbol to express the operation used. The attitude and necessity of attempting <u>to express things more clearly</u> reveal the benefits of thinking that symbolize.
In other words, by writing the equation "3+5=8," one can communicate the understanding that bringing 3 items and 5 items together results in 8 items. This equation succinctly and clearly expresses the idea that this 8 did not come into being through the addition of 6 and 2, for instance.

③ Points to Remember Regarding Teaching

The points to remember regarding teaching are described in the above examples. This section describes how to understand the benefits of encoding, and how to take advantage of these benefits in teaching.
The advantage of using terms and symbols is that one can develop thoughts without the need for returning to or being restrained by the concrete. Furthermore, when, for instance, the range of numbers is expanded to include decimal fractions, the meaning of multiplication is also expanded. In other words, the meaning of the symbol “×” is expanded. When the meanings of concepts or operations are expanded or integrated, the understanding of the terms and symbols that express them must also be expanded or integrated in the same way. The formality of symbols sometimes plays an important role in this expansion and integration.
Thinking that uses terms and symbols in this way is effective when used for the following purposes:

- To express things clearly and succinctly
- To think in an organized fashion, with intellectual rigor
- To generalize thinking

Furthermore, by using terms and symbols, it is possible:
To proceed with formalized thought
When one uses terms and symbols, properly determining the meanings of those terms and symbols, using them in making correct judgments, and acting methodically, are all important. The lower the grade level, the more important it is to fully consider the crucial role played by operations in the regulation of the meaning of terms and symbols.

11) Thinking that Express with Numbers, Quantifies and Figures

① Meaning

Rather than giving children only numerical expressions, and simply teaching them how to process the numbers, it is necessary for them to start at the stage before quantification, and to have them think about how to quantify the information.

What is **Thinking that Express with Numbers and Quantifies?** This thinking takes qualitative propositions and understands them through qualitative properties. Thinking that selects the appropriate quantity based on the situation or objective is thinking that express with quantities. Thinking that uses numbers to express amounts of quantities is thinking that express with numbers. Conversion to numbers makes it possible to succinctly and clearly express amounts, thereby making them easy to handle. These types of thinking are summarized and referred to as “thinking that express with numbers and quantifies.”

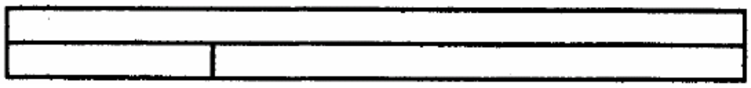
In addition to quantification, thinking that expresses problems with figures is also important.

What is **Thinking that Express with Figures?** This thinking replaces numerical propositions and the relationships between them with figures. Situations, propositions, relationships, and so on are replaced with figures and the relationships between them. This type of thinking is referred to as “thinking that expresses with figures.”

② Examples

Example 1 (Expressing with numbers): Comparison of the two lengths A and B revealed that [A is slightly longer than B]. “Slightly longer” does not tell us the exact difference, however. Therefore, in order to express this difference a little more clearly, consider expressing the extent indicated by “slightly longer” with a number. This leads to learning how to express remainders in measurements with fractions and decimal fractions, as well. In the same way, one-to-one correspondence allows for the comparison of numbers of objects, and thinking that uses numbers to express the extent of differences is also a form of expressing with numbers.

Example 2 (Diagramming): When the 2nd grade problem “you use 25 yen, and have 64 yen left; how much did you start with?” is expressed with a tape diagram as shown below, it clarifies the fact that the answer is to be found with 25+64, rather than through subtraction.



In other words, thinking that expresses problems as diagrams is useful for deciding which operation to use.

③ Points to Remember Regarding Teaching

Quantification sometimes involves situations such as “a noise too loud to sleep.” In this situation, the definition of “too loud” varies from person to person, and the determination of what level of noise is “too loud” cannot be made objectively. By expressing the loudness of a noise quantitatively, it is possible to compare noises, and operations become clear. The judgment regarding the crowdedness of a train is also subjective, and varies depending on the person. Crowdedness can be expressed using numbers, such as 150% or 120% of capacity. This is one example of expressing with numbers. Thinking that considers the number of people per unit area represents another idea of expressing with numbers that can be used to quantify how crowded a train is. Another example of this type of thinking is the use of equality of corresponding angles, as a way to quantify “parallel.” Quantification can be used in many different situations. It is important to expose students to actual situations that evoke an understanding of how subjective and qualitative thinking can be insufficient, in order to teach them the benefits of quantification.

Instead of teaching diagramming by telling students to “express this type of thing as this kind of figure,” it is important to give them diagramming options to select based on the situation, such as line segment figures for problems that involve addition or subtraction, or area diagrams for problems that involve multiplication or division, or number lines for problems that express general relationships between amounts.

Diagrams have the characteristic of appealing to the sense of vision to express phenomena in such a way that they are easier to understand, and so thinking that attempts to actively use graphs and figures (line segment figures, area figures, tree diagrams, number lines, geometric figures, and so on) must be cultivated. The length of the line segments, areas, and so on need not precisely express the sizes of what they represent.

Since figures and diagrams are meant to express problems in a manner that is easy to understand, and are used to grasp the relationships between amounts, they may ignore actual sizes. It is important to consider what is being abstracted in the representation of a figure, based on the particular objective.

Chapter 5

Detailed Discussion of Mathematical Thinking Related to Mathematical Contents

1) Clarifying Sets of Objects for Consideration and Objects Excluded from Sets, and Clarifying Conditions for Inclusion (Idea of Sets)

① Meaning

i) Clearly grasping the object for consideration
This is an important aspect of the idea of sets. For instance, when one counts objects, it is not enough to simply count. It is important to first achieve a solid grasp of the scope of objects to be counted. Also, when grasping a concept such as the isosceles triangle, it is important to determine and clearly indicate the scope of objects under consideration (just one printed triangle, a number of triangles created with sticks, or any triangles one can think of with the presented triangles simply offered as examples).

ii) Consider whether or not objects under consideration belong to a certain set based on names or conditions, with an awareness of the fact that names or symbols are being used to express the set. Clarify which objects do not belong to the set in order to improve the clarity of the original set.

iii) When grasping a set of objects, be aware that there are methods of indicating members, and methods of indicating conditions for entry into the set. Use these different types of methods appropriately.

iv) Maintain as comprehensive a perspective as possible, bringing as many objects as possible together and treating them as the same thing, so that they can all be considered collectively.

v) Thinking that sorts into classifications

Follow these procedures to sort:
1. Clarify the scope of objects to be sorted.
2. Decide upon a perspective regarding classifications that matches the objective.
3. It is important that the perspective is one that places every object into a specific category, with no single object belonging to two different categories, and that objects can be sorted without dropouts or overlapping.
4. Find as many conditions as possible for representing classifications, and consider the value of these classifications.
5. One can sometimes combine a number of categories into larger classifications.

② Examples

Example 1: Teachers sometimes teach students that “a parallelogram is a quadrilateral with two sets of parallel sides facing each other,” and then distribute print-outs showing a parallelogram, saying “what kinds of characteristics does this quadrilateral have?” and “measure the length of the sides, compare the angles, and examine the properties.” Once the students are finished with their examination, the teacher will explain the properties, stating “as you can see, in a parallelogram, the lengths of the facing sides are equal, as are the facing angles.” This type of teaching is absolutely inadequate.

The reason for this inadequacy is that the above properties are not limited to a single parallelogram, but are rather the properties of all parallelograms. Students must be made to consider as many different parallelograms as possible, so that they see these properties as common to all parallelograms. This is why it's important to consider as many different parallelograms (sets) as possible. This is an important point behind the idea of sets. An even more important point is that even if one only considers parallelograms, the possibility remains that other quadrilaterals might also have these properties. This makes it important to take into account objects that are not a part of the parallelogram set for the sake of comparison. This clarifies further the fact that these are the properties of parallelograms. Examining objects that do not belong to a set is another important part of the idea of sets.
Of course, this applies to the meaning of parallelograms as well. After one abstracts the property "facing sides are parallel," coming up with a name will make pinning down this kind of quadrilateral easier. This is also an idea of sets.

③ Points to Remember Regarding Teaching

It's important to pay attention to the kind of set that things belong to, whether they are "objects" such as numbers or diagrams, "problems" such as addition, or the "methods" used to perform these calculations. This provides one with a general grasp, and makes it possible to deepen one's level of understanding.
Comprehending sets makes the conditions for elements in these sets clear, which enables logical consideration in turn.
Commit to classification in each of the various stages listed above.

2) Focusing on Constituent Elements (Units) and Their Sizes and Relationships (Idea of Units)

① Meaning

Numbers are comprised of units such as 1, 10, 100, 0.1, 0.01, as well as unit fractions such as 1/2 and 1/3, and are expressed in terms of "how many units" there are. Therefore, focusing on these units is a valid way of considering the size of numbers, calculations, and so on. In addition, it goes without saying that amounts are expressed with various units such as cm, m, l, g, and m^2, and that tentative unit can be used. Therefore, when one considers measuring the amount of something, it is important to pay attention to the unit. Also, figures are comprised of points (vertices), lines (straight lines, sides, circles, and so on), and surfaces (bases, sides, and so on). For this reason, thinking that focuses on these constituents, unit sizes, numbers and interrelationships, is important.

② Examples

Example 1: How to multiply a fraction by a whole number
A variety of different methods can be considered for multiplying 4/5 × 3. One possible method is to "focus on the unit fraction and see 4/5 as 4 times 1/5" (this is when the idea of units is applied). Based on this, the answer is 4 times 3 times 1/5, or (4×3) times 1/5. This is the same as the following:

$$\frac{4\times3}{5}$$

Example 2: Given the problem "draw a square and a parallelogram with all four vertices on a circle," first focus on the constituent elements of squares and parallelograms. Next, consider

which constituent elements are best focused on, namely the relationships with a circle in this case. Diameters and radii are easy to use when it comes to circles. Consider focusing on the diagonal (constituent element) that seems to have the closest relationship with the elements of the circle (this is idea of units).
Next, think about all the things that can be used with respect to the square or parallelogram's diagonals. Also note that the diagonals of squares have the same length, and that their mid-points meet perpendicularly.
This makes it evident that one should draw diameters that meet perpendicularly. Since the diagonals of parallelograms do not have the same length, however, it becomes apparent that their four vertices cannot be drawn upon the same circle.

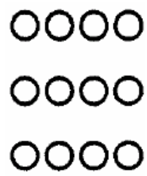
③ Points to Remember Regarding Teaching

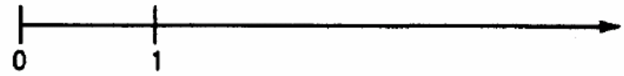
What are seen as the units used for these numbers (quantities, figures)? It is important to ensure that students look for the units (constituent elements) and their relationships that they need to focus on.
Furthermore, one must consider which unit to use, as in the case where tentative unit can be used to examine a width. In the case of "3/4 is what times of 2/3?," 1/4 and 1/3 are the units. Students must be made to see the need to "look for the units," and furthermore to "try changing units to something easy to compare, while considering what should be changed to the same unit (fraction)."

3) Attempting to Think Based on the Fundamental Principles of Expressions (Idea of Expression)

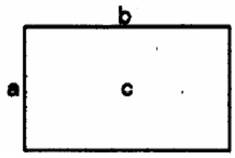
① Meaning

Whole numbers and decimal fractions are expressed based on the decimal place value notation system. To understand the properties of numbers, or how to calculate using them, one must first fully comprehend the meaning of the expressions of this notation system. The ability to think based on this meaning is indispensable. When it comes to fractions as well, one must be able to see 3/2 as a fraction that means a collection of three halved objects, or the ratio of 3 to 2 (3÷2).
It is necessary to consider measurements of amounts based on the definition of expressing measurements with two units, such as when 3 l and 2 dl is written as "3 l 2 dl," as well as the definition of writing measurements with different units, as in the case where "10,000 m^2 is written as 1 ha."
Also, in order to achieve a concrete grasp of the set of numbers, it is necessary to express numbers in a variety of different concrete models, and to take advantage of knowledge about the definition of these expressions.
There is a model referred to as the "number line" that is used for expressing numbers. This involves placing an origin point (0) on a straight line, determining the unit size (1), and using this to correspond numbers to points on the line. Use this model based on the definition of this expression.


(Array figure)



There are many other models in addition to this one. For instance, array or area figures can be used to model a×b=c.



(Area figure)

These figures show a number of circles lined up in a rectangular shape, or a “relationship between a rectangle’s height, width, and area.” Thinking that correctly understands the definitions of these expressions and takes advantage of them effectively, is important.

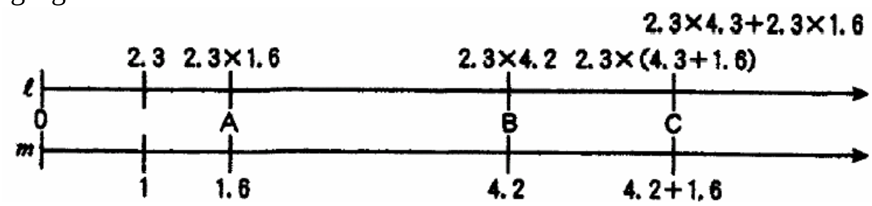
② Examples

Example 1: When 2nd grade students compare the size of numbers such as 5,897 and 5,921, they are basically “thinking based on the meaning of number expressions (the decimal place value notation system).”
The numerical expressions of the decimal place value notation system are based on the following principles:

- Every time ten of the object selected as the unit are gathered, this is expressed with a new unit.
  (Principle of the decimal notation system)
- The size of a unit is expressed by the position of the number that indicates how many of that unit there is.
  (Principle of the place value notation system)

This thinking makes it evident that one can judge in this way: “First, the number with the most digits is larger. If the number of digits is identical, compare the digits of the number starting with the highest unit. The number with the first higher digit is the larger number.”

Example 2: When one attempts to clarify the fact that “the distributive law applies to decimal fractions as well,” it is not sufficient to simply calculate both 2.3×(4.2+1.6) and 2.3×4.2+2.3×1.6, and state that “since both give the same answer, the distributive law applies.” The reason is this is just a single case of arriving at the same answer and the induction that the law always holds based on a single example. No matter how many examples one has, this is still just induction.
Therefore, consider expressing this calculation on a number line. This is idea of expression. The following figure is the result.



(Note: The length of 1.6=OA=BC on m; C on l is 2.3×(4.3+1.6); this is also 2.3×4.3+2.3×1.6)

Once the points 2.3, 4.2, and 1.6 are determined as described above, the point for 4.2+1.6 is found as point C on l by moving 1.6 to the right from 4.2. Therefore, the point on l corresponding to this point is:

2.3×(4.2+1.6)

Also, 2.3×4.2 and 2.3×1.6 are the points B and C on l corresponding to 4.2 and 1.6 on m. Therefore, the point on l corresponding to 4.2+1.6 on m is:

2.3×4.2+2.3×1.6

This is obviously the same point as 2.3×(4.2+1.6). Therefore, the distributive law holds. Since the same can be said no matter how these three numbers are changed, it is proven that the distributive law applies generally.
It is important to consider the meaning of addition and multiplication while using a number line in this way.

③ Points to Remember Regarding Teaching

The principles of expressions based on the decimal place value notation system are used often. The selection of appropriate expression methods for problem solving, such as number lines, line segment diagrams and area figures, is important, as is the appropriate reading of these expressions. It is important to have students think along these lines:

- What kinds of expressions are there? Let's try them out.
- What do they (the expressions) say?

4) Clarifying and Extending the Meaning of Things and Operations, and Attempting to Think Based on This (Idea of Operation)

① Meaning

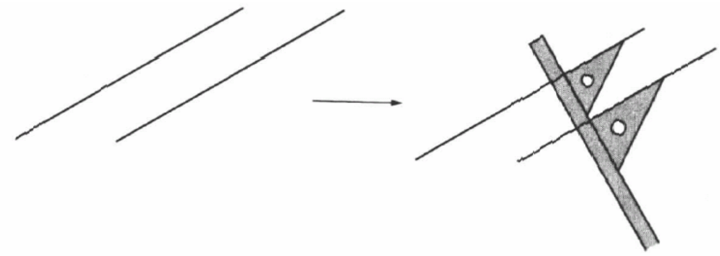
The "things" referred to here are numbers and figures. For instance, what does the number 5 express? How do you clarify the meaning (definition) that determines what a square is? Also, consider numbers and figures based on this meaning.
"Operations" refers to formal operations that are used for counting, the four arithmetic operations, congruence, expansion and reduction (similar), the drawing of figures, and so on. These operations are used to calculate with numbers, to think about the relationship between figures and how to draw them in one's head.
When are computation such as addition used? Since the meaning (definition) is precisely determined, the decision of operations naturally follows from the meaning (definition) of computation, along with the methods and properties of computation. Also, the properties and methods of drawing figures, as well as relationships with other figures, are originally clarified based on the meanings (definitions) of those figures.
As the scope of discussion expands from whole numbers to decimal fractions and fractions, operations on these numbers also become applicable to a wider scope, and so their meanings must also be extended.
Make sure the meanings of things and operations are concrete. It is important to think about properties and methods based on these meanings. These thoughts must follow when one is thinking axiomatically or deductively.

② Examples

Example 1: One must think of considering the meaning of addition when deciding whether or not it should be used to solve a particular problem.
When trying to solve the problem "a basket has 5 bananas, 5 tomatoes, and 4 apples; how much fruit is there?," one attempts to judge it based on keywords such as "altogether" or "total." In this case, however, there are no such keywords, so this method of judgment will not work. It is necessary to consider the problem and judge based on the meaning of addition, and whether or not it applies in this case.
To do this, it is necessary to clarify the meaning of addition by concrete operation with fingers or with a tape diagram.

Example 2: The calculation 12×4 can be solved as 10×4=40 and 2×4=8, which when added together result in 40+8=48. This can be explained in the following way: "12×4=12+12+12+12, which is done by first adding four 10s, followed by four 2s. This can be written as 10×4 and 2×4. Adding both of these answers together can be shown as 10×4+2×4." This conclusion results from first considering the meaning of the multiplication 12×4, and then thinking based on this. The addition of four 10s is also based on the meaning of multiplication, which is "repeatedly the repeated additions," and is expressed as 10×4. In other words, the idea of "thinking based on the meaning of multiplication" is key here.

Example 3: Consider the problem "are the lines in the figure to the lower left parallel straight lines?" One can tell whether the lines are parallel or not by looking at them. If visual judgment regarding whether or not the pair of lines is parallel is accepted, then this problem is solved. Of course, this is not enough. It is important to explain "why one can state that the lines are parallel."
One must consider how to explain one's judgment based on the meaning of "parallel." Thinking in this way causes one to use the property of parallel (as in the diagram to the right, "judgment based on the manipulation using two rulers").



Example 4: In 5th grade multiplication of decimal fractions, the meaning of multiplication must be extended.
The meaning of multiplication is "repeated addition of the same number" up until the multiplication of whole numbers by decimal fractions. In other words, "multiplication, or a×b, is used to add the same number a, b times." In the case of problems such as "how many kg is a 3.4 m bar of iron that weighs 2.4 kg per 1 m?," one can no longer explain the problem as repeated addition of the same number (added 3.4 pieces of 2.4s). In order to explain how multiplication is useful in this situation as well, one must expand the meaning of multiplication.
This can be done on the same number line that we used to express our thinking in Example 2.

③ Points to Remember Regarding Teaching

The most crucial point here is to emphasize thinking that judges and explains based on a solid foundation. Your explanation should inspire students to consider the following:
"Why is that correct?"
"How can we explain this?"
"What can we use?"
"What is the basis of this?"
The meanings of each number and figure act as the base, as well as the meaning of operations such as addition, and the properties of objects such as computation and figures. Thinking that attempts to understand these things well and always uses them, is important.

5) Attempting to Formalize Operation Methods (Idea of Algorithm)

① Meaning

Formal calculation requires one to have a solid understanding of methods, and the ability to mechanically perform calculations based on this understanding without having to think about the meaning of each stage, one after the other. This allows one to conserve cognitive effort, and to easily execute operations. This also applies to measurements and drawing figures.
The mechanical execution of a predetermined set of procedures is referred to as using an "algorithm." Thinking that attempts to create algorithms based on an understanding of procedures, is important.

② Examples

Example 1: When stating the populations of a number of towns, differences of a few hundred people are not problematic and so round the populations to the nearest thousand. Express each population as the nearest thousand, such as 23,000, 24,000, or 25,000. 23,489 is closer to 23,000 than to 24,000, so it would be expressed as 23,000 people.
Since 23,510 is closer to 24,000 than to 23,000, it would be expressed as 24,000 people. Expressing numbers as the closest unit number in this fashion is referred to as "rounding to the nearest number."
Thinking of rounding based on this meaning is thinking based on the meaning of 3). Once one departs from this point, however, and goes on to discover that "this process involves discarding the remainder when the next digit lower than the target position is 4 or less, or rounding up when this digit is 5 or higher," the next step is to convert this method to an algorithm. This has the benefit of allowing one to mechanically apply the method of rounding without considering its meaning each time. This way of thinking is the idea of algorithm.

③ Points to Remember Regarding Teaching

This thinking, which aims to create and execute algorithms, is important. Note, however, that teaching this does not involve first teaching the algorithm, and then simply having students practice using it.
It is necessary to have students think clearly about reasons, and to understand them well. The process of executing algorithms based on this understanding is aimed at saving more effort, and further improving efficiency. This shows the benefits of using algorithms. Teaching is centered on the goal of making students understand these benefits. Students first gain the ability to apply algorithms effectively when they understand their benefits, and this understanding lets them treat any errors that might arise with algorithms by using their own abilities.

6) Attempting to Grasp the Big Picture of Objects and Operations, and Using the Result of this Understanding (Idea of Approximation)

① Meaning

A general understanding of results is effective for establishing a perspective on solving methods or on results, and for verifying results. By attaining a grasp on approximate numbers, amounts, or shapes, or doing approximate calculations or measurements, one can establish a perspective on results or methods, and verify results. This is the idea of approximation.

② Examples

Example 1: When expressing the populations of a number of cities on a bar chart, one uses approximations rounded to the nearest ten thousand, or the nearest thousand people. Given that one knows the largest population is 243,000 people, it is possible to establish the perspective that graph paper with a length of 25 cm can be used, with each 1 mm square corresponding to 1,000 people. Thinking that first attempts to express problems based on approximate numbers in this way, is necessary.

Example 2: One can infer based on analogy with previous calculations that the multiplication of decimals 2.3×4.6 is done by first finding the product 1,058 of 23×46, and moving the decimal point to the correct position. One can also establish a perspective with a rough calculation by treating the numbers as 2 and 3, revealing that the answer is approximately 6 (2×3=6).
This type of thinking allows one to establish the perspective that the solution is probably 10.58.

Example 3: When students do addition of fractions, they tended to make the following mistake:

2/3+3/5=(2+3)/(3+5)=5/8

To avoid this, have them think about establishing a perspective on the result.
3/5 is smaller than 3/3, so the answer must be smaller than 2/3+3/3=5/3. On the other hand, 2/3 is larger than 2/5, so the answer must be larger than 2/5+3/5=5/5=1. This perspective makes it clear that 5/8 is a mistake because it is smaller than 1.

Example 4: In order to create a cube or rectangular parallelpipe, draw a developmental figure of a cube or rectangular solid on a piece of paper. If one attempts to just start drawing with a ruler, the development may end up too small, or may not fit on the paper. To avoid this problem, start out by drawing a freehand approximation first. Next, consider whether the size of the diagram is appropriate and correct. By establishing a perspective with a rough figure in this way, one can conserve cogitative effort, and draw the desired figure.

Example 5: When measuring length or weight, by taking a rough measurement first, one can decide what to use as a ruler or scale based on this perspective.

③ Points to Remember Regarding Teaching

As the above examples show, it is important to have students think about the following:
“Establishing a perspective on amounts.”
“Establishing a perspective on methods to be applied.”
“Is there a large mistake in the answer?”
This way, students will learn to think based on the idea of approximation, and will attempt to use approximate numbers or rough calculations.
Teach students to develop a habit of establishing a perspective before going to work. Even if one achieves an approximate grasp, unless it is used, the effort is wasted. In such cases, students will gradually stop applying this type of thinking.
After one reaches a solution, one must check whether or not there is a major difference between the solution and the approximate size or shape. It is also important to have students consider whether they can discover a new method based on the general understanding of results. For instance, in the above example, once one has come up with the estimation that “2.3×4.6=10.58,” one can take advantage of this. This will give one the idea to go on to calculate 23×46 and to count the places under the decimal point for both numbers and add them together to discover the number of places under the decimal point for the solution. One then proceeds to consider the correct reason for the correctness of this conjecture.

7) Focusing on Basic Rules and Properties (Idea of Fundamental Properties)

① Meaning

Calculation involves rules such as the commutative law, as well as a variety of properties such as “in division, the answer is not changed when one divides both the divided and dividing numbers by the same number.” Also, numbers have a variety of different properties such as multiples and divisors.

Furthermore, figures and shapes have properties such as parallel and equal side lengths, area formulas, relationships between the units of amount, amount properties, proportional/inversely proportional amounts, and other numerous arithmetical or mathematical rules and properties. One must consider finding these, selecting the appropriate ones, and using them effectively.
Thinking that focuses on these basic rules and properties, is therefore absolutely indispensable.

② Examples

Example 1: For instance, when one solves the problem “draw a square inscribed in a circle” (a square with four vertices on a circumference), one takes advantage of thinking that focuses on basic properties, paying attention to what properties there are. One then realizes that it is possible to use the fact that “if diagonals have the same length and bisect each other perpendicularly, then the figure is a square.” In other words, one understands that it is possible to “draw two diameters that cross each other perpendicularly, and connect each of the four resulting intersection points on the circle” to solve this problem.

Example 2: To “draw an axisymmetrical shape,” first attain a grasp of the approximate shape, then think of using the basic properties of axisymmetry. You will then realize that the property of “a straight line connecting two corresponding points is bisected perpendicularly by the axis of symmetry” can be used.

Example 3: One can infer the method of calculating 3/5÷4 by analogy with multiplication, as shown below on the left side. 3÷4 cannot be completely divided, however. At this point, one considers whether or not it is possible to use the basic properties of fractions to perform this division. One discovers that this is indeed possible, as shown below on the right side.

$3/5 \div 4 = (3 \div 4) \div 5$ $\qquad$ $3/5 \div 4 = (3 \times 4 \div 4) \div (5 \times 4) = 3 \div (5 \times 4)$

③ Points to Remember Regarding Teaching

Teach students to always think along the lines of “What types of things can be used?,” “What kinds of properties are there?” and “Which of these is appropriate in this case?”

8) Attempting to Focus on What is Determined by One’s Decisions, Finding Rules of Relationships between Variables, and to Use the Same (Functional Thinking)

① Meaning of Functional Thinking

When one wants to know something about element y in set Y, or the characteristics and properties common to all elements of Y, in spite of the difficulty of clarifying this directly, one first thinks of object x, which is related to the elements in Y. By clarifying the relationship between x and y, functional thinking attempts to clarify these characteristics and properties.

For instance, assume one wants to know the area of a certain circle, but one does not know how big this will be, or how to find it. In this situation, (1) think of something easy to measure that has a relationship with the area of a circle. The length of the radius is easy to measure. Also, when the length of the radius changes, the size of the circle also changes. Once one knows the length of the radius, one can draw a circle based on this length, which determines the area of the circle.

For this reason, one can find out the method of determining the area by considering the functional relationship between the radius and circle area. To do this, collect approximations of circles with areas and radii of different sizes, and infer the rule based on what you find. Also, in order to clearly show this rule, think of how to express it as a formula, create the formula, and use it to determine the area of the original circle.

Functional thinking follows these lines: "I want to think about a certain proposition, but it is difficult to consider it directly. Therefore, instead of considering the proposition directly, I will think about a related, easy-to-consider (or known) proposition. This thinking attempts to clarify the proposition of the problem." Therefore, functional thinking can be seen as "representational thinking."

The practice of this functional thinking, is defined by the following types of thoughts:

(1) Focusing on Dependencies

If a proposition called Proposition A is changed when changing another proposition called Proposition B, and setting B to a certain value (state) also sets the value (state) of A to another corresponding value state, then one says that "A depends on B," or "B and A are dependencies (in a dependent relationship)." The next task is clarifying the rules that state how A is determined based on what B is set to, or how A changes when B is changed. First, when it is difficult to directly consider a certain Proposition A directly, think of another Proposition B that is in a dependent relationship with Proposition A, and is easier to consider. Using this second proposition in this way is "focusing on dependencies."

(2) Attempt to Clarify Functional Relationships

This could also be referred to as "attempting to clarify rules of correspondence."

Once dependencies are clear, it is next necessary to think about how to clarify the rule (f) of correspondence between the mutually dependent propositions A and B (f). This rule tells us how A changes when B is changed, or what A will be once B is set to something.

Once the rule is found, this can be used to determine A based on the value of B, or what value B must be set to in order to get a certain value from A. For instance, once one knows that the area of a circle depends on the radius, one can then look for the rule that connects the radius and area. If one learns that this rule (functional relationship) is

area = radius^2 × π, (note: $\pi \fallingdotseq 3.14$)

then one can use this to plug in say, a radius of 10, resulting in area = 10×10×3.14, or 314. Given an area of 314, this gives a radius of 314÷3.14=100=10×10, or 10.

What kinds of thinking are used to find these kinds of rules?

In order to find the rule, try changing the radius to various different lengths and find the corresponding area. The following types of thinking are required to do this:

i. Idea of sets

Radii and areas can be set to a variety of different values. In other words, one must be aware of the sets to which values can belong. For instance, there are sets of whole numbers and decimal fractions.
In general, the subject of consideration will belong to a certain set. It is therefore necessary to be aware of which set this is.

ii. Idea of variables
One must consider what happens when various different elements of this set are plugged in. When it comes to the area of a circle, try setting the circle's radius to 3 cm, 5 cm, 10 cm, 15 cm, and so on.

iii. Idea of order
Simply trying various random values at this point would make discovering the rule difficult. Sequential values such as 5, 10, 15, and so on are preferable for testing circle radii. This is referred to as "idea of order."

iv. Idea of correspondence
Change the values sequentially in this way, determining the corresponding values, in an attempt to discover the correspondence between the pairs of values, and the rule governing this correspondence.

As one sequentially changes one of the variables in this way, one often attempts to consider how the other variable is changing, by trying to discover the rule behind this, or discovering the commonality between how the two variables correspond in value in this way. This is how one focuses on changing and correspondence.

② Coming Up with Ways to Express Functional Relationships

When discovering a functional relationship as described above, consider how to express the relationship of the two variables to make it easier to discover. It is important to come up with an appropriate method.
When finding the rule of a function, also consider how to express this rule so that the relationship is easy to understand and use. Idea of expression is also important.
Teach the students the benefits of this kind of thinking so that they actively apply it, in order to cultivate functional thinking.
Although there are also cases where only one or two of the types of thinking described above are used to solve a certain problem, in many cases, all of these types of thinking are used together.

③ Educational Value of Teaching Functional Thinking

(1) Cultivating the Ability and Attitude to Discover
Functional thinking is representational thinking, as shown above, and is used to clarify things. Therefore, this can also be referred to as one type of heuristic thinking. By actually focusing on dependencies, one can discover and clarify problems. By discovering rules in order to clarify dependencies, one can use this to solve problems. This is therefore a powerful type of thinking for discovering methods for solving problems. By judging that a problem depends on conditions, one can change some of the conditions of the problem and discover new problems with a related type of thinking (the "what if not?" technique). Therefore functional thinking is a valuable type of thinking for cultivating the ability to discover, as well as the attitude of discovering.

(2) Cultivating the Ability and Attitude to Mathematically Grasp Phenomena

For instance, in order to solve the problem of crowdedness as an everyday proposition arithmetically/mathematically, consider how to place the problem on an arithmetical/mathematical stage. To do this, consider what determines the crowdedness. In other words, by focusing on the dependencies, one can view crowdedness as a function (ratio) of area and the number of people. Realizing this allows one to precisely process the problem. Cultivation of the power to place a variety of different propositions and problems on an arithmetical/mathematical stage in this way, as well as abilities and attitudes for grasping situations mathematically while focusing on dependencies and using functional thinking to clarify functional relationships, serves a vital purpose. This ability to grasp situations mathematically is one form of the scientific method.

(3) Cultivating Inductive Thinking
This is obvious. When solving a problem, collect various data and use it to discover general rules. Inductive thinking uses these rules to solve problems. Focusing on functional relationships and clarifying them obviously plays an important part in inductive activities.

(4) Deepening the Understanding of Various Learned Matters
Functional thinking is used to understand numbers, computation, figures, and matters in other areas better, and to develop them further. It goes without saying that it works to deepen the understanding of these matters as well.

9) Attempting to Express Propositions and Relationships as Formulas, and to Read their Meaning (Idea of Formulas)

① Idea of Expressing as Formulas

Thinking based on appreciation of expressing problems with the following types of formulas, which actively seeks to use these formulas, is important.
i. Formulas can express propositions and relationships clearly and succinctly
ii. Formulas can express propositions or relationships generally
iii. Formulas can express thinking processes clearly and succinctly
iv. Formulas can be easily processed in a formal manner themselves

② Reading the Meaning of Formulas

Work to read and actively utilize formulas in the following ways:
i. Read the specific situations or models to which the formula applies
ii. Read the general relationships or propositions
iii. Read the dependencies and functional relationships
And furthermore:
iv. Read the formats of formulas

Chapter 6

Detailed Discussion of Mathematical Attitudes

Although we have discussed two types of mathematical thinking (the meanings of mathematical thinking regarding mathematical methods, and of mathematical thinking regarding the substance of mathematics), there is another type of mathematical thinking referred to as “mathematical attitudes” that drives the aforementioned types. Therefore, these “mathematical attitudes” are also a type of “mathematical thinking.”
This chapter considers each “mathematical attitude.”

1) Attempting to Grasp One’s Own Problems or Objectives or Substance Clearly, by Oneself

① Meaning

For instance, the correctness of formulas one has created is not something that one must have another person recognize, but rather something one must determine and recognize through one’s own ability. In general, when it comes to arithmetic and mathematics, the correctness of a solution is based on the meaning of the original problem and the four arithmetic operations, which can be judged independently. The correctness of a solution is not something that is determined by the authority of a vote held among the children, or based on the authority of the teacher’s assertion that it is correct. Arithmetic and mathematics are characterized by the independence of judgment, and independent judgment is easy in this field. This is why arithmetic/mathematics is an appropriate subject for the cultivation of a desire and attitude aimed at learning things independently.
The attitude of seeking precision in one’s own problems, objectives, and matters, as well as the attitude of trying to solve problems through one’s own ability, are both crucial.
To this end:

(1) Attempting to have questions
If one accepts what is there, or what is given, without any doubts whatsoever, then this attitude will not be possible, and one will not be able to make any new discoveries.
Questions such as “why is that?” or “is that really correct?” lead to the discovery of new problems, as they clarify the objectives one must seek.
Even if one is given a problem, for instance, this does not mean that all one has to do is to solve that problem. One must also question whether or not all of the given conditions are necessary for that problem, whether or not the solution can be achieved with just the given conditions, thereby clarifying the problem and enabling the discovery of new problems.

(2) Attempting to Maintain a Problem Consciousness
Students must be inspired to want to take control of problems for themselves, and to solve problems with their own power. Once a person strongly feels that a problem is their own, they can then be expected to take action on their own. When they run into problems due to incertitude between themselves and the environment around them, students must have a solid awareness of the fact that this is their own problem, and attempt to solve their own problems in the appropriate form.

(3) Attempting to Discover Mathematical Problems in Phenomena
To cultivate this attitude, it is not enough to simply solve given problems and develop them further. One must discover problems for oneself, and attempt to pose new problems.

② Examples

Example: Even if a student solves a problem that can be solved simply by counting items, if the teacher tells the student to “count these things,” then it cannot be said that the student has solved the problem. The student will have acted based on instructions without understanding why he/she is counting and arriving at the solution. Students must attempt to think of a variety of different types of solutions and to achieve a clear awareness of the problem based on these potential solutions. Students will realize that they must clearly determine the conditions regarding the range of items that must be counted to solve the problem. In turn, this will lead them to understand the appreciation of the idea of units and sets, which aim to clearly determine what must be counted, and the scope of the problem.

2) Attempting to Take Logical Actions

① Meaning

It is important to cultivate feelings that demand that one makes judgments based on solid grounds, and that one reflects upon or considers whether or not one has skipped any steps in one’s thought processes.
Also, one must not think about various individual things in isolation, but rather consider their relationships with other things. One must try to think of connections, or try to make connections.
The following types of thinking are important for nurturing this attitude:

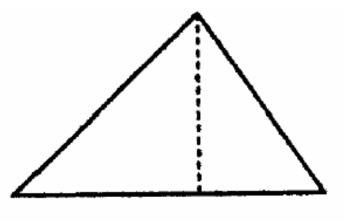
(1) Attempting to Take Actions that Match the Objectives
Objectives must be clearly grasped. No matter how inductively or deductively one thinks, if this does not match the objectives, then one cannot say that one’s actions are logical. Furthermore, even while one takes actions, until one arrives at a solution, it is always necessary to maintain a clear grasp of the objectives, verifying that the approach is being used in the correct way. It is necessary to occasionally reflect upon the objectives and make corrections when needed.

(2) Attempting to Establish a Perspective
To take logical actions that match one’s objectives, it is desirable to establish a general perspective on results. It is also desirable to establish a general perspective on the solving method as well. If one just suddenly starts taking actions, one might very well take illogical actions that do not match the objectives at all, or one might make a major mistake without realizing it. Establishing a perspective based on this is a manifestation of the attitude of thinking logically.

(3) Attempting to Think Based on the Data that Can Be Used, Previously Learned Items, and Assumptions
Clear objectives are sometimes expressed in the form of a clear understanding of what is being sought. For this reason, clarify what you have been given in terms of data and conditions, and what can be used. Think about the data and conditions that are available for use, and take advantage of previously learned, applicable lessons. This is necessary for one to think logically.

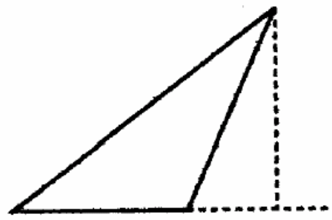
② Examples

Example 1: Assume that children have explained that in the case of a triangle such as the one in the figure to the right (1), area = base side × height ÷ 2.



(1)

Next, explain that even in the case of a triangle such as the one in the figure to the right (2), where the dotted line indicating height hits a line extending from the base, the above formula still holds.



(2)

At this point, it would not be desirable to treat the method in (2) as if it were unrelated to (1), as if one had completely forgotten the method in (1).
In order to think a little more systematically and logically, consider whether or not (2) can be thought of in the same way as the previously learned (1) because it resembles it, or whether one can use the result of (1) to explain (2) as well through analogical or deductive thinking.

3) Attempting to Express Matters Clearly and Succinctly

① Meaning

It goes without saying that as one proceeds with thinking, it is necessary to clarify the matters under consideration as much as possible, and to express them clearly and succinctly. Failing to express matters clearly and succinctly can confuse things and result in misunderstandings. Attempting to clearly grasp matters and express things clearly and succinctly is an important attitude. Therefore:

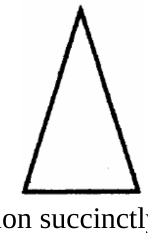
(1) Attempting to Record and Communicate Problems and Results Clearly and Succinctly
The clear and succinct recording and communication of matters evolves into thinking that attempts to abstract and to use appropriate symbols, and to use numbers and quantities for expressions.

(2) Attempting to Sort and Organize Objects When Expressing Them
Sorting and organizing are also necessary attitudes for expressing things clearly and succinctly.

② Examples

Example 1: After learning the unit dl, a student uses it to measure the dimension of containers. In case the measurement results with a remainder, if one simply states that Basin A is slightly larger than 3 dl, and Basin B is also slightly larger than 3 dl, it's still not evident which basin is larger. Also, the sizes of each basin are not clear.
It is necessary at this point to take the attitude of wanting to express these sizes clearly. This leads to thinking aimed at quantifying the amount of each remainder, and to thinking aimed at picking and using a new unit. The student thus learns the benefits of expressing values with fractions or decimal fractions.

Example 2: When explaining the location of a line that is the axis of symmetry in an isosceles triangle, there are cases when the student remains seated and points to the triangle and says, "If you draw a line straight from that point to that line, it's the axis of symmetry." The teacher accepts this, drawing the triangle's axis of symmetry and repeating back "you mean, if you draw a straight line from this point to this line at a right angle, this is the axis of symmetry."



In this case, it is obvious that the child is not attempting to express this proposition succinctly and clearly, and the same goes for the teacher. Since students have learned all of the terms, vertex, base, perpendicular, and straight line, they should be taught to say "a straight line

drawn perpendicularly from the vertex to the base is the axis of symmetry." For this reason, instead of picking up on the real intent behind a child's ambiguous expressions and explaining things on the blackboard as describe above, a teacher should help the child understand how ambiguous their expressions are by asking questions about the ambiguous points. Students can be taught to think of more accurate and concise expressions in this way. This is a method of cultivating the right type of attitude, by which they understand and rediscover the importance and convenience of terms, and understand the benefits of thinking that symbolizes.
This is the general attitude one must take to teach students that terms and symbols are necessary for expressing propositions clearly and succinctly.

4) Attempting to Seek Better Things

① Meaning

In our lives, there are many things that must be learned, and many problems that must be solved. To process these things well, one must think of how to use the least possible effort and thought to process the largest number of things possible. It goes without saying that mathematics seeks this, as do other sciences, and it is natural for people to think in this manner.
If one is caught up in examining various individual things separately, then it is necessary to focus on an extremely large number of matters and expend a great deal of energy. By refining methods, substance, and ways of thinking further, a person can increase their strength so that they can work over a wider range. In other words, searching for more refined and beautiful things is economy of thought. Seeking economy of thought and what is beautiful can also be seen as referring to the same things in different ways. By seeking what seems better and more beautiful one can bring together many things and consider and process them collectively, thereby conserving thought and effort. This is achieved by summarizing many different propositions to form a concept, by summarizing separate methods to become general rules, forming basic conceptual principles, and creating systems. This is what mathematics and other sciences do.

(1) Attempting to Raise Thinking from the Concrete Level to the Abstract Level
Concrete thinking uses the current problem to proceed with thinking that matches the matters at hand, whereas abstract thinking abstracts and forms general concepts without directly handling the matters at hand, using instead the abstract and generalized words to proceed with reasoning. By solving these matters, abstract thinking raises solutions to more general targets and methods. Therefore, it is important to take this kind of attitude in order to seek that which is better.

(2) Attempting to Evaluate Thinking Both Objectively and Subjectively, and to Refine Thinking
It is necessary to continue working to seek more comprehensive and better laws and methods, so that one can come up with higher order concepts. To do this, one must nurture the attitude of correctly evaluating one's own thinking and results and those of others, and of seeking further refinements.
In arithmetic and mathematics, it is important to nurture an attitude of seeking what is better. As described above, "seeking what is better" involves:

(3) Attempting to Economize Thought and Effort
It also involves:
(i) Attempting to find that which is more beautiful
(ii) Attempting to find that which is more certain

(iii) Attempting to find that which is different and new
(iv) Attempting to find that which is more organized and straightforward

② Examples

Example 1: Have children create a times table by themselves. It goes without saying that this will involve using the meaning of multiplication, which is the repeated addition of the same number. For instance, the 4s row is created in this fashion:

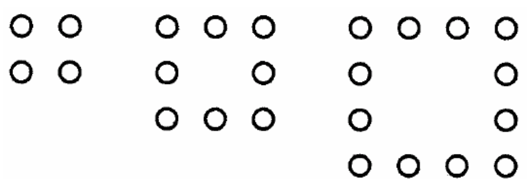
$4\times1=4$
$4\times2=4+4=8$
$4\times3=4+4+4=12$
$4\times4=4+4+4+4=16$

When one does this, it becomes apparent that repeating addition in this way is a hassle, and one thinks about ways to economize more effort. Reconsider the method and induce the following:

$4\times3=4+4+4$
$=8+4$
$=12$
$4\times4=4+4+4+4$
$=12+4$
$=16$

By generalizing this, one realizes that 4×5 can be derived by adding 4 to the previous answer. Furthermore, it can be inferred by analogy that this rule $a(b+1)=ab+a$ can be applied to other rows as well. Verify and use this.
In other words, by focusing on the functional relationship between multiplier and products, one can induce rules and use them to develop functional thinking. Next comes generalization and integration of each row. The point is to inspire students to take the attitude of wanting to conserve thought and effort by experiencing the hassle of creating each row.

Example 2: Arrange "Go" stones in a square pattern. How many stones are in a square with 50 stones on each side?



It is important to use inductive thinking, by creating or drawing squares with two stones on each side, then three, then four and so on, in order to discover the rule that defines the relationship between the number of stones on one side and the number in the entire square. However, one wants to examine and explain the discovered rule in a more practical manner than going over each number one at a time.
Since one wants to develop the level of concrete thinking to abstract thinking, one can use deductive thinking to reveal that when one stone is added per side, the total number of stones increases by four, or that if one side has a stones, then the entire square has $4(a-1)$ stones.
This is one case of the development of inductive thinking into deductive thinking.

Chapter 7

Questions for Eliciting Mathematical Thinking

Teaching should focus on mathematical thinking. Teachers need to first think of how they can help children appreciate and gain the ability to use mathematical thinking. When children get stuck, rather than helping them directly with useful knowledge and skills, teachers must prepare a way to teach the mathematical thinking required to elicit the knowledge and skill and moreover to teach the attitude that leads to this thinking methods. Also, this assistance must be of a general nature, and must be applicable to many different situations. Assistance should take a form that is frequently helpful when one focuses upon it. This is because this kind of assistance is useful in many different situations. By repeatedly providing it, a student can grow accustomed to this type of mathematical thinking. This kind of assistance is not something taught directly, but something that should be used by children themselves to overcome problems. Therefore, this assistance should take the form of questions.

It goes without saying that the goal of teaching based on these kinds of questions is for children to gain the ability to ask these questions of themselves, and to learn how to think for themselves.

Questions related to mathematical thinking and attitudes must be posed based on a perspective of what kinds of questions must be asked. This must be considered in advance. Questions must be created so that the problem solving process elicits mathematical thinking and attitudes. The following offer a list of question analyses designed to cultivate mathematical thinking, based on a consideration of these kinds of questions. In other words, this question analysis list is comprised of questions derived from the main types of mathematical thinking used at each stage of the problem solving process.

The A questions on this list deal with mathematical attitudes, with the stage indicated as "A11" and so on. Questions related to mathematical thinking related to mathematical methods are marked with M, and questions related to mathematical ideas are marked with I. Types of thinking corresponding to the question are given in parentheses ( ).

[List of Questions Regarding Mathematical Thinking]

<Problem Formation and Comprehension>

Questions Regarding Mathematical Attitudes

A11 What kinds of things (to what extent) are understood and usable? (Clarifying the problem)
A12 What is needed to understand, and can this be stated clearly? (Clarifying the problem)
A13 What kinds of things (from what point) are not understood? What does one want to find? (Clarifying the problem)
A14 Does anything seem strange? (A questioning attitude)

Questions Regarding Thinking Related to Methods

M11 What is the same? What is shared? (Abstraction)
M12 Clarify the meaning of the words and use them by oneself. (Abstraction)
M13 What (conditions) are important? (Abstraction)
M14 What types of situations are being considered? What types of situations are being proposed? (Idealization)
M15 Use figures (numbers) for expression. (Diagramming, quantification)
M16 Replace numbers with simpler numbers. (Simplification)
M17 Simplify the conditions. (Simplification)
M18 Give an example. (Concretization)

Questions Regarding Thinking Related to Contents
I11 What must be decided? (Functional)
I12 What kinds of conditions are not needed, and what kinds of conditions are not included? (Functional)

<Establishing a Perspective>

Questions Regarding Mathematical Attitudes
A21 What kind of method seems likely to work? (Perspective)
A22 What kind of result seems to be possible? (Perspective)

Questions Regarding Thinking Related to Methods
M21 Is it possible to do this in the same way as something already known? (Analogy)
M22 Will this turn out the same thing as something already known? (Analogy)
M23 Consider special cases. (Specialization)

Questions Regarding Thinking Related to Contents
I21 What should one consider this based on (what unit)? (Units, sets)
I22 What seems to be the approximate result? (Approximation)
I23 Is there something else with a similar meaning (properties)? (Expressions, operations, properties)

<Executing Solutions>

Questions Regarding Mathematical Attitudes
A31 Try using what is known (what will be known). (Logic)
A32 Are you approaching what you seek? (Logic)
A33 Can this be said clearly? (Clarity)

Questions Regarding Thinking Related to Methods
M31 What kinds of rules seem to be involved? Try collecting data. (Induction)
M32 Think based on what is known (what will be known). (Deduction)
M33 What must be known before this can be said? (Deduction)
M34 Consider a simple situation (using simple numbers or figures). (Simplification)
M35 Hold the conditions constant. Consider the case with special conditions. (Specialization)
M36 Can this be expressed as a figure? (Diagramming)
M37 Can this be expressed with numbers? (Quantification)

Questions Regarding Thinking Related to Ideas
I31 Think based on units (points, etc.). (Units)
I32 What unit (what scope) should be used for thinking? (Units, sets)
I33 Think based on the meaning of words (words used to express methods, or methods themselves). (Expressions, operations, properties)
I34 Try following a predetermined procedure (calculations). (Algorithms)
I35 What is this (formula or symbol) expressing? (Formulas, expressions)
I36 Can I express this in a formula? (Formulas)

<Logical Organization>

Questions Regarding Mathematical Attitudes
A41 Why is this (always) correct? (Logical)
A42 Can this be said more accurately? (Accuracy)

A43 Can this be said more simply and in a manner that is easier to understand? (Clarity)

Questions Regarding Thinking Related to Methods

M41 Can this be said in such a way that it also applies to other times (any time)? (Generalization)

M42 Can you explain how this is right (sometimes it will be incorrect or not hold true)? (Deduction)

M43 What grounds were there for thinking this? Can you explain this based on what you know? (Deduction)

Questions Regarding Ways of Thinking Related to Ideas

I41 Reexamine (attempt to explain) based on the meanings of known words (properties, methods). (Expressions, properties, operations)

I42 Express more clearly with figures (formulas). (Using formulas or figures)

I43 Can this be summarized with a simpler drawing method (calculating method)? (Algorithms)

I44 Focus on units and reexamine the problem based on them. (Units)

<Verification>

Questions Regarding Mathematical Attitudes

A51 Can this be said more simply? (Economizing thought and effort)

A52 Is there a better method? Can this be done better and more simply? (Better methods)

A53 Is there a way of summarizing and stating this in a more straightforward manner? (Better methods)

A54 Is there another method? (Better methods)

A55 Can new problems be discovered? (Newer things)

Questions Regarding Thinking Related to Methods

M51 Can this be summarized? Is there anything similar or identical? (Integration)

M52 Is there something that appears the same, which I already knew? Can this be seen as a special case of the same thing? (Integration)

M53 Is it possible to look at this in another way? (Development)

M54 What happens if the conditions are changed? (Development)

Questions Regarding Thinking Related to Ideas

I51 How can the conditions be changed? (Functional)

I52 What relationships are there? (Functional)

I53 What can be said about what needs to be done to solve this? (Algorithms)

I54 What can be understood from formulas (what kind of problems can be created)? (Reading formulas)

Special Observation Class – Sixth Grade Mathematics Class Teaching Plan

Theme: **Shifting from Discovery of Patterns in a Numerical Table to Sum of Natural Numbers**

Instructor: Shigeo Katagiri, Honorary chairman, Society of Elementary Mathematics Education

Class: Sixth Grade Room 1, Kawasaki City Hiyoshi Elementary School

1. Objective

To teach the importance of **inductive thinking, deductive thinking, developmental thinking and idea of reading about the meaning of formulas** through discovering and using patterns seen in numerical tables

2. Class Development

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Teacher	Students	(K) Mathematical Thinking(MT); (H) Evaluation of MT ® Note
Task 1		
T: (Above chart) In this chart, 1+100 is 101. Are there other pairs whose sum is 101?	C: (After some thought) 12+89, 23+78, 34+67… They all add up to 101.	® Distribute chart and worksheet.
T: What two numbers make a pair? T: Yes, numbers facing across the central point, right?	C: Numbers facing each other.	(K) Inductive thinking
Task 2		
T: Are there other pairs that add up to 101? T: If you find other pairs, mark them with the same sign. Try to find as many as you can.	C: (Marking pairs independently) C: (Children mark many pairs adding up to 101.)	® Make sure different pairs are not marked using the same sign.

Appendix 1

T: If you have found some pairs, come up and mark the table in the front of the room.		
T: What can we find from this chart? Write down what you have found.	(Some children may have become aware of the pattern of the pairs.)	(H) Inductive thinking
T: Let's talk about what you've learned.	C: There may be a lot more. C: Each number may have a partner that makes a total of １０１.	(H) Inducing.
T: It sure looks that way. Can you explain why?	C: (I don't know.)	
T: For example, if one number is called □, how do you find its partner?	**C: The partner of □ is 101－□. And since 101－□＜101, the partner is sure to be in the chart.**	(K) Encourage deductive thinking.
Task 3		
T: From this pattern, can you tell me "What is total of the numbers from 1 to 100?"	C: I don't know.	(K) Encourage expanded thinking, helping them find new problems.
T: (Hint) How many pairs add up to 101?	C: 101x50=5050	
T: Let's express the solution in a formula. Let's write it down.	C: (100+1) x 100 ÷ 2	® Have children write formulas using the figures they know (1 and 100).
Task 4		
T: What would be the total of the numbers from 1 to 60?	C: I don't know.	
T: From 1 to 10?	C: 1+2+3・・・up to 10 is 55.	
T: Can we apply the formula for the numbers up to 100?	C: If we replace 100 with 10…	
T: How about for up to 60? T: (Teach children how to apply the formula)	C: (Children try to read the meaning of the formula)	(K) Help children learn to apply formulas.
Summary **Let's summarize how we made our discovery.**	**C 1 : First we found a pattern from a few answers.** **2: Then we thought the reason that the found pattern was true.** **3: Then we applied a known formula and found the sum of other numbers.**	