MATHEMATICAL THINKING LIKE ANGULAR STONE IN THE UNDERSTANDING OF REAL WORLD PHENOMENA

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Mathematical thinking is a constituent element of genuine mathematical activity. A central aspect of the mathematical activity consist to construct a mathematical model of a reality that we want to study, to work with this model and to interpret the results obtained in this work. In this report we present how mathematical thinking is defined in Chilean curricular documents, the mathematical modelization like a key window to consider mathematical thinking, and how it is developed into a LEM[1] Lesson.

MATHEMATICAL THINKING IN CHILEAN CURRICULUM

In Chile, there are three levels of compulsory instruction: Preschool, Elementary and Middle. Each of these levels has their own curricular frame. Besides the curricular frames, we have other documents with a greater level of concretion: the *Study Programs*. In them we can find direct references to the mathematical thinking, which we present in a synthetic way:

- Mathematic teaching must promote the development of ***logical thinking***, analysis, deduction and precision, competence to construct problems from reality and solve them, and to formulate and understand models of a mathematical kind.
- Teaching will have to contribute to a better performance of the people in daily life, by the way of using concepts and mathematical skills that allow them to reinterpret the reality and to solve daily problems of the family, social and labor areas, contributing at the same time to establish a language for the comprehension of the scientific and technological phenomena.
- What is wanted is to promote the development of ***ways of thought*** that it make possible for children to process information about the reality and to deepen their knowledge of their own reality in ways of mathematically thinking.
- It is necessary that the students establish a relationship in the study of the arithmetical operations in the classroom and its application in daily social practices. This will allow them to approach problems in the school in which they will use the above-mentioned operations to extend and to specify their knowledge of reality.
- Students have to appreciate that mathematics can be a great tool in the understanding of reality**.**

[1] **LEM** it is a strategy to support primary schools in the mathematical curriculum implementation. It has been developed by the Ministry of Education and the University of Santiago de Chile.(Gálvez (2006)

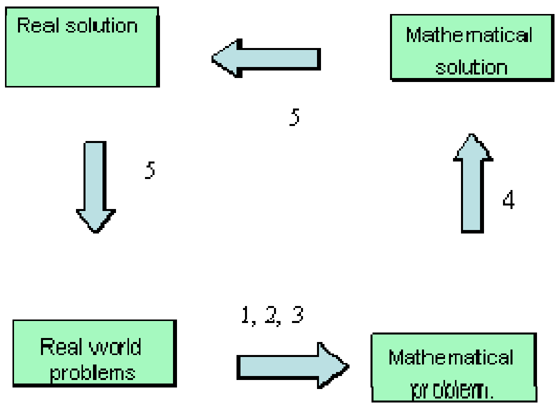
- To recognize that a wide range of problems can be expressed and solved in terms of the four elementary arithmetical operations. These operations are considered as ***mathematical models***, using simple algebraic expressions.
- To identify the operations of addition, subtraction, multiplication and division can be used to represent a wide range of situations and allowing them to determine unknown information from available data.
- In the field of the four operations, the work culminates by considering the operations as ***mathematical models*** that allow students to approach to problematic situations, which require combinations of these operations.
- The Chilean Middle Instruction is placed in the perspective of the right of all people to develop their capacity to think and to interpret phenomena in a mathematical way. The role of instruction is to facilitate people's incorporation, in an informed way, to a technified society that is in constant change.
- To develop intellectual skills, in specific relative to processes of abstraction and generalization, formulation of conjectures, proposal of argumentative linking and the use and analysis of ***models*** that allow the persons to describe and to predict the behavior of some phenomena in diverse contexts.
- To develop abilities in ***mathematical reasoning***: to formulate conjectures, to establish relations and conclusions; to organize and to link mathematical arguments; to demonstrate properties; to recognize numerical, algebraic and geometric regularities.
- To apply the process of formulation of mathematical models to the analysis of situations and to the resolution of problems in the real world.

A KEY WINDOW FOR CONSIDERING MATHEMATICAL THINKING

We will focus on the key window denominated **mathematical modelization**. We understand that there are many others ways to approach the subject, but they are all somehow related.

A central aspect of the mathematical activity consists of constructing a mathematical model of the reality that we want to study, to work with this model and to interpret the results obtained in this work in order to answer the questions initially raised. A great part of the mathematical activity can be identified, therefore, with the activity of mathematical modelization. (Chevallard et als. 1997, p 51)

We can represent the cycle of mathematization that has been described through following scheme (Jan de Lange, 2006):



1. Problem situated in reality
2. Identification of the relevant mathematics
3. Gradually trimming away the reality
4. Solving the mathematical problem
5. Interpret the meaning of the mathematical solution in terms of real world.

It is in this going through the cycle where there appears successive opportunities to develop the mathematical thinking that, according to Gustave Choquet (1980), the mathematical thinking advances by cycles, each one of which is formed in four phases: observation, mathematization, deduction and application. Each one of the phases is a necessary stage for the restructuring of the brain.

Considering that this cycle is on the mathematicians' central task, we could ask ourselves: Will it be possible to initiate the students in this process? A good mathematical education would have to develop the mathematical thought of the student, and the mathematical modelization is one of the privileged ways.

There are authors like Krygowska (1980) who indicated that mathematics would have to be applied to natural situations, in which there appear real problems, and to solve it is necessary the use of the mathematical method. The mathematization of a situation has its initial scene in the border between two dominions: the one of the real world and the dominion of the mathematical one. Since it leaves from the real world, its beginning will be in this dominion, is that where the problem is formulated obviously using their own conceptual categories of that dominion and using its language. In this first stage, the reality is not yet mathematized. The mathematization properly so begins with the construction of a representation of the initial situation, that is still constructed within the considered dominion and it is expressed in the language of this dominion. Then it is ready for the mathematical description. What does it means to mathematize something? What does one mathematize and how? Joining with Weinzweig (1980) we can say that:

> "One mathematizes a situation by imposing a structure on it. This serves to organize the situation into a more concise and precise form. This provides a more succinct and clearer description of the situation, which enables us to deal more effectively and more easily with the data, to transform them and get new information. In this way we develop procedures for solving specific problems in this particular situation which are simpler, more algorithmic and hence simpler to uses".

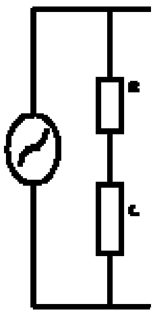
Instead of seeing mathematics as subject matter that has to be transmitted by the teacher, Freudenthal stressed the idea of mathematics as a human activity. *Education should give students the "guided" opportunity to "re-invent" mathematics by doing* *it. This means that in mathematics education, the focal point should not be on mathematics as to closed system but on the activity, on the process of mathematization. The idea of mathematization clearly refers to the concept of mathematics as an activity, which can best be learned through to mathematical activity".*

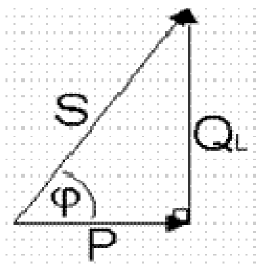
This idea is remote to think that the students are mere receivers of mathematical concepts already done. They must be considered like protagonists of their own process of learning, in which they occupy the techniques and tools that they have and those that are being constructed by them, into a teacher guided process.

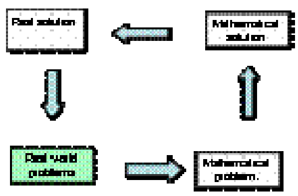
Then we are going to present an example taken from real classroom. We will see in it how a teacher and their students go through the mathematization cycle: A mathematical teacher, Fernando Pavez[2] detected that his butcher neighbor receive an additional charge in its electrical bill due to an incorrect to *power factor*[3], and he needed somebody help him.

Then, we can say that they were in the real world problem box, in the mathematization cycle scheme. Then he talked about this situation to his 2° level middle school pupils. These students that knew basic principles of electricity began to make a simplified representation of the electric butcher shop situation.

This meant that they set up the following electric diagram composed of an electric alternating current source, a resistor *R* and a coil *L*. These three elements are described in one electric phasorial diagram:







As we have seen, the mathematization properly so, it begins with the construction of a primary scheme of the initial situation, that is still constructed within the considered dominion and it is expressed in the language of this dominion. Then they have not yet taken it to a mathematical model. This object continues being described in the language of the initial dominion, but all already he is ready for the mathematical description.

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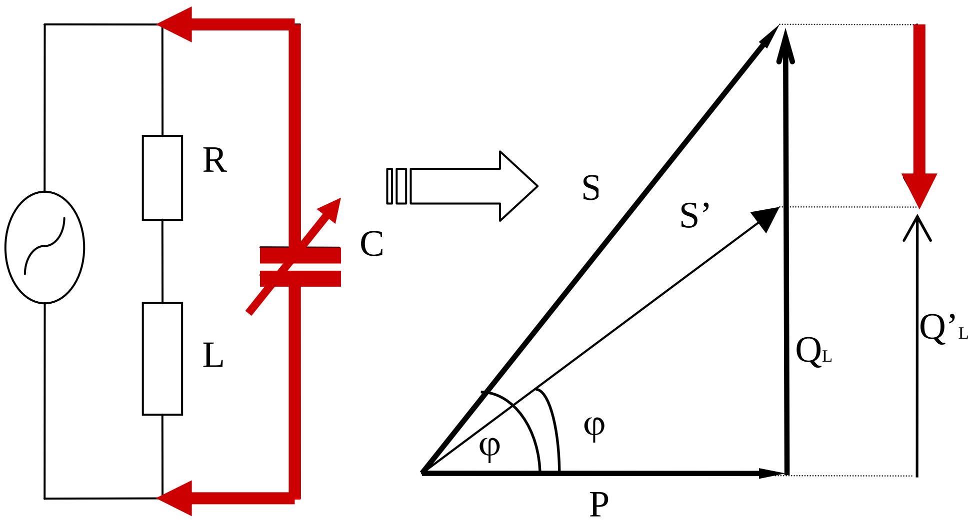
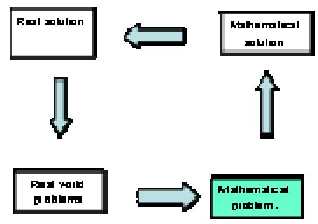
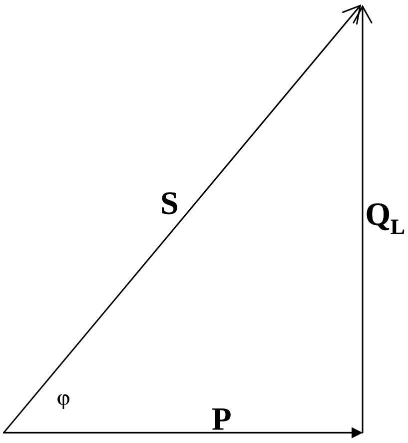
[2] Mathematical teacher of "San José" Middle School. Requinoa. 6th Region. Chile
[3] The power factor is an electrical indicator.

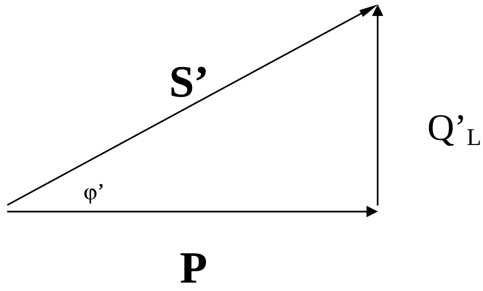
The mathematical description of that situation is based on a rectangular triangle that relates three types of powers: Active power (P), Reactive Power (Q) and Apparent power (S). In the butcher shop, the problem is that Q component is too large, and it must be diminish. The electric solution is adding an external condenser to the butcher shop electric circuit.





Now they have the mathematical problem (they were on the 2nd box of mathematization cycle). The students were dedicated to solving the mathematical problem, and once they found the solution they were already in the mathematical box solution.





P= 9.9 kW

Power factor= 0.82

cosφ = 0.82

φ = cos^{-1}0.82 = 35°

tg 35°= Q_L / P = Q_L / 9.9

Q_L =6.93 kVAR

P=9.9 kW

Needed Power factor = 0.93

cos φ' =0.93

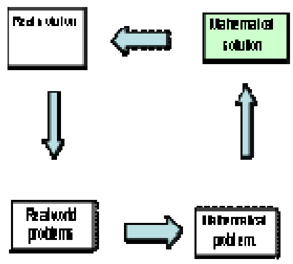
φ' = cos^{-1} 0.93 = 21.6°

tg 21.6°= Q'_L / P = Q'_L / 9.9

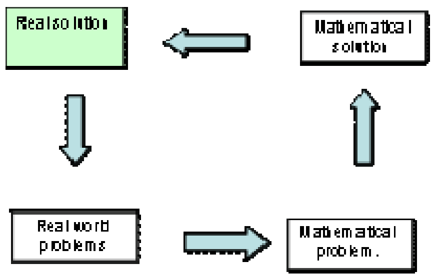
Q'_L =3.96 kVAR

We want to reduce the reactive power Q_L from 6.93 kVAR to 3.96 kVAR

The difference 2.97 kVAR must be provided by an specific condenser



Finally, they interpreted the results in the real context of the problem. They asked themselves the following question: if exists a condenser that could develop the calculated power (2,97 kVAR) and that was possible to install in the electrical board of the butcher shop?

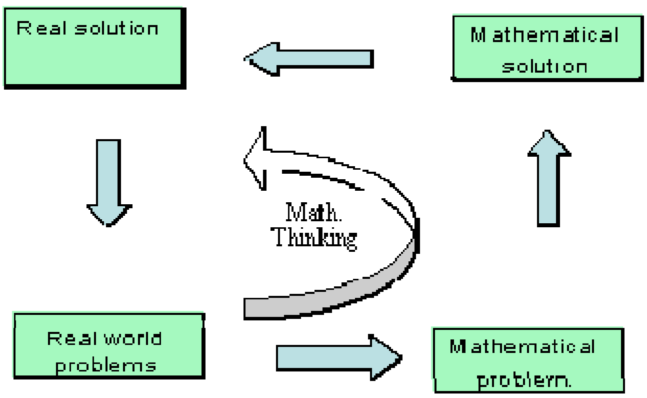


This solution was indeed later implemented. We have seen how a teacher has lead a process of mathematization with his students with full mathematical (and electrical reasoning) Starting off with a real problem, the students adapted a mathematical structure in an electrical problem (Pythagorean relation between three values of electrical power) and they operated in that model. What they made was to diminish one of the cathetus of the rectangular triangle with the purpose of obtaining the power factor demanded by the electric company. This reduction of “the reactive cathetus” (electrically speaking) was obtained with the addition of a condenser, whose power had been calculated.

Once that they have the model, they could even to experiment in it: What would happen if the value of the condenser were increased? The experimentation in the model allows them to continue developing mathematical thinking.

DEVELOPING MATHEMATICAL THINKING THROUGH A LEM STRATEGY LESSON

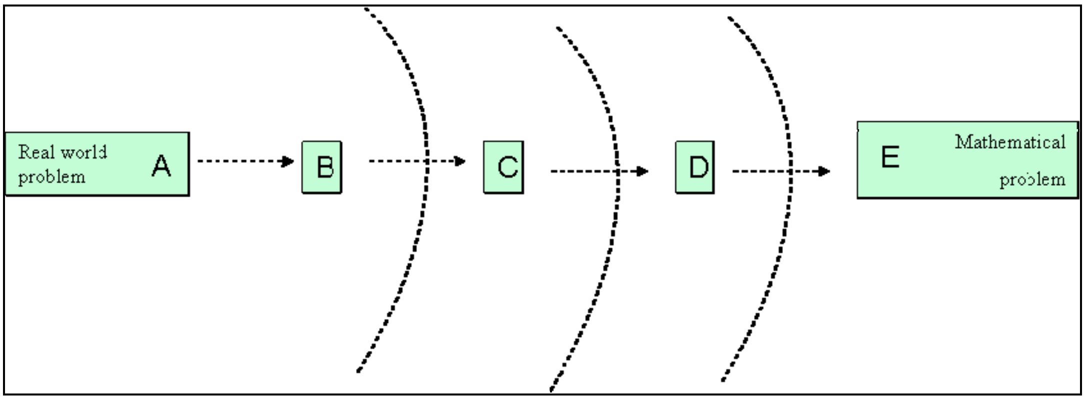
The mathematical thinking is a constituent element of the mathematical task. An activity of problems solving, or mathematization, cannot exist without a maintained mathematical thinking in each one of its phases. The teacher must develop the mathematical thinking for their students through out the lessons. Mathematical thinking is an inherent component of the mathematization cycle. Then we can analyze mathematical thinking along this cycle. Let us remind of mathematization cycle:



From real world problem to mathematical problem

In the LEM Lessons the teachers must direct the study process that allows the students to go through this cycle. One of the crucial moments of this process is the passage from the real world situation towards the mathematical model. This step is not automatic neither it takes place in spontaneous form, nor exist short cuts to make it the more direct with a simple anticipated presentation of the model that we are looking for to construct. This movement from the real world to the mathematical world means to cross through successive zones that not follow the traditional pattern: *from simple to complex*. The logic of the design is to be raising successive challenges that the children can approach this with the techniques they have. The teacher, like a director of this study process, will modify certain conditions that will make fail the techniques already used in a certain zone and the necessity to have new techniques will arise. This progress allow them to mathematize a situation initially *not mathematized*, putting a mathematical structure where there was not.

Schematically, this movement through zones can be visualized like this:



As a way to exemplify the previous situation we are going to describe these aspects, based on a LEM lesson that deals with the construction of the mathematical model of the **division** (3rd level of elementary school).

A very common situation with children is equally sharing of objects. For example in this lesson the initial situation considers to distribute 20 candies into 4 friends. The children could carry out the distribution without using an explicit mathematical notion: the children put themselves in a row and one of them is giving one each one in one first round and repeat the rounds until there are not candies left. This is the **Zone A** in the scheme above.

In the **Zone B**, the teacher provides to each pair of children 20 pieces and 4 glasses. The children are represented by the glasses and candies by the pieces. (In this moment we have a first separation of the real situation). They are throwing the pieces in each one of the glasses. Once finalized the distribution, they count the pieces in each glass. The sharing technique that is a concrete form, still serves.

In the **Zone C** the teacher challenges students to distribute 30 pieces in 5 glasses, with the condition that they use the glasses. They now, will be covered and that there will be only a small groove by which to introduce the pieces. On this condition the children, once they have distributed the pieces in the glasses, will not be able to count them. Perhaps the children could throw one piece at a time and to register the Nº of rounds. This process is slow. The professor can request that after each round they say how many pieces they have distributed and how many they have left to distribute: of the 30 pieces distributed 5 in one first round with which they have 25 left. Again to distribute another piece in each glass, with which they have 20 left. Soon the process is repeated and there are 15left, and after 10 left, then 5 and finally none. That is to say, by means of successive subtractions and counting of the number of conducted rounds, the number of pieces in each glass can be calculated. This procedure is already arithmetic and it is associated to the subtraction, but it is precarious

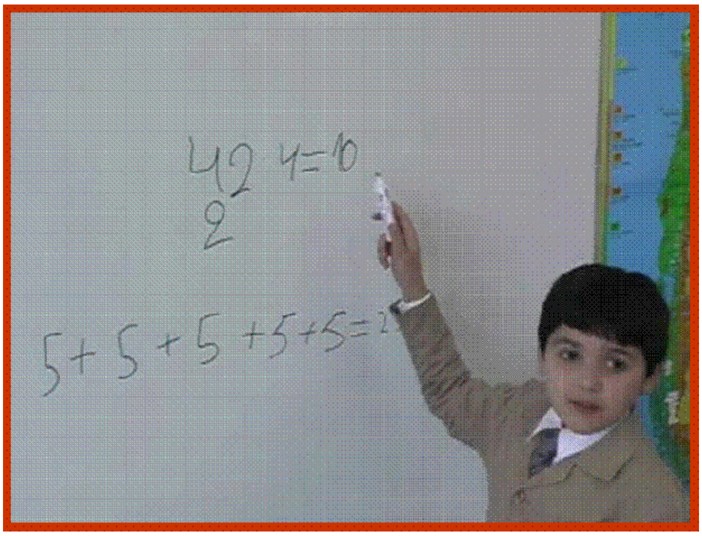
In the **Zone D** the teacher requests them to distribute 30 pieces in 6 glasses with the restriction of distributing them in a single round, putting one piece in each glass at a time. The children will be able to make several insolvent attempts: they put 3 pieces in each glass, but they exceed in 12 pieces. They put 4 in each glass and they exceed in 6 pieces. Finally, they put 5 in each glass and they do not exceed any piece.

In the **Zone E** now they will have neither pieces nor glasses available. The teacher will request them to distribute 42 pieces into 4 glasses. Now the children must anticipate the number of pieces that correspond to each glass and to indicate if there ere any remaining. The way to do that is through the division **42 : 4**.

In this stage they have put a mathematical structure (**division**) on another less efficient structure (**successive subtractions**). This situation is in accordance with Weinzweig who said that *one mathematizes a situation by imposing a structure on it. If a structure was already present (*consecutive subtractions*), a new structure (*the division*) was put onto the first.* The process does not finish here. Since once conducted the mathematical operation it will be necessary to interpret the numerical result in terms of candies and friends to give to each friend its candies. What one did was to construct a numerical model of the situation that does not require to physically having either the children or the candies present; it was enough with having certain previous information, pencil and paper.

To plan a **Lesson** of the division between natural numbers at an elementary level requires us to ask:

- In what type of situations could they use the division in this level? What type of situations could be obtained in which the students produce an initial mathematical technique?
- In what cases could we put on approval techniques of division in order to make them evolve towards more effective techniques?
- What aspects of this activity do they have to justify, to systematize, and to consolidate?
- How to articulate and to integrate with multiplication and how to continue this construction about the division in the following level of schooling?



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