DEVELOPMENT OF MATHEMATICAL THINKING IN HONG KONG SCHOOLS

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HOW MATHEMATICAL THINKING IS DEFINED IN THE HONG KONG CURRICULUM DOCUMENTS AND THE MATHEMATICS LESSONS?

The Hong Kong Primary Mathematics Curriculum was developed at the following years (1967, 1973, 1983, 1990 (TOC), and 2000). The present secondary mathematics curriculum was developed in 1999.

According to the Hong Kong Primary Mathematics Curriculum 2000, it has the following Aims:

1. Stimulate the interest of pupils in the learning of mathematics;
2. Develop pupils' understanding and acquisition of basic mathematical concepts and computational skills;
3. Develop pupils' creativity, and their ability to think, communicate and solve problems;
4. Develop pupils' number sense and spatial sense, and their ability to appreciate patterns and structures of number and shapes;
5. Enhance pupils' lifelong learning abilities through basic mathematical knowledge.

The ability to think is included in the aims of the curriculum and the attitude that students are capable of "independent thinking and persistence in solving problems" is one of the objectives of the curriculum. It is advocated in the curriculum guide that pupils should learn pleasurably through various learning activities and develop their imagination, creativity and thinking skills. These thinking skills include

- Inquiring
- Communicating
- Reasoning
- Conceptualizing
- Problem Solving

For the secondary school mathematics (S1 – S5), the document "Five High Order Thinking Skills" in the document. The Hong Kong Mathematics Curriculum defines five high order-thinking skills in the mathematics secondary curriculum. These are the same as defined in the primary curriculum: problem-solving skills, inquiring skills, communicating skills, reasoning skills, and conceptualizing skills.

However, the document remind the teachers that there are no simple and clear cut definitions for high order thinking and the five high order thinking can be arranged under several categories such as metacognitive skills, critical thinking. These can be overlapping in a sense.

Having said that, the common technique of posing open-ended problems in mathematics classes is usually advocated and practised in the classroom. It is believed that these five skills are not easily isolated. And though these skills can be taught in isolation, the popular way is to incorporating them into content areas.

WHAT IS A KEY WINDOW FOR CONSIDERING MATHEMATICAL THINKING?

Mathematical thinking involves inducting reasoning, deductive and analogical reasoning. Inductive reasoning is the generalization of specific cases, a process in which information of the same members in a sets are generalized so that every elements of the sets will shared the same information or character. Deductive reasoning is the reasoning from general principles. The third kind of reasoning is analogical reasoning, process of reasoning by similarities.

To solve a problem, students can draw on their knowledge and develop new mathematical understandings. The window of mathematical thinking in proposed in Hong Kong follow through the reference of Polya’s of problem solving skills. The strategies adopted in the skills are: understanding the problem, devising a plan, carry out the plan and look back. In the process, conjecture and examining the process is important.

The author will divide mathematical thinking into 3 areas.

The first one is more marco, this include generalization, abstraction, modeling, conjecture. The second is logical thinking, such as classification, reverse thinking, counter example, specification, induction and deduction. The third area is specific mathematics skills, such as substitution, completion of square, determination of coefficients.

The author will use the process of conjecture and verification as a basis for mathematical thinking. The reason is mathematics knowledge start with conjecture and verification. This is suitable in the primary school mathematics. For secondary mathematics, the focus will be on correspondence and proof. Both process of thinking will be based on mathematical structure.

The aim for the reasoning is to construct mathematical structure, which is part of the problem solving. Structure is the set of object or variables with relation of the variables and a new product of the structure. Structure can be represented as a number pattern, and also can be represented as in closed form.

HOW CAN WE DEVELOP MATHEMATICAL THINKING THROUGH THE LESSONS?

The process of development of mathematical thinking in the classroom:

- Demonstration, the teacher will solve a problem and hence express his thinking process before the class.
- Then the teacher will provide another problem and invite the whole class to think for 10 minutes (individual or in groups) and then the teacher will solve the problem with the class together. The students will be invited to talk about their thinking and finally the teacher will also talk about his "finding".
- The teaching includes a good choice of mathematical problem so that students can investigate the structure through their thinking.

The most important elements in the process are the selection of suitable mathematics problem. These problems are suitable in term of level, able to have open solutions and have a mathematical structure. All problems are related to the basic operation learned in the curriculum so that students can benefit basic mathematics thinking.

USING PATTERN AND NUMBER PROPERTIES

Question：	Process：
Find the unit digit of 3^{1997}.	As the pattern is 3, 9, 7, 1, 3, 9, 7, 1, …. And 1997 has reminder 1 when divided by 4. The unit digit of 3^{1997} is 1.

Fill in the number in the column			
20×30 = 600	30×40 = 1200。	40×50 = 2000。	50×60 = 3000。
19×31 = 589 21×29 = 609	29×41 = 31×39 =	39×51 = 41×49 =	49×61 = 51×59 =

Question:

Refer to the following expression; find out how and when the two products can be worked out when the numbers are reversed in multiplications.

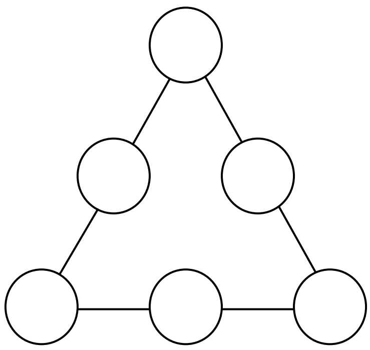
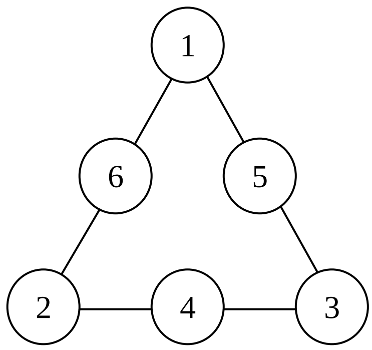
A
$$\begin{array}{r} 36 \\ \times\ 21 \\ \hline 756 \end{array} = \begin{array}{r} 63 \\ \times\ 12 \\ \hline 756 \end{array}$$

B
$$\begin{array}{r} 31 \\ \times\ 23 \\ \hline 713 \end{array} = \begin{array}{r} 13 \\ \times\ 32 \\ \hline 713 \end{array}$$

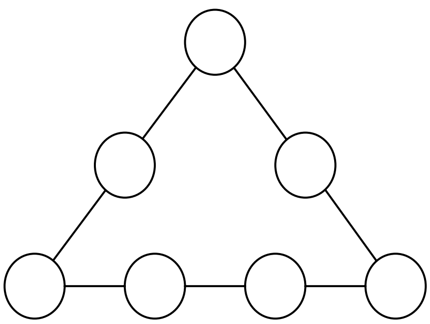
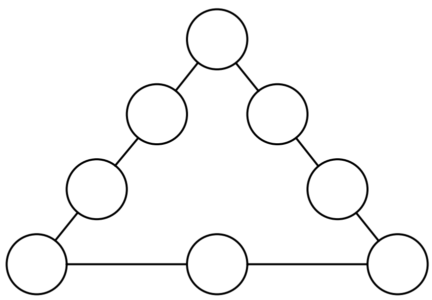
Question: Find the product of 17×12, by knowledge of □□×□	
12 = 3×4 17 × 3 = 17 × 3 × 4 = 17 × 12 =	17 × 2 = 17 × 10 = 17 × 12 =

USING DEDUCTION THINKING

If 1 + 2 + 3 + … + 100 = 5050, find the answer of 2 + 4 + 6 + …+ 200 = ? 3 + 6 + 9 + …+ 300 = ? 101 + 102 + 103 + … + 200 = ? 202 + 204 + 206 + …. + 400 = ?	If $[1^2 + 2^2 + 3^2 + \ldots + 100^2 = P]$, find the answer of $[2^2 + 4^2 + 6^2 + \ldots + 200^2]$

Question：	Process：
Fill in the number 1, 2, 3, 4, 5, and 6 in the circles so that the sum on the three sides of the triangle are the same. 	Students need to find out the relations of the corner number and the sum. They need to find out that the sum of the three corners is a multiple of 3. Below is one solution. 

The investigation will carry on so that students can pose their questions. The following are the questions posed by students. The whole class will investigate the problem together.

Extension Question 1：	Extension Question 2：
Fill in the number 1, 2, 3, 4, 5, 6, and 7 in the circles so that the sum on the three sides of the triangle are the same. 	Fill in the number 1, 2, 3, 4, 5, 6, and 7 in the circles so that the sum on the three sides of the triangle are the same. 

USING LOGICAL DEDUCTION

<table>
<tr><td>Question：</td><td>Process:</td></tr>
<tr><td>Using 1、2、3、4、5、6、7、8, so that the sum of column and rows are even numbers.
<table>
<tr><td> </td><td> </td><td> </td><td> </td><td>Even</td></tr>
<tr><td> </td><td> </td><td> </td><td> </td><td>Even</td></tr>
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<table>
<tr><td>1</td><td>2</td><td>4</td><td>3</td><td>Even</td></tr>
<tr><td>5</td><td>6</td><td>8</td><td>7</td><td>Even</td></tr>
<tr><td>Even</td><td>Even</td><td>Even</td><td>Even</td><td></td></tr>
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USING LISTING CASES:

Question:
A food shop sells noodle, rice and congeese, with the following sidebar: fish ball, beef ball, squid ball, and sausage. How many different combination of food can the shops offer?

Many students response with answer 12, which they multiply 3 by 4. However, there are more to the solutions as you can choose 2 sidebar, 3 sidebar or 4 sidebars.

List out

Single Sidebar: 12, Double sidebar: 18, Three sidebar: 12, Four sidebar: 3

Total combination = 45

Students are encouraged to use symbol and some student can list by using the following tables to get the answer 45.

Fish ball《A》，, Beef ball《B》、squid ball《C》、sausage《D》

noodle《1》、rice《2》、congeese**《3》**

Single sidebar

A1 A2 A3	B1 B2 B3	C1 C2 C3	D1 D2 D3

Double sidebars

AB1 AB2 AB3	AC1 AC2 AC3	AD1 AD2 AD3	BC1 BC2 BC3	BD1 BD2 BD3	CD1 CD2 CD3

Three sidebars

ABC1 ABC2 ABC3	ABD1 ABD2 ABD3	ACD1 ACD2 ACD3	BCD1 BCD2 BCD3

Four sidebars

ABCD1 ABCD2 ABCD3

Total is 45.

USING INVERSE REASONING

> Question
> Add a bracket in the following expression to make the equality valid.
> 1 + 2 × 3 + 4 × 5 + 6 × 7 + 8 × 9 = 303.

We have 1 + 2 × 3 + 4 × 5 + 6 × 7 + 8 × 9 = 141, we need to group some number together and multiply. As 303 is not divisible by 9, 8×9 should be isolated.

Using 303-8×9= 231, and 231 is multiple of 7, 231÷7 = 33. And 1 + 2 × 3 + 4 × 5 + 6 = 33. Hence (1 + 2 × 3 + 4 × 5 + 6) × 7 + 8 × 9 = 303.

USING ELIMINATION (UNIQUENESS)

Question Using the number 1,2,3,4,5,6,7,8,9to fill in the boxes □ , so that the expression are valid. Each number is used only once.	
(□ × □) + 6 = 30	(4 × 6) + 6 = 30
□ ÷ 4 × 5 = 10	8 ÷ 4 × 5 = 10
2 + (□ × □) = 20	2 + (9 × 2) = 20
□ + (7 ÷ □) = 7	0 + (7 ÷ 1) = 7
□ + 30 ÷ □ = 13	7 + 30 ÷ 5 = 13
12 ÷ □ × 2 = 8	12 ÷ 3 × 2 = 8

USING SYMMETRY THINKING

Question:
Using the number 1, 2, 3, 4, 5, 6,7and put them into the circles, so that the sum of all the number in the circle is the same.

There are a few approaches in building up the thinking.

The first one is by trial and error or by listing all the possible combination.

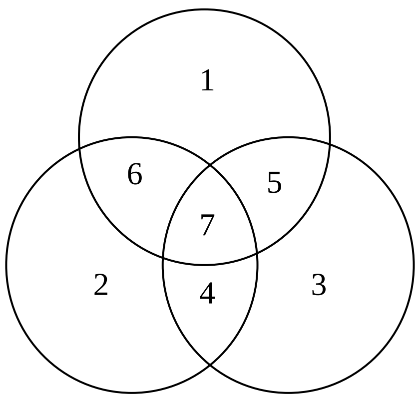
The second is by using symmetry in numbers.

The third one is to give the students one answer of the question, and then ask then to think of other solution base on the given answer.

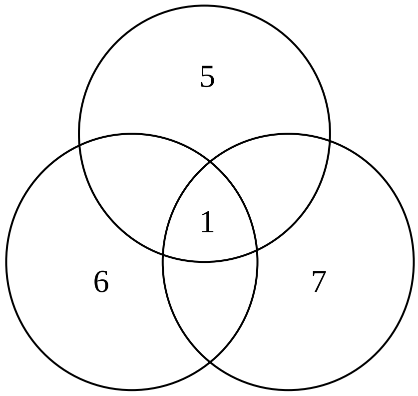
We give the thinking of the second approach.

By using symmetry, the three numbers 1, 2, 3 should have equal “status”. We may put them at the far out. And as 7 is the largest one, we can put it in the middle. The remaining three numbers 4, 5, and 6 will be put in places where the sum can compensate each other.

The following is the answer.



After that, the class is required to discuss whether there are any answers. By reversing the thinking, the three largest number is put in the far out, and the smallest number 1 is put in the middle. Then the rest of the 4, 5, and 6 will be put in by adjustment of the sum.



CLASSROOM TEACHING APPROACH

Using a set of 2 to 3 questions to investigate the structure of the problem. First with the more difficult question, and then a simpler version if they could not solve the first question.

Question 1：	Question 2：
Using 1,2,3,4,5,6 to fill in the boxes to make the smallest answer □ □ □ – □ □ □	Using 1,2,3,4 to fill in the boxes to make the smallest answer □ □ – □ □

Question 1：	Question 2：
Using 1, 2, 3, and 4 to fill in the boxes to make the largest product. □ □ × □ □	Using 1, 2, 3, 4, 5, 6 to fill in the boxes to make the largest product □ □ □ × □ □ □

Question 3：	Question 4：
Using 1, 2, 3, 4, and 5 to fill in the boxes to make the largest product. □ □ □ × □ □	Using 1, 2, 3, 4, 5, 6 to fill in the boxes to make the largest product □ □ □ □ × □ □

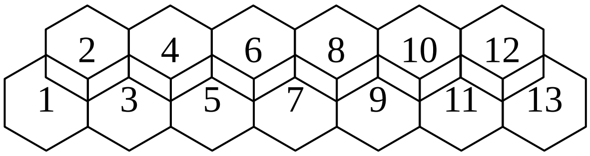
When it comes to Secondary level, the same thinking is adopted. The following are examples quoted from the document.

Example 1:

Soft drink cost \$5 per can and chicken wings cost \$8 each.
There are n people and each has x cans of soft drinks and 2x chicken wings.
The total cost is \$273; find the number of people and the number of soft drink and chickens wings.

Example 2 (Inquiring Skills and Reasoning Skills)::

If a bee one needs to move from cell number 1 to a cell of larger number, how many ways are possible?



For example, it is hope that students can obtain the number of paths needed. For example, T(1) = 1, T(2) = 2, and T(n) = T(n-2) + T(n-1) for n > 2. The sequence of the paths is a Fibonacci sequence and satisfy by T(n) = $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$.

Reference

Curriculum Development Council (2000), Mathematics Education Key Learning Area, Mathematics Curriculum Guide (P1 – P6).

Mathematics Section, Education Department (2001), Fostering Higher Order Thinking Skills, Teaching package on S1-S5 Mathematics.