DEVELOPING MATHEMATICAL THINKING IN CLASSROOM

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In relation to the three questions for the specialist session, firstly, the historical shift of mathematical thinking in Japanese curriculum is mentioned. Secondly, three windows for considering the development of mathematical thinking are described: Learning how to learn at the Problem Solving Approach, Mathematical Gap for setting mathematically considerable situations, and evaluation activity for the teaching value of mathematical thinking. Thirdly, an evidence of how teacher can develop mathematical thinking is descried. Based on the evidence, the importance of developing mathematical value is concluded.

ANSWERING QUESTION 1: MATHEMATICAL THINKING ON CURRICULUM REFORMS IN JAPAN

At the age of Japanese Mathematics in Edo period (until almost 140 years ago), mathematical thinking was not used as the same word of today. Thinking (Takumi) is deeply related with devisal technique and applauds. Japanese introduced European mathematics on the translated name on Sugaku (Chinese uses same character as for the academic subject, now). From the influence of curriculum reform in UK and Germany at the beginning of last century, Japanese secondary school textbooks and national curriculum standards had begun to enhance *Functional Thinking* (Kansu Shisou: Today, Shisou uses only in philosophy) in relation to the curriculum integration among arithmetic, algebra and geometry at the secondary school mathematics in 1910s. Before World War II, Functional thinking had been used for developing thinking in mathematics which is not clearly described as teaching contents on arithmetic, algebra or geometry. Elementary school curriculum and national textbooks had begun to enhance the mathematical thinking with the word of *Mathematical Science* (Suri Shisou) in 1920s and enhanced mathematical problem solving. In the World War II, secondary school curriculum completed integration of curriculum up to calculus with the process of *Mathematization* (Sugaku-ka) for enhancing mathematical activity.

On the re-established school system after World War II, the mathematics curriculum on compulsory education enhanced problem solving with *reflective thinking* on the social context in classroom. At the beginning of 1950s, mathematical Appreciation (Yosa) and Beauty (Utsukushisa) were used in a part of curriculum explanation documents for enhancing the value of mathematics. For more clarifying mathematical activity and for structuring mathematics curriculum consistency, the word Mathematical Thinking (Sugaku-tekina-kangaekata: Kangaekata is a new word for Shisou/thinking) was introduced in 1950s. In 1960s reform, *Evolutions of mathematical ideas* in teaching-learning process of mathematics curriculum were embedded in the spiral curriculum of mathematics with the words of (re)Integration and (re)Development

(Togo-Hatten). At the end of 1980s, Appreciation was re-enhanced for clarifying the value of mathematical thinking and learning how to learn or how to inquire mathematics, and with recent reform at the end of 1990s, this emotional value was more enhanced with more humanity by the word of Enjoyable (Tanoshisa).

Keywords for mathematical thinking have been changing but keeping the position of core target in national curriculum. Mathematical thinking usually described as the invariant ways of thinking aimed in mathematics education. Against its importance on the curriculum, there are no detail descriptions about each keyword in national documents. The freedom of interpretation was supported to develop the theories of mathematic education. Authorities had used to write different commentaries from their perspectives. Mathematical thinking is used to be explained with 'see as' and mathematical ways of thinking. Satoshi Kodo (1983) categorized it into mathematical ways of thinking, mathematical ideas and methods, and Shigeo Katagiri (2004), and so on.

On the other hand, teachers have been using the mathematical thinking as a kind of mathematical ideas' diversity in different students' solutions. In teachers' description, mathematical thinking is usually used as a meaning of children's mathematical ideas. In the case of elementary school teachers, they feel difficulty to use the analytical keywords on mathematical thinking which are clearly defined by mathematics educator. They usually prefer children's usage in mathematics for focusing on teaching 'learning how to' with the appreciation on the development of mathematical ideas by children in the classroom.

ANSWERING QUESTION 2: WINDOWS TO DEVELOP MATHEMATICAL THINKING

Three conditions to fix windows for developing mathematical thinking are given as following :

- Firstly, the learning how to learn or developing mathematics in the problem solving approach[1] on Japanese classroom
- Secondly, planned mathematical gaps in the approach for setting the mathematically problematic situation for reflecting on mathematical experience and getting didactics from mathematical experience on how to think and communicate. (See Appendix 1 as for example of theory for teaching Approach on Problem Solving in Japan)

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[1] In 'before it too late' with the quoting 'Teaching Gap', it illustrated Japanese mathematics classroom as follows: Teachers begin by presenting students with a mathematics problem employing principles they have not yet learned. They then work alone or in small groups to devise a solution. After a few minutes, students are called on to present their answers; the whole class works through the problems and solutions, uncovering the related mathematical concepts and reasoning. The similar approach was existed even before WWII but it spread all over Japan in 1980s on the word Problem Solving Approach. In the Lesson Study meeting at elementary level, teachers usually plan the lesson on this format but there are a number of variations and the structure of the planned lesson is not as same as the practice. It is a good format to plan the lesson in the case of novice teachers. Leading teachers prefer their way depending on the aim of practice and do not like to manage the lesson with in the fixed format.

- Thirdly, Teachers’ evaluating activity for teaching is used to teach value of what kinds of activity or way of thinking are important, enjoyable and what students can learn from others in classroom., how these kinds of experience are useful. These evaluating activities are represented with Appreciation

In relation to how to learn, Shigeo Katagiri has enhanced teachers’ question for eliciting mathematical thinking from children. Masami Isoda (2004) categorized three perspectives on teacher’s position of questioning to enhance children’s mathematical thinking: first perspective is mathematical question for stimulating students mathematical activities, second is pedagogical questions depending on the flows of problem solving approach for driving his/her lesson, and third is children’s questions learned from the teacher’s first and second ways of questioning among classroom.

In relation to mathematical gap in the approach, Masami Isoda published a book (1996, Appendix 1) from the view point of planning the problem solving approach with dialectic discussion in classroom based on the theoretical background of Conceptual and Procedural knowledge in the case of Mathematics (1986).

Isoda and Warashina described how children learn ways of dialectic reasoning such as ‘if’ in the classroom communication. Their theoretical background is the Vygotskian perspective which explains development of mathematical thinking as internalization of communication (Wertsch, 1991). They clarified the development of children’s mathematical thinking in the syncopate communication.

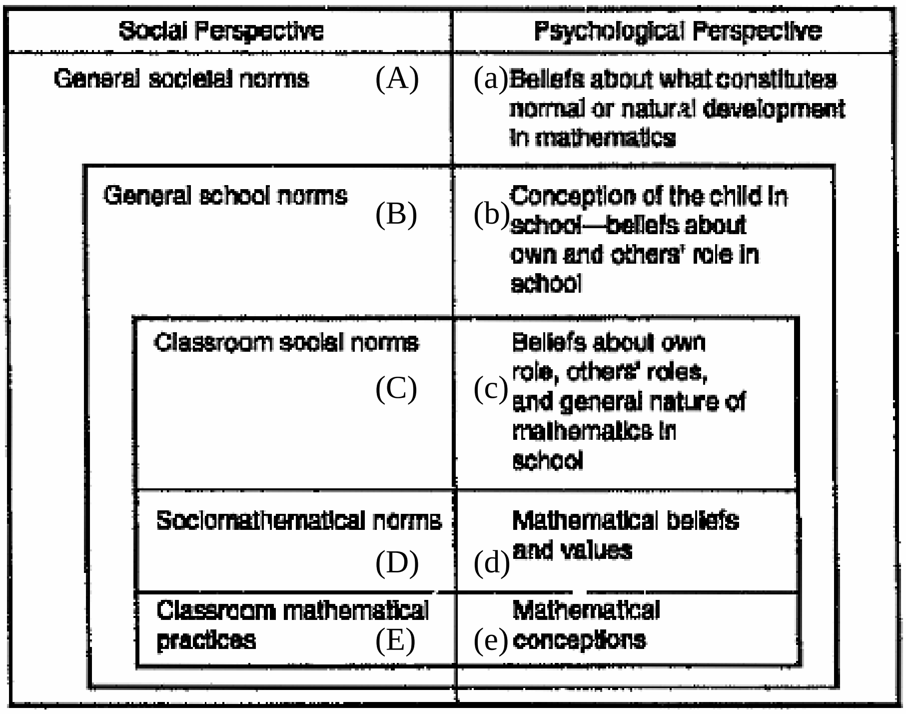

Figure 1. Social and Psychological Perspective for Observation (Paul Cobb 1996)

A case how children learn mathematical thinking by teacher’s evaluating activity in classroom is illustrated by Isoda (1999) with the theoretical perspective of Emergence by Cobb (1996, figure 1).

ANSWERING QUESTION 3: A CASE STUDY OF THE DEVELOPMENT OF MATHEMATICAL THINKING

This case study illustrates how children learn mathematical thinking from teacher's evaluation. It is second grade classroom about the multiplication by Teacher, Atsutomo Morii at Utsukushigaoka Elementary School in Sapporo City. 6th class in 13 lesson hours of the first unit of multiplication[2], and children already learned the meaning of multiplication and constructed 5 times multiplication in the unit. In the last class, children explored "5 times multiplication table" and wrote down the "secrets (properties)" of the 5 times table on cards (one secret per card). The class was then conducted with the results of this activity mutually organized and arranged by the students, with the plan then calling for application of the experiences of learning the 5 times table to the 2 times table.

See and read Appendix 2 for knowing how a child, Kumi, developed her thinking and attitude depending on the teacher's evaluation in teaching.

Referring to figure 1, let us analyze how Kumi's thinking was fostered through this process. At the 6th class session, and at the 8th class session that followed, Kumi possessed her own ideas (e). But she developed attitude toward categorization, a kind of mathematical thinking (e). In the 6th class, she only made comparisons when she put her own cards on the blackboard. At the 8th session, she continued comparative examinations of the cards put up subsequently by her classmates. That comprises a change in her awareness of her own role in the mathematics class (c), and an expansion of the consciousness of making comparisons on her own volition (d). When this is viewed not just in terms of Kumi but from a social perspective of fostering skills for everyone participating in the class, it may also be viewed as a stance of establishing rules for mathematics learning methods (E).

Kumi's growth is one of the fruits of the teaching and evaluation of Mr. Morii as a teacher, who possesses the following aspiration as the theme of fostering students capable of learning on their own: "I want to work through the creation of situations in which the need exists for individual students to consciously compare their own ideas against those of others, thereby fostering students who will think on their own." This aspiration of Mr. Morii is unconsciously supported by the students as the classroom norms (rules) for learning mathematics through this type of teaching and evaluation, and comes to be shared in common as a belief of the individual students. The students then strive through the assigned activities to evolve rules into even more effective means. The structural approach that the implementation of mathematical activities produces even better rules from the view point of mathematical value, with those better rules then acting to further deepen the implementation of mathematical activities is recognized in these results.

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[2] In Japanese, 5x3 means '3times5' in English. In grade 2, multiplication is taught in two units. At the first unit, meaning, definition and 2 times and 3 times are taught and at the second unit, multiplication table is developed and explored.

CONCLUDING REMARKS

In Japanese elementary classroom, children learn how to learn through the teacher's evaluation for teaching. Here, teacher's evaluation is aimed to improve children's understanding. Normally, teacher does not say simple questions expecting such as examiner's yes or no answer based on his/her mathematical knowledge. Even if teacher says yes or no, it has the meaning to develop children's mathematical thinking.

References

Hiebert, James edited (1986). *Conceptual and procedural Knowledge: the case of mathematics.* Hillsdale, NJ:Lawrence Erlbaum Assoc Inc.

Masami, Isoda (1999). Evaluation in Teaching process, *Ikiru Chikarawo Hagukumu Sansu Jyugyou no Souzou*. Tokyo, Japan: Nichibun (in Japanese, English translation of its part is sited on Appendix 2 on this paper)

Masami, Isoda edited (1996). *Developing Lesson on Mathematics Problem Solving Approach: Conflict and Appreciation on the dialectic among Conceptual and Procedural Knowledge*, Tokyo, Japan: Meijitosho (in Japanese, English translation of its part is sited on Appendix 1 on this paper)

Paul Cobb(1996). *Constructivist, Emergent, and Socio Cultural Perspectives in the Context of Developmental Research*, Proceedings of the Seventeenth Annual Meeting, North American Chapter of the International Group for the Psychology of Mathematics Education(Vol.31,No.3&4, pp.3-29)

Shigeo, Katagiri (2004). *Mathematical Thinking and How to Teach it, Meijitosyo.*

Satoshi, Kodo (1983). Problem Solving and Mathematical Thinking, *Tsukuba Journal of Educational Study in Mathematics* (Vol.2, pp1-8) (in Japanese)

Wertsch, James V. (1991). *Voices of the Mind : a socio cultural approach to mediated action*, Harvard University Press

Where do mathematics problems come from at elementary school classrooms?

Dialectic discussion beyond contradictions in the classroom on the Problem Solving Approach

Library data for your reference:

Masami ISODA edited (1996), Problem-Solving Approach with Diverse Ideas and Dialectic Discussions: Conflict and appreciation based on the conceptual and procedural knowledge, Tokyo:Meijitosyo Pub. (written in Japanese)

多様な考えを生み練り合う問題解決授業：意味とやり方のずれによる葛藤と納得の授業づくり
タヨウナ カンガエ オ ウミ ネリアウ モンダイ カイケツ ジュギョウ：イミト ヤリカタ ノ ズレ ニ ヨル カットウト ナットクノ ジュギョウズクリ
礒田正美編著
東京：明治図書出版, 1996.4

Brief Introduction for the English Translation of the book (Isoda, 1996 in Japanese)

This book was written to support elementary school teachers in Japan who plan the lesson based on the Problem Solving Approach, which is a famous approach for teaching mathematics worldwide. According to the theory of mathematics education on developing lesson plan or textbook sequence, mathematics educators usually consider the sequence of mathematical contents, various situations including real-life, or mathematics and representation for the process of abstraction. For example, one of those embedded in the textbook based on the ‘Model of, Model for’ framework by the Freudenthal Institute is the ‘Mathematics in Context’ that includes the process of the Situation, Model and Form through Mathematization. It aimed to support students’ reality of reasoning and activity in mathematics.

There are various textbooks in the world. Each country’s textbook is based on its curriculum. However, most of them do not treat students’ misunderstanding directly. On the other hand, the curriculum standards and textbooks are shared in Japan. Teachers’ guidebooks explain expected children’s answers for each problem and include how the teachers can treat children’s misunderstanding in classes based on their experiences in doing Lesson Study.

Many elementary school teachers and some mathematic educators in the world believe that mathematics problem comes from real situations. It is true but based on Japanese tradition. Isoda (1996) showed an alternative idea, that problematic situations for children really emerged from special occasions in lessons on the curriculum sequence. In the Japanese Problem Solving Approach known from 1960s, problematic situation is defined as an unknown when compared with what was already learned before. Thus, problem posed on the sequence of teaching on planned curriculum enables children to learn mathematics based on what they learned before. What is theoretically new in this book, which was published in the 1990s are the following: This book described the source of problematic situations in the process of extension on the curriculum sequence. It explained the development of conceptual and procedural understanding through the learning of mathematics based on the curriculum extension sequence. It likewise explained the dialectic way of discussion (Neriage in Japanese), a discussion that expects other’s perspective.

Chapter 1

The Structure of Lesson Based on Problem-Solving Approach that Produce Diverse Ideas and Promote Developmental Discussions: Focus on the Gap between Meaning and Procedure

Prepared for the theory of understanding mathematics on the lesson process

Masami Isoda

At an introductory lesson on adding dissimilar fractions that teaches students how and why they should perform calculations like 1/2 + 1/3, Children who do not know the meaning of 1/2 *l* or 1/3 *l* cannot objectively understand the meaning of the word problem. Children who are not proficient in the procedures of reducing fractions to a common denominator that they have already learned will likely struggle with solving problems like this. Teachers are surely well aware of the importance of the meanings and procedures (including form and way of drawing) learned over the course of problem-solving lessons.

In the Problem Solving Approach as a well known Japanese Lesson style, children are challenged to solve a big problem based on what they already learned. This chapter will use specific examples to show that previously learned meanings and procedures (form and way of drawing) help elicit diverse ideas among children. Next, it will describe methods of creating lessons that support children's learning through eliciting of diverse ideas and developmental (dialectic) discussion. It is based on the notion that it is precisely when people are perplexed by something problematic that they develop their own questions/tasks, truly have an opportunity to think, can promote their learning, and reach a point of understanding. The following aims to shed new light on the true significance of this notion.

1. It goes well! It goes well!! What?

In Japan many teachers have experienced the following situation: you finished a class feeling confident that the lesson went well and you believed that your students understood the material, but the students ask "What? I don't understand" in the next class. Their comments clearly indicate that they had not well understood the material previously presented even if they said they had understood it at that time. This is precisely the secret to problem-solving approach, i.e., eliciting diverse ideas and promoting developmental discussions. First, let us examine this secret by taking a look at a fourth grade class taught by Mr. Kosho Masaki, a teacher at Elementary School attached to the University of Tsukuba (*Sansuka: mondai kaiketsu de sodatsu chikara*, Toshobunka 1985).

1-1. Fourth Grade Class on Parallelism Taught by Mr. Kosho Masaki

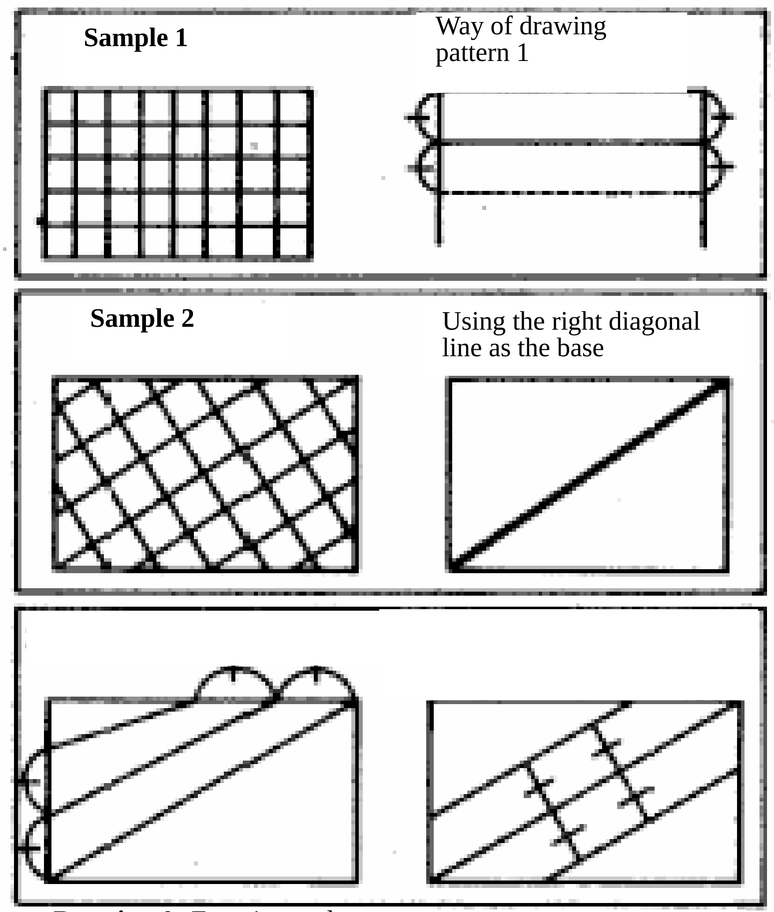
To introduce parallelism, Mr. Masaki started by drawing a sample lattice pattern. The following process shows how students develop the idea of parallelism:

<u>Task 1. Let's draw the sample 1 lattice pattern</u>

All of the children were able to draw the lattice pattern by taking points spaced evenly apart along the edges of the drawing paper and drawing lines between them. "It went well!"

<u>Task 2. Let's draw the sample 2 lattice pattern</u>

The children begin to draw the pattern based on a diagonal line moving upward to the right. What kind of reactions do the children have? The results are varied and depict several different strategies. However, they can generally be categorized into ways as shown in Drawings A and B.



Drawing A: Even intervals along the edges

Drawing B: Even intervals from the line

<u>Developmental Discussion: "What? What happened in Task 2?"</u>

Mr. Masaki explained his problem solving approach as follows: even children who robotically completed task were asked why they were able to draw the pattern in Task 1 but not the pattern in Task2, and try to find ways by which they can draw lines that will reproduce the pattern shown in Samples 1 and 2. Because students saw that others came up with results different from their own and everyone developed confidence from their ability to draw the pattern, the students began asking one another "How did you draw that?" and "Why did you think you could draw it by doing it that way?" They found it necessary to discuss their results. They began to distinguish and explain. It is through this developmental discussion that they were able to put the name 'parallel' to what they had learned based on what they had learned from others.

When children become aware of the unknown – in other words, there is a gap in their knowledge – they become confused and think "something is wrong." This is then followed by a sort of conflict, leading to the questions "What?" and "Why?" Furthermore, when children enter developmental (dialectic) discussions and are faced with ways of thinking that are unknown to them (knowledge gaps with others), it also causes conflict, forcing them again to ask "What?" or "Why?" Here again, they have to compare their way of thinking with others', evaluate it again by themselves and discuss their findings with other children. In this sequential flow, children make use of what they previously learned to turn the unknown into newly learned knowledge (a new understanding). This is the problem-solving approach discussed in this book based on the conflict and understanding.

Here, one must ask why then that all the children felt the drawing in Task 1 had "gone well," but in Task 2 two distinctly different types of drawing appeared. The reason lies in the diverse ways of thinking that appear in the sequence of tasks. In the next section of this chapter, we will clarify this using the terms "conceptual or declarative knowledge" and "procedure (form and way of drawing)." And based on the terms, the sequence of tasks are analyzed again.

1-2. Looking at Mr. Masaki's Class in Terms of Meaning and Procedure

Meaning (here, Conceptual or declarative knowledge) refers to contents (definitions, properties, places, situations, contexts, reason or foundation) that can be described as " ~ is…" For example, 2+3 is the manipulation of ○○←○○○. The meaning can also be described as: "2+3 is ○○←○○○," and as such explains conceptual or declarative knowledge. In Mr. Masaki's class, this method can be used to explain as follows: "The sample model is parallel." and therefore describes the meaning, which subsequently becomes the foundation of creating conceptual or declarative knowledge regarding parallel of the sample model.

Procedure (here, Procedural knowledge) on the other hand refers to the contents described as "if…., then do…" This is the procedure used for calculations such as mental arithmetic in which calculations are done sub-consciously. For example, "if it is 2x3, then write 6" or " if it is 2+3, then write the answer by calculating the problem as ○○←○○○." This is procedural knowledge.

By doing this, you may say, "Oh, I see, the meaning is merely another expression of the procedure, that's why they match." Yes, that is true for those who understand that they do match. However, people do not immediately understand that they match. Even if they know that the sample models are graphs of parallel lines (conceptual knowledge), this does not mean that they can draw them (procedural knowledge). On the other hand, even if people can draw (procedural knowledge) of parallel lines, it does not mean that they understand the conceptual meaning (properties, etc) of parallelism. Cases when conceptual and procedural knowledge do not match are not only evident in mathematics classroom, but also in other facets of everyday life. For example, despite knowing their alcohol limit (conceptual knowledge), there are cases when people drink too much. Furthermore, it is this mismatch and contradiction that becomes the catalyst for the process in which people encounter a conflict, experience reflection, deepen their knowledge and gain understanding.

Let's get back to Mr. Masaki's class. At first glance, the way of drawing pattern 1 in the first task appears to be a general method for drawing figures. However, from the perspective of the ways shown in Drawing A and B in task 2, it seems that the children confused the two procedures shown in the box. Even if the children produce the same answer, the ways they understood the problem, how they acquire the conceptual and procedural (form and way of drawing) knowledge, and the use of that understanding and knowledge are diverse.

> Way of drawing 1: **Procedure a**
>
> →Way of **Drawing A**; Task 2
>
> If you want to draw the model, draw lines spread evenly apart from the top edge of the paper.
>
> Way of drawing 1: **Procedure b**
>
> →Way of **Drawing B**; Task 2
>
> If you want to draw the model, draw lines spread evenly apart.

Based on analysis of the ways shown in drawings A and B, Masaki's class is described by the conceptual and procedural knowledge.

> **The gap between the Sample model (conceptual knowledge) and the way of drawing (procedural knowledge) : Meet the Conflict**
>
> - Thinking "hold on, I can't draw this using procedure a; the lines cross over if extend, but as shown in the samples, the lines do not cross.
>
> - "Why was I able to draw Sample 2 pattern using procedure B and not procedure A?"
>
> **Reviewing the way of drawing (procedure) and revising and reconsidering the semantic interpretation of the Sample model which acts as the foundation of the drawing method.**
>
> - How did you draw that? Why did you think it would go well if you did it that way?
>   - Reason (coming from semantic interpretation of the Samples); lines in the Samples are all evenly spread apart, so they don't cross over.
> - "I tried to draw the lines spread evenly apart, but they crossed over. How should I do it?"
>   - How do you properly draw lines spread evenly apart? By using the correct drawing method, which makes right angles and alternate-interior angles latent.
>
> **Elimination (bridging) of the gap between the semantic meaning and way of drawing (procedure): to the coherent understanding**
>
> Taking into meaning (even spreading of lines, no crossing-over, and characteristics of the right angle, corresponding angle and alternate-interior angle), designation (definition) of parallel and drawing method (procedures including the equal spread of lines, the right angle, corresponding angle and alternate-interior angle).

Within the developmental discussion process, procedure b, in which lines are drawn equidistantly at all points, works for both Samples 1 and 2. In contrast, procedure A, in which the lines are drawn from the top edge of the paper, clearly works for Sample 1, but does not work for Sample 2. Because Sample 1 is contrasted with Sample 2, the meaning of equal spread of lines is connected to the method of drawing with attention on the lines equidistant at all points, the right angle, corresponding angle and alternate-interior angle. As a result, the basis (meaning) of why that way of drawing was attempted, is explained by the children's comments.

Naturally, Mr. Masaki anticipated and expected to encounter undifferentiated schematic interpretations and drawing methods on the part of the children, and as such planned his classes accordingly. The teacher does not start by teaching the meaning and way of drawing parallel lines he is familiar with, but in fact starts by teaching at a level which assumes that children have not yet learned the word "parallel." The teacher tries to make use of previously learned methods of drawing parallel lines (procedures) that the children already know. By confirming previously learned knowledge, the teacher instills a sense of efficacy through leading children to a successful completion of the task. Following that, the teacher then makes the children face the difficulties of questioning "what?" at times when it does not go well. Due to the conflict that arises, children then ask about the meaning of the parallel lines. The teacher aims to have the children create their own reconstruction of the method of drawing and the meaning, using what they already know as a foundation.

Dialectic Structure of Mr. Kosho Masaki's Parallel Class

Confirming Previously Learned Knowledge Situation: Task 1

"It goes well"—**sense of efficacy**

Even if gaps in meaning and procedures exist, they do not appear here.

Different Situation from Previously Learned Knowledge: Task 2

There are children who show gaps in their understanding of meaning and procedure and some who don't.

"What?" – **conflict**

Developmental (dialectic) discussion by questioning new meanings and procedures

Acquisition of a **Sense of Achievement** by Overcoming the Conflict and Proceeding **through Understanding**

Looking back, it can be seen that the mechanism presented on the right is embedded in Mr. Masaki's class. As it indicates, the class is structured in such a way that the children proceed from a feeling that everything is "going well" to suddenly asking "what?". This transition serves as the context in which a diverse range of ideas appears regarding how the children have understood the problem and what type of meanings and procedures they have acquired. This class is indeed a type which solves problems through developmental (dialectic) discussion and makes use of a diverse range of ideas by overcoming the conflict of "what?", homologizing previously misaligned meanings and procedures, and finally reaching a stage of understanding.

2. Reading the children's diverse range of ideas through meaning and procedure (form and way of drawing)

For the planning of the lesson on the Problem Solving Approach, it is necessary to anticipate the diversity of children's responses and plan the developmental discussion for studying the target of the lesson. This section shows the ways of reading and anticipating children's ideas using the words "meaning" and "procedure (form and way of drawing)." Theory of conceptual and procedural knowledge in mathematics education by J. Hilbert (1986) was well known and in Japan. Katsuhiko Shimizu applied the similar idea in the classroom research (1986). The meaning and procedure for the lesson planning theory had been developed by Isoda (1991) as an adaptation of the cognitive theories to the evolutional development of mathematics ideas through lessons.

To begin with, we would like the readers to read once more the above-mentioned explanation of meaning and procedure, and do the following exercise.

EXERCISE 1

Which do the following correspond to meaning or procedure?

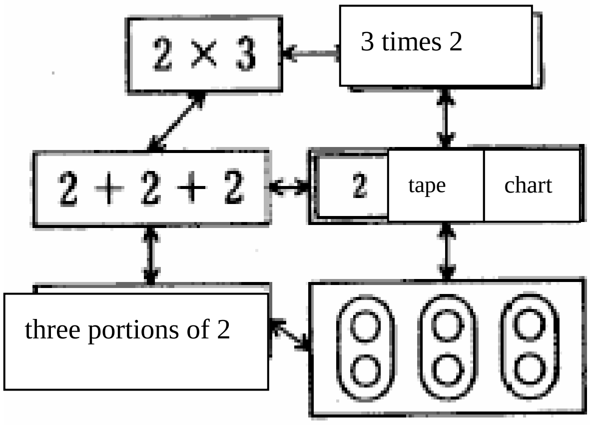
① Reduction to the common denominator <u>refers to</u> finding the common denominator without changing the size of the fractional numbers.

② <u>In order to</u> compare the size of the fractional numbers, either reduce or double the fractional number size.

③ <u>In order to</u> solve the division of fractional numbers, take the reciprocal of the divisor and multiply.

2-1. What is meaning? What is procedure (form and way of drawing)?

a. What is meaning?



Meaning (conceptual knowledge) can be expressed as "man <u>is</u> a wolf," for example. Of course, a man is a human being, however by likening man to a wolf and changing the way of saying it, one can make a sentence that aims to express the meaning of "man." The example given previously "2+3=○○←○○○" gives a concrete example and changes the way it is said to express the meaning. The mathematical expression "2x3=2+2+2" is also a meaning. This is a paraphrase, too. These paraphrasings are not only referring to a concrete example but also referring to what is already known. Incidentally, the meaning of multiplication that students learn in the second grade can be summed up as shown in the figure above. The characteristics of the meaning are seen in the fact that a number of elements are connected like a net, and as such, we as teachers think that children can understand the meaning in more diverse ways when we are able to paraphrase like this. The important thing regarding diverse expression is that the meaning is in fact picked out and expressed through such paraphrasing.

As an answer to the problem "How many *l* and *dl* is 1.5*l*?", a student replied: "Before, we learned that 1*l* is 10 *dl*, and that 1 *dl* is 0.1*l*. If I use that, 1.5*l* is 15 parts 0.1*l*. 10 parts 0.1*l* is 1*l*. The remaining 5 parts are 5 *dl*. So, 1.5*l* is 1*l* and 5*dl*." When that child explained the basis of her reasoning, we as teachers can see that the child has made a deduction and explained it based on the meaning.

b. What is procedure?

Procedure (procedural knowledge, form, way of drawing, method, pattern, algorithm, calculation, etc.) can be expressed as follows: "if the problem is the division of fractional numbers (recognizing conditional situations), then take the reciprocal of the divisor and multiply." The first characteristic of the procedure is being able to process automatically, unwittingly, and suddenly. However, proficiency (in other words, practice) is necessary. When answering the question how many *dl* are in 1.5*l*, in a case where student quickly answers "1*l* 5*dl*," and if the student instantly follows the rule "if *l* is paraphrased as *l* and *dl*, then focus on the position of the decimal point and think of *l* as coming before that, and *dl* coming after it," then one could acknowledge that the student is using the procedure. Being able to solve a problem instantly like this by using procedure means that we have come to a stage where we can find a solution without having to spend a lot of time deducing meaning, which in turn brings us to the point where we can devote more thought to shortening thinking time (e.g. short-term and working memory). Another characteristic of the procedure is that it produces new procedures such as the complex grouping of the four operations, as seen in the example for division in vertical notation (long division) whereby numbers are composed (expecting quotient), multiplied, subtracted and brought down (to next lower digit)." If each procedure is not acquired, it is difficult to use complex procedures that incorporate some or all of them. In other words, if one becomes proficient, it doesn't matter how complex the grouping of procedures are, as one will be able to instantly use them. Simplifying complex deductions and being able to reason a complex task quickly means that one is able to think about what else should be considered.

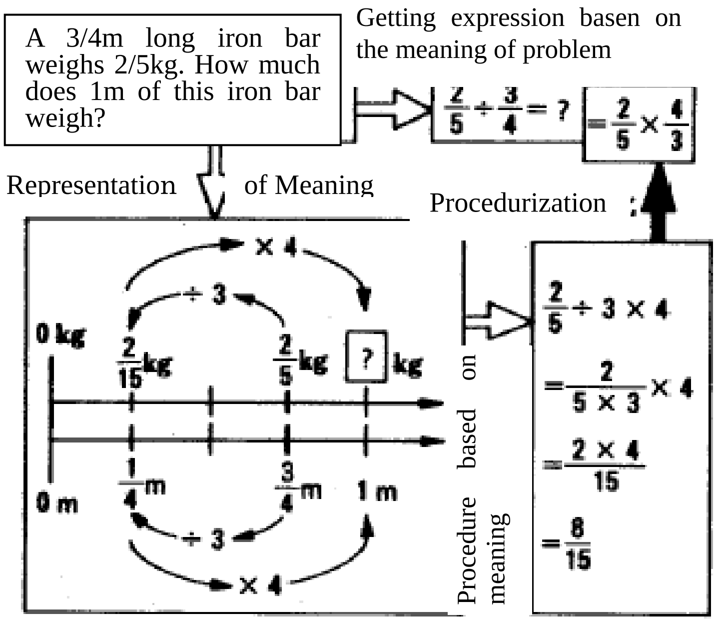
C. The relationship between meaning and procedure

As was shown in the method of drawing and the meanings of the patterns in Mr. Masaki's class, there are instances when the meaning and procedure match (no appearance of gaps, consistently use) and other instances when they do not match (appearance of gaps, inconsistency). There are times when the meaning and procedure contradict each other and times when they don't. Moreover, from the curriculum/teaching-learning sequence perspective, these two instances are linked as follows.

Procedures can be created based on meaning (the procedurization of meaning, in other words, procedurization from concept). For example, when tackling the problem “how many *dl* are in 1.5*l*?” for the first time, a long process of interpreting the meaning is applied and the solution “1.5*l* is 1*l* 5*dl*” is found. Additionally, this can be applied to other problems such as “how many *dl* are in 3.2*l*?” with the answer being “3.2*l* is 3*l* 2*dl*.” Not before too long, children discover easier procedures by themselves. Simultaneously, children realize and appreciate the value of acquiring procedures that alternate long sequential reasoning to one routine which does not need to reason..

There is a remarkable way to shorten the procedure from known concept and procedure. In the case, “<u>if</u> the problem is the division of fractional numbers, <u>then</u> take the reciprocal of the divisor and multiply” is shown in the diagram below. Using the previously learned concept of proportional number lines, the meaning of calculation is represented and the answer is produced depending on the representation. From the result of representation, the alternative way of calculation ‘take the reciprocal of the divisor and multiply’ is reinterpreted so that it can be produced simply and quickly from an expression of division. As a result, children reconstruct a procedure that can be carried out simply and quickly by reconsidering the result based on meaning. Even in the case of multiplication 2 times 3, it is 3+3 = 6 as a meaning, but as a procedure, 2x3 alternates the memorized result of 6. This remarkable way is also the procedulization of meaning. Many teachers believe that the procedure should be explained based on the meanings but the alternative is preferred because it is much simple and easier. Based on the value of mathematics, which is simplicity, we finally develop procedure based on meaning.

Meaning becomes the foundation for getting procedure. The significance of procedure such as faster, easier become clearer with the contrast of diverse mathematical ideas and difficulty of long reasoning. Thus, when diverse concepts of meaning are produced previous knowledge is deviced in the struggle to derive the answer. By debating the diverse concepts of meaning, students can make clear <u>the meaning from the viewpoint of recognizing conditions of applicable situations, which uses procedure to determine when that procedure should be used. Procedure has the ‘if, then’ structure. For recognizing the meaning of ‘if’ as conditional situation, it is necessary to discuss the situation that we can not use.</u>



The above is an example of how procedures can be created based on meanings. However, the reverse can also be achieved: meaning can be created based on procedure (meaning entailed by procedure, in other words, conceptualization of procedure). Let us consider this notion from the perspective of addition taught in the first grade and multiplication taught in the second grade of school. In the first grade, like in the operation activity where "○○○←○○" means 3+2, children learn the meaning of addition from concise operations and then become proficient at mental arithmetic procedures (the procedurization of meaning). At that point, calculations such as 4+2+3 and 2+2+2 are done more quickly than counting, which is seen as a procedure. Further, in the second grade, comparing with several additional situation, only repeated addition problems lead to the meaning of multiplication. It is here where the specific addition procedure known as "repeated addition" is added as part of the meaning (meaning entailed by procedure). The reason such situations become possible is that children become both proficient at calculations and familiar enough with the procedure to do it instantly as well as the meaning of situation. Children who are not familiar with the procedure resort to learning addition and multiplication at the same time, which in turn makes it more difficult for children to recognize that multiplication can be regarded as a special case of addition.

For the people who well know the meaning and the procedure, they use it as one thing or like compatible jacket[1]. From curriculum sequence and its teaching-learning perspective, meaning and procedure have mutual relations developing with each other. Due to the fact that meaning can become procedure and vice versa, it is impossible to separate one thing as meaning and the other as procedure without the decision maker of distinguish conditions. This book is aimed to support the teachers who will plan the lesson. It is up to the teacher to decide what is meaning and what is procedure in each class in compliance with the actual situation of the children and the classroom objectives.

2-2. Using meaning and procedure (Form and way of drawing) to anticipate children's ideas

In the problem solving approach, teachers anticipate childrens' ideas for planning to develop their ideas based on what they already knew. Meaning and procedure support anticipations[2].

a. Knowing meaning and procedure even allows you to anticipate the children's incomplete ideas

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[1] The metaphor is as same as Sfard but the idea itself developed until 1991 working with elementary school teachers.

[2] In the case of Japan, curriculum standards are fixed and textbooks are distributed from the government. One of the basic curriculum sequence and textbook contents sequence are expansion. Depending of this situtation, teachers can share children's' response through the Lesson Study and teachers guidebooks, and at the same time, they can anticipate children's reasoning and the process of discussion.

Some months after learning how to divide fractional numbers, children are asked "why does that happen?". A lot of children answer as "because you turn it upside down and multiply" (procedure), even if they could answer with the meaning when they first learned it. This indicates that they lose the meaning in exchange for procedural proficiency (Proceduralization of meaning). Here, I would like readers to answer Exercise 2, with children who tend to forget the meaning in mind.

> **Exercise 2** A third grader with previously learned knowledge to quickly solve the problem "1.5*l*=1*l*+5*dl*" is asked the following: "4.2*m*=how many *m* and how many *cm*?"
> Anticipate the child's reaction (rewording single denominate numbers as multiple denominate numbers).

A procedure that a child becomes proficient in is something like swimming or riding a bicycle; it is not easily forgotten, but meaning does not stay in one's consciousness unless it needs to be used. The most common answer to the above exercise by children, as expected, is "4.2m=4m+2cm." In the third year, students are taught to handle as far as the first decimal point in small numbers. Therefore, when learning, children are usually only faced with units of 1/10 such as in *l* and *dl*, or *cm* and *mm*. When learning, children who become able to quickly answer 1.5*l*= 1*l* + 5*dl* only experience the situation where that procedure is applicable. As a result, they become unable to make semantic judgments on when that procedure can be used.

The correct procedure "If do…, then…" will always produce the correct result as long as the conditional "if" part of the semantic judgment is correct. However, if children only experience applicable instances they overgeneralize the meaning and become unable to make a correct judgment. As a consequence, many children who use this so-called "quick/instant" procedure may use it in instances where it does not apply.

Notice should be paid to the fact that this quick response procedure is not only something that the teacher has taught, but rather is an extremely convenient idea that the children arrived at on their own. Even if this concept is invalid, children will not realize it as long as they continued to be presented with tasks which do not show the weaknesses of the invalid concept. For example, even if children from Mr. Masaki's class, completed the first task using an invalid concept, the underdeveloped nature of the concept would not become apparent until it was applied to another task. Therefore, what the teacher should first recognize a child's idea created on his or her own. From there, the next step is to deepen that idea by investigating whether or not that idea can be generalized to other tasks. This is the challenge for teachers.

b. Gaps between meaning and procedures appear in extending situations

As presented at the beginning of this chapter, the steps "It goes well! It goes well! What?" are important. As long as everything goes well and is applicable in the end, the gaps between meaning and procedures will not become a problem. In such a situation, children are not faced with a difficulty situation; they are within the range of previously learned knowledge, and have not challenge an unknown fact, yet. However, situations when something does not go well or when there is a need to close a knowledge gap is indeed where true discoveries and creations exist. When a person considers “what?” in a situation, this indicates issues should be given genuine thought. An example of when things do not go well is the “extending situation.” In an extending situation[3], the gap between meaning and procedure appears as diverse ideas. Here, let us look at the example of the expansion of a procedure from whole numbers to decimal numbers.

Example ① shown on the right is an over-generalized idea that can be seen in the decimal number calculation. It is usually explained as misunderstanding the meaning of a place-value.

①	②
2.3	23
+ 1.25	+ 125

Why does this type of idea appear? It is because when calculating whole numbers in vertical notation ②, the proper procedure is to write the numbers so that they are aligned on the right side. Example ① indicates the whole numbers procedure that was previously learned was applied. Having only experienced the calculation of whole numbers, the child is aware only of the procedure of aligning numbers on the right. It can further be stated that the child has learned the procedure of right alignment through her experience of learning whole number calculation in vertical notation.

The diagram at next page illustrates the process of the expansion of application of the whole number procedure. With regard to the introduction of whole numbers in situation I, the procedure for aligning decimals matches the meaning of a place-value (arrow A). When children become accustomed to this procedure, they forget the meaning of a place-value and become proficient in quickly aligning to the right (II). In the sphere of whole numbers, the meaning of a place-value is not contradicted even if numbers are aligned to the right (arrow B). However, when children apply this procedure to decimal numbers (III), it contradicts the meaning of a place-value as shown in ① (arrow C). Therefore, when children are faced with an instance when the procedure does not apply, they become aware of the gap and must once again come back to the meaning of a place-value. Then, they apply the procedure to both whole numbers and decimal numbers, and they become aware of the procedure of aligning decimal numbers as a procedure in accordance with the meaning of a place-value.

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[3] Expanding is a basic principle of Japanese curriculum and textbook sequence in mathematics. Thus, overgenelization by students itself is within teacher’s anticipation. The examples, here, may not be special even if in other countries.

Situation	Meaning Procedure	Explanation	Appropriateness
I Introduction of calculation in vertical notation using whole numbers	Decimal notation system meaning (A) ⟺ Write 23 + 5; (B)	The meaning of a decimal notation system is based on the procedure of keeping decimal points in alignment. (The meaning and procedure match)	Appropriate
II Becoming proficient in whole numbers	(C) ? (Forgotten) Align to the right and write	When children become proficient, they no longer need to think about the reason they follow that procedure. As a result, the procedure is simplified from the alignment of the decimal points to one of right-side alignment.	Valid
III Application of decimal numbers	(No meaning) Align to the right and write	The procedure for whole numbers is generalized for decimal numbers.	Inappropriate

Obviously, many children solve decimal number calculation in vertical notation by understanding the meaning of place-value. As such, the number of children who resort to the right-side alignment procedure is small. From the perspective of meaning and procedure, however, the mechanism in which gaps in meaning and procedure occur tells us in short that there is a necessity in the teaching process to separate meaning and procedure into the following three categories.

I) Deepening meaning: No appearance of gaps between meaning and procedure
"It goes well!"
II) Gaining an easy-to-use procedure from the meaning: Gaps are unrecognizable.
"It goes well!!"
Children become accustomed to easy-to-use procedures that work and many of them become unable to recognize the meaning.
III) Situation where easy-to-use procedures do not work: Awareness of gaps
"What?"

Children's levels of comprehension are by no means uniform in the process of learning. Comprehension develops differently in each child. While there are children who are no longer aware of meaning because they have become accustomed to using quick and easy-to-use procedures, there are also children who are aware of meaning and use it as a basis for the procedures. Because the conditions vary, a diverse range of ideas involving previously learned knowledge appears in situations (extending situations) (III) when easy-to-use procedures do not work.

The problems considered in Mr. Masaki's class and in exercise 2, a practice of rewording from a single denominate number to multiple denominate numbers, are the examples of extending situations. In an extending situation, the procedures and meanings that have been established will not work, which means that they will need to be reconstructed. Taking the above decimal number calculation in vertical notation as an example, the meaning of a place-value works, but the right-side alignment procedure needs to be revised. Accordingly, the meaning of a place-value needs to be reviewed, and the procedures used need to be revised to ones that align the decimals positioning accordance with the proper place-value notation. In short, as an educational guidance situation, III can also be described as follows:

III') Reviewing of meaning and revision of a procedure: Elimination of gaps

2-3. Diverse ideas can be classified by meaning and procedure

Up to this point, we have focused on the most extreme over-generalized ideas (misconceptions) to indicate the occurrence and elimination (bridging) of gaps between meanings and procedures. Naturally, in actual classes a diverse range of ideas will surface, including correct and wrong answers. In order to plan developmental discussions, it is necessary to anticipate the type of diverse ideas that will most likely appear. Here, let us treat the students' ideas as observations. For example, Mr. Hideaki Suzuki's 5th grade class looks at division with numbers containing 0 in ending places, at the Sapporo City Public Konan Elementary School. This class, as was the case with Mr. Masaki's class, first confirms previously learned knowledge of division when there is no remainder (task 1) and then moves onto the target content, which has yet to be learned: division when there is a remainder (task 2). The objectives of this class can be confirmed in the following chat showing the class flow.

Task 1. Known problem to confirm the previously learned procedure and the meaning that forms its base: Previously learned task.

When children who have knowledge of basic division work out the equation, 1600÷400 is done, the following is reviewed:

$$\begin{array}{r} 4 \\ 400\overline{)1600} \\ \underline{1600} \\ 0 \end{array}$$

A. Take away 00 and calculate: procedure

B. Explain A as a unit of 100 (bundle): meaning

C. Substitute A for a 100 yen coin and explain: meaning

Task 2. Unknown problem that presses for application or expansion of the previously learned meaning and procedure: Target task.

The target problem presented is 1900÷400, which presents a problem for some children and not for others as to how to deal with the remainder. As a result, the following ideas appear.

a) **Answer to the equation using a procedure in which the meaning is lost.**

Apply A and make the remainder 3. Because the meaning is detracted, the children do not question the remainder of 3: Half of the class

b) **Answer to the question when procedures have ambiguous meanings.**

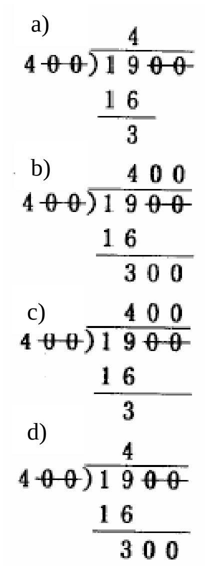
Using A and B, the remainder was revised to 300. However, because the meaning was ambiguous, it was changed to 400: Several students.

c) **Answer to the question when the procedure is ambiguous.**

A was used, but here a different procedure was selected by mistake. No students question the quotient 400: Very few students

d) **Answer to a question that confirms procedural meanings.**

Using A, an explanation of the quotient and remainders from the meaning of B and C.



Why do answers differ?

Where did you get lost? What did you have a problem about?: A reminder of conflict through solving an exercise using your own ability.

By reviewing the solution process, the base meaning is reconfirmed and the procedure for dealing with remainders is learned.

Situation: confirming what they have already learned
“It goes well” – **sense of efficacy**
Mutual confirmation of meaning and procedure
Even if gaps in meaning and procedure exist, they do not appear here
Situation: different from what they have learned before—Conflict
What?: **the unknown** due to an awareness of the gap with what they have already learned
Some students experience such gaps in meaning and procedure whilst some do not.
What?: **Surprise at the difference in ideas** with other students and reflection on one’s own ideas.
Developmental discussion that correctly redefines meaning and procedure
Acquisition of a sense of achievement, appreciation, by overcoming conflict and proceeding through to understanding

Firstly, the students grapple with Task 1, which they have learned before. The teacher links this task directly to Task 2 in the target content of the class, keeping the children’s solutions in mind. This is done by asking students to confirm the procedure for the division in vertical notation, and asks them why it is not a problem to do this (meaning). Simultaneously, the teacher makes sure the children are able to explain the procedure and meaning. Following that, the children tackle target Task 2, which requires them to deal with remainders. In Task 2, a variety of ideas (a-d) appears among children who are doing the work without knowing the meaning, and children who are confirming the meaning while working on the task.

The objective this time is to have a developmental discussion regarding the place-value of the remainder being adjusted to the place-value of the dividend.

Here, it is important to have readers understand that the above mechanism is fixed in the class. It is noteworthy to mention that even if meanings and procedures are previously confirmed, there is a diverse range of ways to process and implement that comprehension. As such, a variety of ideas appear. The starting point in the creation of diverse ideas lies in ways to process and utilize individually.

When categorizing the variety of ideas (a-d) by meaning and procedure, the following types can be created. Following types are categorized when we posed the extending task after the reminding task which is already known.

Type 1. Solutions reached through the use of procedures without meaning: Prioritize procedure without meaning type

This is the above-mentioned idea a). It refers to an idea reached through consideration without much attention to the meaning, even though the correct procedure (calculation) is applied. There are students who change their ideas by recalling the meaning after having been asked to explain or listening to others students' ideas. However, most students substitute meaning with procedure and when they are asked for an explanation they usually reply by describing their procedure, saying "I did this, then I did that." Prioritizing the procedure means that the students do not give careful consideration to the meaning; rather they tend to use quick procedures.

*In the case of already known task, and if we applied correct procedure, the answer must be appropriate but now we are discussing in the case of extending task.

Type 2. Solution reached through the use of procedures with meaning: Prioritize procedure with confused or ambiguous meaning type

This type is composed of ideas b) and c). These students have the intention of confirming the meaning of the calculation procedure, but their idea <u>includes their own semantic interpretation</u>. Therefore, when getting to the core of their idea, it is found that their idea is one that <u>contradicts</u> the meaning and procedure they have previously learned. As a result, there are many instances in which their idea brings about confusion and unease.

Type 3. Solution reached through the use of procedures backed by meaning: Secure procedure and meaning type

As shown in d), when a solution reflects the appropriate meaning and has been learned as a procedure, there are no contradictions between the procedure and meaning.

Usually, when people are faced with a task they are unfamiliar with, the first thing they do is to test existing quick-to-use proficient procedures. This is what is referred to as the "prioritize procedure situation." If people believe that they can grasp the appropriate meaning, then it is the "prioritize procedure without meaning type." In actual fact, there are many children who react to an unfamiliar task by prioritizing procedure without giving any careful thought to the meaning. If they investigate the meaning when asked if the procedure they chose to implement is appropriate, due to their confusion and concern, they are categorized as students applying "prioritize procedure with confused or ambiguous meaning type." In contrast, a careful student who tackles a problem by always investigating the meaning and making sure there are no gaps will produce a result that has a secure procedure and meaning.

Although not shown in the above example, other ideas such as the following are also recognized.

Type 4. Solutions through meaning only: Prioritize meaning without procedure (or confused) type

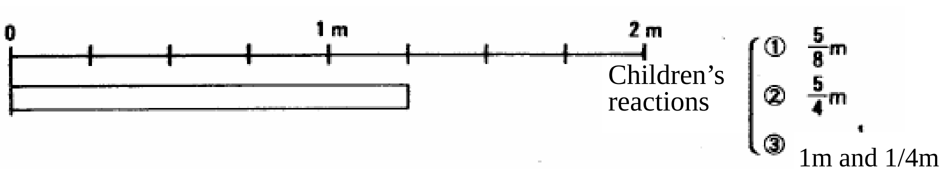
This type is seen when the procedure cannot be used appropriately or the student is not yet proficient in using it. Consequently, the solution is gained through thinking mainly about the meaning. For example, a case where the student cannot calculate 1600÷400, but can answer if asked to solve "You have 1900 yen. You buy as many 400 yen pencil cases as you can. So…"

Type 5. Situation of inability with insufficient the meaning and procedure: No meaning and procedure type

It is particularly important for teachers to keep in mind those students who are unable to solve a problem (Type 5). In the case of Type 4 children, they usually give many possible reactions in the class, but in many cases there are no result when it comes to tests. In the case of 1st and 2nd graders, many children of Type 4 gives possible answers if they know the meaning well, but in the case of higher graders, they will meet difficulties. When elementary school students reach the 5th and 6th grades as well as when they enter junior high school, there is an increase in textbook and course materials that require the procedurizations of meanings, and so it is important to be aware that some students of Type 4 will be falling into type 5 without proficiency of procedure.

Now we would like readers to tackle the following problem regarding the meaning and procedural knowledge children from Mr. Katsuro Tejima's class possess.

Exercise 3. The following is used in the introduction of fractional numbers for 4th graders. When asked to answer using fractional numbers for the length of a piece of tape, children's responses fall into one of three different types. Please explain what the children were thinking.



Answers to Exercise 3

The meaning of fractions as previously taught in the third grade was the some parts of the equivalent divisions on the whole, and in the case of the fraction of quantity, "2/3m is the same as 2 parts of three equivalent divisions of 1m". Fractions of one meter are learned only when the measurement is less than 1m. The previously learned procedure tells students always to divide the whole number evenly and that the numerator never exceeds the denominator.

1. 5/8m: the procedure was applied by making 2m one unit. This method is consistent to the procedure already learned, however the students did not realize the contradiction that the value

obtained is smaller than 1. Accordingly, it fell into the "prioritize procedure without meaning type."

2. 5/4m: this answer was gained under the quick assumption that there were three parts, each of which was 1/4m, the total length would be 3/4m, so if there are five parts, the length should be 5/4m (generalization of procedure). This contradicts the meaning and procedure children were previously taught, in which a numerator is smaller than denominator. If the student felt uneasy in this instance, they would fall under the "prioritize procedure with confused meaning type." If the student used the diagram to establish that 3 parts of 1/4m becomes 3/4m and so 5 parts becomes 5/4m (meaning), but were then confused as to whether they could write (procedure) that way because they had previously learned that the fraction can not exceed the denominator, then those students would fall under the "prioritize meaning without procedure (or confused) type."
3. This reaction shows that the students regarded the length as 1m and 1/4m that was obtained by subtracting 1m from the total. As there is no discrepancy with what was previously learned, these students would fall under the "secure procedure and meaning type."

This book focuses on lesson planning by teachers, and as previously mentioned, teachers ought to decide what the meaning and procedure are in their class material, and should provide appropriate educational guidance in accordance with their teaching plan. Even if children fall under the same type, their actual understanding, thought processes and how they deduce what the meaning and procedure are, differ depending on the individual and the situation.

Before the lesson, it is necessary for teachers to prepare the teaching material and plan the lesson on the planned curriculum sequence. In aiming to support lesson planning, this book has provided the above-mentioned types as part of the teaching material research carried out by the teacher. The teacher will be able to prepare the following in accordance with the categorization by types: anticipate what kind of ideas will emerge from the students based on what they have previously learned; plan well-devised instruction content for the class based on those diverse ideas; and create ways of facilitating the instruction so that the students can realize what they don't understand and experience the joy of understanding. By being able to anticipate the causes of the children's possible confusion and their ideas, the teacher will be able to conceive beforehand how they should develop their explanations and discussions. The categorized types provided are for the teacher to use in order to plan the lesson for conceptual development at the extending sequence based on what the students have previously learned.

3. Planning for a Lesson with Developmental Discussion and Diverse Idea

This section will incorporate what has been covered in previous sections and will demonstrate how to implement the wide range of ideas children conceive and how to apply developmental (dialectic) discussions in the lesson. As we already discussed, the developmental discussion is planned at the special occasion of teaching sequence. If the curriculum or textbook sequence including the expansions of mathematical ideas, we can expect the contradictions which will inevitably happen. In the problem solving approach, we are aiming to develop mathematical communication as well as mathematical conceptual development. Thus, in this book, we affirmatively set these contradictions as the object of discussion in the mathematics classroom.

3-1. Instruction planning in which a wide range of ideas appears by taking advantage of knowledge gaps

Here, the "third grade decimals" lesson conducted by Ms. Junko Furumoto (Sapporo Midorigaoka Elementary School) will be used as an example. When teaching fourth grade decimal lessons, it is known that children tend to over–generalize when it comes to rephrasing single denominate numbers and multiple denominate numbers, as shown previously in Exercise 2. Ms. Furumoto recognizes this over-generalization as a gap that appears due to an expansion of procedure children have developed for dealing with numbers with only one decimal place to numbers with two decimal places. Accordingly, she has created the following lesson plan to take advantage of this gap to add depth to her lesson on decimals.

Lessons	Stage
1st class: In what situations are decimal numbers used? The existence of decimal numbers. **2nd class:** How much juice is there? The need for decimal numbers (meaning). 1/10*dl*=0.1*dl*: decimal numbers are used to express amounts smaller than one unit (meaning) **3rd class:** Let's make a numeric line based on 0.1: The size of decimal numbers	**It goes well! The meaning and procedure match.**
4th class: Let's get decimal numbers to introduce themselves: Practice of large/small, and amount (meaning and procedure). "I am 2.8. I am a number made up of two 1s and eight 0.1s." **5th class:** How much is 3.7cm or 1.5*l*: Practice of single and multiple denominate numbers. Rewording the expression of single and multiple denominate numbers (meaning and procedure).	**It goes well!! Procedurization, loss of meaning, or no loss of meaning.**
6th class: There are two pieces of string, one is 4.2m and the other is 4m10cm. Which one is longer?	**What? The occurrence of gaps.**

The first five lessons, each of which is one hour long, are designed to deepen the students' understanding of the meaning of the first decimal place. In particular, the fourth and fifth hours focus on procedural proficiency (form) in terms of semantic interpretation. Up until this point, the approach of the instruction is standard. The sixth hour (class) is planned to cause students to wonder, "What?" A diverse range of ideas appears as some children try to apply quick, easy-to-use procedures while others consider the problem using their understanding of decimal numbers based on the example 0.1 equals 1/10 of 1. It is planned this way so that conflict will occur. Furthermore, this conflict is used to get children to re-evaluate the meaning of a place-value, including those children that did not understand the meaning of decimals accurately in the first place.

The sixth class unfolds as follows:

- Preconception: It's 4.2m! It's 4m 10cm!
  How should I compare them?
- The units are different, so if I don't align them, I won't be able to compare them.
  What should you do so that you can clearly find out which is longer?
- For children who can't solve this problem by themselves, the teacher makes them realize that they should use diagrams or the numeric value line they have previously learned.
  a) 4.2m=4m20cm, so… b) 4.2m=4m2cm, so… (majority of the students)
  c) 4m10cm=4.1m, so… d) 4m10cm=4.10m so…
- **Conflict:** a) vs. b), c) vs. d). Is 0.1m 10cm or 1cm?
- **Returning to the meaning:** By converting the units to meters
  (Using diagrams and number lines) 10cm is 1/10 of 1m, so it is 0.1m
  4m10cm=4.1m<4.2m
  By converting the units to cm
  0.1m times 10 equals 1m, so it is 10cm
  4.2m=420cm>410cm=4m10cm

  If the units are different, then compare them by converting them (procedure)

  reaching understanding

A wide range of ideas appear in answers a to d. Students chose answers a and c based on the meanings they learned up to the fifth class: "0.1m is 1/10 of 1m" ("secure procedure and meaning type"). Answer b may be applied when the quick procedure in the fifth lesson doesn't work ("prioritize procedure without meaning type"). Answer d may be "prioritizing procedure with confused or ambiguous meaning type" if the students are confused as to why a contradicting expression they do not understand appears. This is due to the fact that even if they consider the quick procedure 4.2m=4m + 2cm as correct and write 4.2m=4m2cm, they also have to write 4.10cm for 4.10m. Another case is that students wrote 4.10, because 1/10 of 1m is 10cm. If the students are confused as to whether they can write 0 in the second decimal position, then it could be said that they fall under the "prioritize meaning without procedure (or confused) type."

After the gap in ideas has been confirmed[4], the class moves onto encouraging students who

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[4] The difficulty to understand other's ideas is that each of them is deduced from the reasoning on the different presuppositions depending on different understanding. For understanding

chose answer d with a question about 4m10cm being 4.10m when 4.2m=4m2cm, to consider the problem in the context of answer b, in order to return to the meaning of decimals they had previously learned, which is 0.1= 1/10 of 1. Through discussion, the quick procedure is revised and the procedure for converting the units becomes clear. Furthermore, the understanding of the meaning of decimals, which observes a place-value of numbers, such as 10cm=0.1m, is deepened.

It is noteworthy to mention that even though the first five hours of lessons have placed heavy emphasis on amounts and meaning through the use of specific examples and number lines, a large number of children will choose answer b. As previously mentioned, when adults learn a quick procedure, they will first try to use that procedure. Children are no different. When children become aware of easy-to-use procedures, many children are unable to recognize the semantics of the pre-requisite “if…” of the procedure (in ‘if…, then…’ structure). Ms. Furumoto’s children would not have acquired even the easy-to-use procedures sufficiently without attending the sixth class. Accordingly, the aim of the sixth class is to deepen children’s knowledge regarding procedures that convert units and the meaning of a place-value in decimal numbers by continuing to detect insufficient understanding and revising it.

The diagram below shows a summary of the sub-unit construction mentioned above, focusing on meaning and procedure.

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each other, it is necessary to reason based on others’ presuppositions or to find the good presuppositions which may deduce the other’s ideas.

I) Constructing meanings

1^{st} – 5^{th} class: Matching meaning and procedure. No gaps become apparent.

Specific amounts, number lines and diagrams are used to learn that 10x0.1 amounts to 1 (meaning).

II) Constructing easy-to-use procedures with meanings as the base

Part of the 4^{th} class: the following quick rewording is taught, "2.3 is made up of two 1s, and three 0.1s."

Part of the 5^{th} class: Becoming proficient in procedure. Some students begin to lose the meaning of the procedure.

5.3cm=5cm3mm, 2.7*l*=2*l* 7*dl* can be reworded quickly.

III) The situation of easy-to-use procedures not working: Extending situation

The meaning is reviewed and the procedure is revised

6^{th} class: the gap is exposed between the solution brought about from the procedure whose meaning has been lost and the solution that reflects the meaning. Then conflict occurs, leading to a review of the meaning of the procedure and a revision of the procedure itself. Through this, a new understanding is achieved.

Children who apply the procedure from the 5th class. 4.2m = 4m2cm Meaning loss of the 1st to 5^{th} classes.	vs Conflict ↓	Children who solve the problem using the meaning learnedfrom the 1^{st} to 5^{th} classes. 4.2m = 4m20cm 4m10cm = 4.1m

The meaning of a place-value in decimal numbers is reviewed and recognized. Then the procedure for rewording denominate number expressions is revised.

* The discussion structure of section III includes the Hegelian meaning of dialectic process through the sublation. We will discuss this later.

The climax of the sub-unit construction is section III. Here, we will discuss the process of reaching III. Firstly, in section I, procedures are learned while being mindful of meaning. In section II, an easy-to-use procedure is acquired. As students become proficient in this procedure, some of them lose the necessity to consider the meaning. In III, they are faced with instances in which the easy-to-use procedure does not work.

At the stage of solving problems by themselves before the whole classroom discussion, each child may come confused because the easy-to-use procedures do not always work. When they participate in developmental discussions, conflict arises regarding the difference in ideas held by other students. By experiencing that conflict, the meaning as a "basis to back up procedure," which many students lost in section II, is once again recognized with a higher form of generality and the procedure is revised.

The following describes the mechanism of sub-unit construction in more general terms.

The mechanism of sub-units in which diverse ideas appear

I) Deepening the meaning: when meaning and procedure match — **It goes well!**

Here, meaning is deepened by being matched to procedure.

II) Constructing easy-to-use procedures based on meaning — **It goes well!**

Here, the "procedurization" of meaning is developed and students become proficient in easy-to-use procedures. At that time, some students fail to remember the original meaning. Even in such cases, however, the procedure will continue to result in a correct answer and no gaps in understanding will become apparent. Therefore, students experience no confusion.

III) A situation when easy-to-use procedures do not work;

A review of meaning and a revision of procedure — **Extending situation: What?**

Gaps in understanding are exposed when some students use a procedure without being mindful of the meaning and others correctly solve the problem because they remain aware of the importance of the meaning of a calculation. This causes conflict and after reviewing the meaning and the procedure, a new level of understanding is reached.

As these cases show, due to the fact that the loss of meaning that accompanies procedurization advances slowly, it is not always possible to differentiate between I and II. The problem lies within the question of how to work out the climax in III. In other words, how teachers teach enable children to overcome the conflict. Looking back on the examples, the following two points must be necessary conditions:

A) Posing task which ill understanding will appear as different answers.

Tasks should be presented in a way that there will be a conflict between children who cannot care about the meaning in acquiring the easy-to-use procedure in the session II and those who can still care about the meaning in mind. In order to do this, tasks must be presented in which students will get stuck or contradiction when both of them apply easy-to-use procedures in extending situations without any care, and may find their own idea which should be changed or reconsider the meaning.

B) Preparation of meaning that will function as the ground for developmental (dialectic) discussion and a base for understanding

For overcoming or Hegelian sublation of the conflict among difference of ideas, it is necessary for the children to understand the meaning of section I because this meaning can be used as the ground for developmental discussion.

In fact, because conflict arises by A, or in other words, children encounter results completely different from their own, they are able to ask “What? Why?” This allows them to reflect on their own ideas and take part in developmental discussions as they compare their ideas with those of others. Additionally, the mutual result from this confrontational developmental discussion makes the children produce an answer explaining why they arrived at different answers. In the developmental discussion, part B is also necessary. This is because if the children cannot understand others, if they cannot stand other’s ideas, or if they cannot reproduce other’s ideas, their discussion has no common ground as basis to argue on and talk at different purposes. If they have the ground for discussion, they can imagine what others are saying.

When children actually ask each other “why?”, the children who resorted to the easy-to-use procedure (“prioritize procedure without meaning type”) can do nothing but answer “last time 1.5*l* was 1*l* and 5*dl*, right? So I did it the same way for 4m2cm,” or “You do not make 4m10cm 4.10m (in other words, “you do not write it that way”), right?” Next, children who correctly applied the meaning to the solution begin to talk about the basis (meaning) of the procedure by saying “0.1m is 1/10 of 1m, right?” By working out the difference in the meaning of a place-value between l dl from last time and the relationship between meters and centimeters the meaning becomes clear. The children who only applied the easy-to-use procedure, and were not conscious of the meaning, become able to reproduce the correct results. Children who are satisfied with the meaning discussed are then able to revise their own ideas.

3-2. Planning a one-hour class with confirmation of previously learned tasks to reinforce children's knowledge and target tasks

The method indicated for the sub-unit construction is also useful for planning a one-hour class. That is, as previously discussed, it demonstrates how to structure a lesson that involves previously learned and target tasks. Here, we will explain Mr. Katsuro Tejima's (Joetsu University of Education) introduction to fractions for fourth graders by way of meaning and procedure, and will show the mechanism of his lesson structure (Ref: "Kazu-e-no Kankaku Wo Sodateru Shido," Elementary School, University of Tsukuba).[5]

Third grade

3/4 …….The three parts of the four equal division of whole.

3/4m……This is composed of the three parts of the four equal division of 1m.

Extending → **Extending**

Fourth grade

1 and 1/4m…This is 1m and one part of the four equal division of the other 1m.

5/4m……This is five parts of four equal division of 1m???

3/4……Three times of 1/4 unit

(Three times of a quarter)

5/4…….Five times of 1/4 unit

(Five times of a quarter)

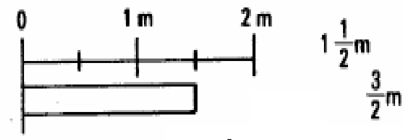
→

5/4m…..Five times of 1/4 unit

(Five times of a quarter)

→Collection of unit fraction

Firstly, Mr. Tejima focuses on that the meaning of fraction learned in the third grade is revised when improper fractions are taught in the fourth grade (see diagram above). As "five parts of four divisions of 1m" makes no sense, it is necessary to teach students about the



Is the tape 3/4m long?

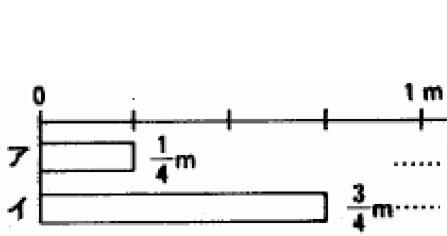
[5] The example is given based on the past curriculum standards. Fraction is introduced as a relation of the context on parts and whole at grade 3. Mixed fraction, Improper fraction, , Proper fraction, and Unit fraction are taught at 4th grade.

way of looking at improper fractions as a collection of unit fractions. Also, Mr. Tejima tries to utilize the gap between meaning and procedure that occurs in the children's thinking. In the third grade, even if the meaning of "3/4m is 3 parts of four equal division of 1m" is studied, there are children who learn it as the procedure of "if it is 3/4m, then take three of the four equal divisions of the whole" because they only learn in the case of the division of whole. As a result of applying the procedure, 2m is seen as the whole and the answer is given as 3/4 (with 'm').

He practiced the following structure of a single class that incorporates previously learned tasks and target tasks. The aim of the class is to bridge the gaps between meaning and procedure children hold and to clarify misconceptions about the meaning of fractions.

Previously learned task 1: The teacher shows the children a 1m long piece of tape and divides it into four parts in front of them. He asks them "how long is each part?"
C1: 25cm, C2: 0.25cm, C3: 4/100m, C4: 1/4m

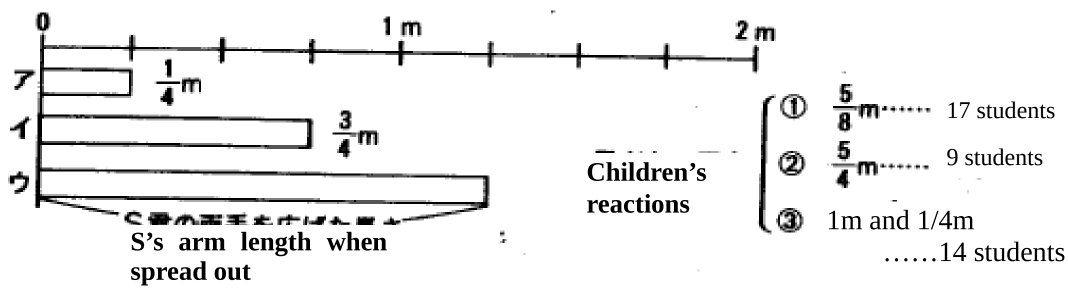
Previously learned task 2: After confirming that the length is expressed as the fraction 1/4m, the teacher then says, "Today, let's express the length of this tape in fractions." He then cuts the tape into two pieces: 1/4m and 3/4m. And, as shown below, the teacher then says, "How can we express lengths A and B in words? First, let's think of A as 1/4m."



C5 This is one part of four divisions
C6 This is one part of four divided parts from 1m.

…… This is one part of four evenly divided parts from 1m.
…… This is three parts of four evenly divided parts from

Target task: Next, the teacher takes out a piece of tape measuring 125cm. He then says, "T, the length of this tape has a connection to the human body. What do you think it is?" Following that, the teacher develops the discussion by saying "C, the length of both arms spread out." "It is an actual fact." He then says to the children, as indicated in the diagram C below, "When S spreads his arms out, the length is over 1m. How can we say this length?"



The developmental discussion unfolds through a debate about tasks 1 and 2.

C9 I think 5/4m is strange.

C10 It's five parts of the four divisions of 1m.

C(to C9) "That's right."/ "I disagree."

C11 I disagree. If you take the 1m away, 1/4m is left. 1m equals 4/4m, so if you put them together, it's 5/4m.

C13 5/4m is strange because even though 1m was split into 4 parts, the numerator is bigger than the denominator.

C14 There are one, two, three, four, five of 1/4 meters, so it's 5/4m.

C15 If it were 5/8m, then it would mean it was the fifth part of 8 evenly divided parts of 1m, but then it becomes smaller than 1m, which is strange.

Summary

If it is 5/8 of 2m, then that is correct.

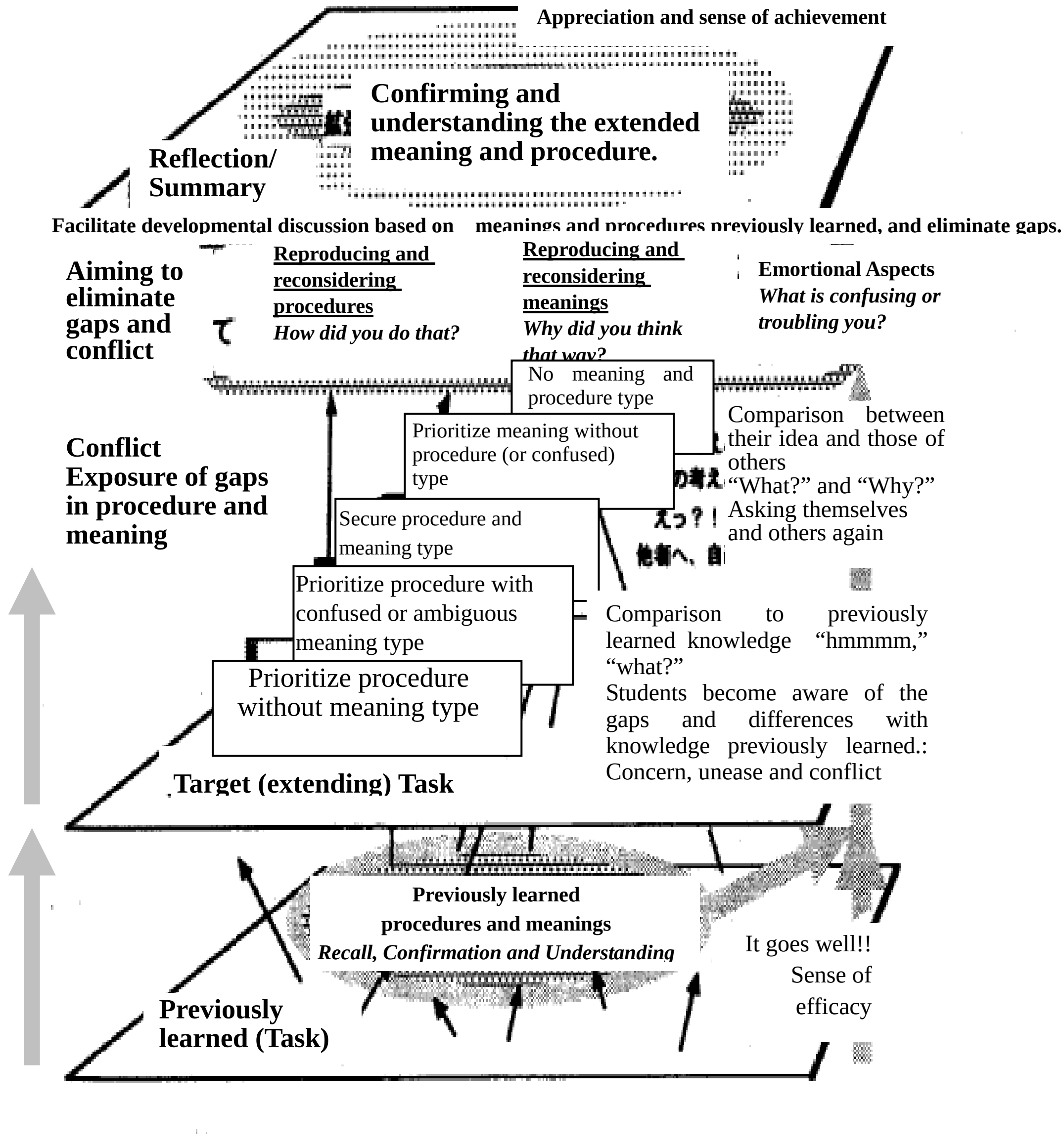
If 5/8m is written with 'm', then it becomes smaller than 1m, which is strange.

The five times the 1/4m tape length, so 5/4m is ok.

The above is an overview by Mr. Tejima. What would have happened if the teacher had begun the class by skipping the review of previously learned material and immediately use the target task? Since the target task is an extension of the previous material, a wide variety of ideas would appear. The developmental discussion would have gotten out of control and continue in the same way if children had not shared the grounding meaning of Task 1 (See, Isoda, 1993).

Mr. Tejima knows that many children come up with the answer 5/8m before his planning. The goal of this class is to make students aware of a new meaning of the collection (times) of a unit fraction, so that they may serve as a basis for a procedure known as improper fraction representation, which will be covered in the next lesson. To that goal, it is necessary to emphasize to the children the idea of collecting a number of unit fraction 1/4 m. (Children do not know the Unit fraction). At the same time, it is also necessary to revise the misunderstanding of 5/8m, which comes about from thinking of volume fractions as segmental fractions. In order to revise this idea, Mr. Tejima reminds children to consider the volume in Task 1 and asks the children if they can confirm that 25cm=0.25m=1/4m. In Task 2, he reviews the accurate definition of volume fractions, confirms it as well as tests it in the target task by placing in order (in odd number sequence) the 1/4 m and 3/4m in the tape diagram on a number line. By being able to create this flow of context, it becomes easy to be aware of “how many 1/4m parts” there are, such as in the answer of 5/4m. Furthermore, the idea of 5/8, which was obtained without a meaning, is “5 parts of 8 even devision of 1m.” This was obtained by applying the previously learned definition of fraction numbers. Children will realize that 5/8 is smaller than 1m. Here, counter-examples are effective: “5/8m should be smaller than 3/4m, so it’s not right.” The developmental discussion was successful, because the meaning and procedure that form the basis of discussion had been confirmed in Task 1 and Task 2 before considering the meaning and procedure in the target Task 3.

As for the conclusion, Mr. Masaki’s parallel lesson, Mr. Suzuki’s division lesson and Mr. Tejima’s class can all be summarized to be consisted of the mechanism shown below.



In order to conduct the lesson which includes such a mechanism, the following work (A-D) is necessary for the planning.

A) Investigate which stage of extension this class constructs within the curriculum sequence, and what kind of changes are necessary regarding the procedure and meaning to achieve the class goals.
B) Consider what types of target tasks are necessary to extend the material.
C) Anticipate what kind of reactions and gaps in meaning and procedure will appear when the children in your class tackle the target task learned in a previous situation.
D) Prepare tasks that review previous material to determine what needs to be covered in terms of meaning and procedure in order to perform the target task. This will also allow the creation of a base for developmental discussion, which will examine what the ground of meaning is necessary for the elimination of gaps that appear during the target tasks.

If the class is conceived from the flow of unit structure, the previously learned task portion can and often is put into the most recent class in consideration of the above.

3-3. Developmental discussion to eliminate (bridging) gaps

Upon reflection, developmental discussion takes place with the aim of eliminating conflict caused by gaps.

Developmental (Dialectic) discussion that eliminates gaps in diverse ideas

The reactions of children who are no longer aware of the meaning of the procedure	⟺	**The reactions of children who remain aware of the meaning of the procedure**

Children who are aware of the meaning of the procedure and children who are not aware of it contradict each other. Here, the discussion develops based on the ideas and concerns of children who have an ambiguous understanding of meaning or procedure.

In order to eliminate contradictions and gaps, it is necessary for children to persuade to revise other's ideas.

If what has been discussed so far is taken into consideration, it is conceivable that developmental discussion will progress in a direction of expectation if the following two points can be done.

1) Developing "Hmmm" and "Why?" in the consciousness

When children are solving new problems by themselves, they become concerned and uneasy and think "is it ok to do it like this?" This concern and uneasiness are manifested in children's feelings when they find gaps in the meanings and procedures of previously learned tasks. However, once the children have successfully answered the task question, they feel better and forget such kinds of important feelings. If children lose the desire to eliminate concern and uneasiness from within themselves, they are not able to understand the complex ideas of others. Moreover, they are not able to stand other's position and revise their own ideas by sharing their opinions with their classmates. Children who "prioritize procedure with confused or ambiguous meaning" and "prioritize meaning with ambiguous procedure," often display this type of concern and uneasiness. Therefore, the use of such concerns and uneasiness makes it easy to connect to the benefits of developmental discussion.

2) Sharing understandings of meanings which will serve as the basis for the developmental discussion

Mutual differences in procedures are exposed as gaps during the developmental discussion. In order to eliminate those gaps, children must talk about the meanings of the basis for each other's procedures by asking, "why did you think that way?" Additionally, if they do not share or understand each other's interpretation, they cannot revise their own procedures.

Using the above two points as a premise, the following two points can be shown as measures to set up and summarize developmental discussion.

A) Searching for a mutually recognized meaning to enable children to share a logical explanation as a base.

B) Indicating which idea is correct to allow contradictions to arise.

It is fundamental for a developmental discussion to be planned for A. However, it is not easy for the children to share meanings. This is because it is difficult to respond when listening to another person's comments. If students quarrel, a proper debate becomes hard to establish and those involved cannot break away from their own ideas and affirmations. Here, the following teaching skills become necessary (Kimiharu Sato, 1995).

- When different ideas are shown, give children time to reconsider why they think their idea is appropriate, so that they can explain why they think that way.
  Example: Get children to write down their ideas regarding why they think that way.
- Develop the points of confusion and concern as points of discussion in order to organize them within the developmental discussion.
  Example: Ask children to comment on their points of confusion and concern.
- Organize the points of discussion so that arbitrary comments do not cause the developmental discussion to get out of hand.
  Example: “Try to say that again,” “Hold on, I understand what he/she said,” “That’s good. Can someone translate it?” “Well, the points of discussion are on different planes now. Let me reorganize the problem.”

Using these teaching techniques, the teacher encourages children to find a meaning that everyone is satisfied with and ideas can be presented logically based on this meaning. In such a developmental discussion, B usually becomes necessary. In the first part of B, presuming that “the other person is right” is a necessary condition for considering the other person’s perspective. In other words, what is the premise used for consideration so that children will reach such a result? In order to reach this result, children are required to conceive what premises the other children are basing their ideas on. However, it is not an easy task to reproduce another person’s ideas. In actual fact, when performing a task which exceeds the ‘if’ conditions of a procedure that works, it is not uncommon that more than half of the children misconceive the problem and use a procedure without any meaning. Among those children, some answer the way they do because they are unable to understand the reason for that meaning and seek to understand its basis. In that case, even if they listen to another person’s explanation, they cannot agree with the other person’s idea due to the fact that they are unable to understand what the other person is talking about, because they cannot understand the premise on which that person’s idea is based. When this happens, first it is necessary to make the children aware that failing to take the premises into account will cause confusion. The good way to persuade is that the person temporarily accepts the other’s idea even if it is very different from his/hers, continues to use the idea in another case, and then indicates that it will contradict what they already learned before (the latter half of B). This is the Socratic dialectic method used since ancient Greek times, and is the origin of the reductio ad absurdum (reduction to absurdity) in terms of mathematics. In simply words, it is an indication of a counterexample. If the other person does not understand it as a counterexample, it is not effective. Accordingly, the following section examines two methods that are effective in making counterexamples.

a. What if A's idea is correct?

Here is an example. Mr. Hidenori Tanaka, a teacher at Sapporo Municipal Ishiyama-Minami Elementary School, is teaching fifth graders the addition of different denominate number fractions using 1/2 + 1/3. Among the students, some of them give the answer as 2/5. This answer shows a student in the "prioritized procedure without meaning type: The students merely added the fractions together, top and bottom, without understanding the meaning. Additionally, some of the students advocated the mistaken meaning by arguing (○●)+(○●●)=(○○●●●) ("prioritizing procedure with confused or ambiguous meaning type"). For children who think this explanation is correct, it shows a lack of understanding of fractions, since it is impossible to add fractions together which are in different units. For that reason, even if the children were able to understand their classmate's explanation using a diagram, they would not understand why a classmate would say their own diagram explanation was wrong. What disproves their misguided understanding is the rebuttal, "so, have you ever added up denominators before?" According to this procedure, 1/2+1/2=2/4=1/2, and as the students see it, (○●)+ (○●)= (○○●●). Looking at it this way goes against what has been previously learned. Accordingly, this type of refutation, which is not a straight denial of that person's idea, uses their answer as an opportunity to critique their way of thinking, and is therefore quite convincing.

b. Facilitating awareness through application of tasks in different situations and examples

The excelent aspect of asking, "what if A's idea is correct?" is that it makes use of a procedure without meaning. In doing so, it focuses on the contradiction in procedure that the student has used rather than the meaning he or she does not understand. The use of A's procedure allows him or her to realize his or her own misconception of the procedure. This is the same method seen in Mr. Tejima's class. There are also times, however, when the contradiction needs to be indicated in new examples. Here, we present an example of this method – the third grade fraction class run by Ms. Mikiko Iwabuchi, a teacher at Sapporo Municipal Kitasono Elementary School in Sapporo. In this example, a shift from segmental fractions to volume fractions is planned.

1st class: Halves…dividing equally…introduction of fraction segmentation using 1/2.
It goes well!
2nd class: "Let's make 1/4." Using fraction segmentation. *It goes well!*
The teacher asks children to make a 1/4 size piece of colored paper and tape to send to their sister school, Astor Elementary, for its music festival.
3rd class: "Let's make 1/4m." quantities' fraction introduction. *What?*
The teacher wants the students to cut a 1/4m length of tape to send to their sister school's student festival. Make sure the measurement is right.

A) The original size of the tape can be any size, so if the whole length is not given, it is not set. two children "prioritize the procedure with confused or ambiguous meaning type."
B) 4m is divided evenly, each piece is 1m. sixteen children "prioritize procedure with ambiguous or no meaning type."
C) One piece from 1m is divided evenly (25cm). Nineteen children "secure procedure and meaning type."

4th class: "Let's make 1/2m." quantities' fraction introduction (continued from the 3rd class).

In this lesson plan, the meaning of segmental fractions is used as a base to define quantities' fractions. This definition creates a limit in a shift of the meaning from "*n* parts of the *m* equal division of the whole" to "*n* parts of *m* equal division of the unit quantity." Up until the second class, students have only studied segmental fractions, so there are discrepancies in the semantic interpretation of the answer as 1/4m in the third class. The results are wide ranging.[6] Debate arises among students, and as expected, conflict is seen between students who chose answer B and students who chose answer C. In particular, as 1/4m is read as "1 of 4 parts"m in Japanese, it is easy for students to arrive at the idea that the number is four times the standard of 1m. As an idea to support C, a student claimed "it should be shorter than the original length" to make use of the meaning studied in segmented fractions. Another is the indication expressed in the comment "if 1/4m =1m, you should say 1m, otherwise it's strange." However, because the meaning of 1/4m is undefined and discrepant, the students who are listening will not be able to make sense of it. Therefore, in the fourth class, the students are asked about the case of 1/2m. If B is correct, 1/2=1m and 1/4m=1m, and so you would have "1/2m=1/4m," which again is strange and a debate centering on "it should be shorter in the order of 1/2m, 1/4m, 1/10m," would occur from the perspective of what was learned about segmental fractions. In other words, a conclusion that answer C is correct can be reached because the meaning and logic of segmental fractions studied in the second class does not match answer B from the first class.

Based on the above discussion, the second chapter will show the practice of developmental

6 Here, when the meaning matches the definition, it is classified as a "secured meaning type," however, as this is at a stage before definition, it does not mean that others are misconceptions.

discussion classes that lead to the creation of diverse ideas. In chapter 3, we will analyze lesson planning and deployment techniques of teachers in the Elementary School affiliated with the University of Tsukuba.

Notes & References

For readers who will use the theory in this book as an academic research, the following is an explanation of the research path, its position in mathematics education, as well as the reference materials used in making this book.

In the mid 1980s, It can be said that the theoretical framework for problem-solving approach, as it is now known in Japan, has already developed. In fact, actually, the contents provided here do not differ much from the research that has been done after constructivism became a significant issue for debate in the mid 1980s. Furthermore, as far as teaching practice is concerned, the level of classes run by teachers with problem-solving techniques in Japan are considered to rank at the top, even from the perspective of constructivists. For example, Dr. J. Confrey (vice-chairperson of International Group for the Psychology of Mathematics Education in the 1995, this book was written), who leads the sublation of radical constructivism and social constructivism, has given a high evaluation of this paper as a constructivist piece in terms of the classes and ideas presented in it.

However, in the early 1980s and 1990s, there was a gap. For example, in the early 1980s, the discussion of diverse ideas was in terms of the diversity of correct ideas the with open-ended problems. One factor that changed that trend was research on understanding. This chapter has been written to include the method to describe the phases of understanding – conceptual knowledge and procedural knowledge theory – as part of research on understanding, as well as to show the theoretical aspects of the problem-solving class and teaching practice of teachers from Sapporo. Also, the following papers act as a framework for this chapter.

Masami Isoda (the author), *“Katto to Nattoku wo Motomeru Mondai Kaiketsu Jugyo no Kozo,”* Riron to Jissen no Kai Chukan Hokokusho, 1991

I have studied much from the following researchers in order to acquire my theory:

Toshio Odaka & Koji Okamoto: *Chugakko Sugaku no Gakushu Kadai.* Toyokan Publishing Co., Ltd., 1982

Tadao Kaneko: *Sansu wo Tsukuridasu Kodomo.* Meijitoshoshuppan Corporation, 1985

Katsuhiko Shimizu: *Sugaku Gakushu ni Okeru Gainenteki Chishiki to Tetsuzukiteki Chishiki no Kanren ni Tsuite no Ichi-kosatsu.* Tsukuba Sugaku Kyoiku Kenkyu, 1989 (co-authored with Yasuhiro Suzuki)

Katsuro Tejima. *Sansuka, Mondai Kaiketsu no Jugyo.* Meijitoshoshuppan Corporation, 1985

J. Hiebert. *Conceptual and Procedural Knowledge: The Case of Mathematics.* LEA, 1986

Toshiakira Fujii. *Rikai to Ninchiteki Conflict ni tsuite no Ichi-kosatsu.* Report of Mathematical

Education, 1985

The originality of this book lies in the following areas: using a descriptive method of children's understanding as class material and class conception; the applied research methods; and, under the theme of intelligence viability, the occurrence and resolution of problem situations due to gaps in procedure and meaning that come about from extending and generalization. Dr. J. Hiebert, who is known as the leading authority in conceptual and procedural knowledge theory has appraised the theory as being appropriate and states that the above-mentioned areas have not been documented before.

Below are the references and contents that could not be included in the book although they too are worthy of use in this context.

Author's material:

Sansu Jugyo ni okeru Settoku no Ronri wo Saguru, Kyoka to Kodomo to Kotoba. Tokyo Shoseki Co., Ltd., 1993.

Miwa Tatsuro Sensei Taikan Kinen Ronbun Henshu-iinkai-hen. *Gakushu Katei ni Okeru Hyougen to Imi no Seisei ni Kansuru Ichi-kosatsu, Sugaku Kyouikugaku no Shinpo.* Toyokan Publishing Co., Ltd., 1993.

Sugaku Gakushu ni Okeru Kakucho no Ronri – Keishiki Fueki to Imi no Henyo ni Chakumoku Shite. Furuto Rei Sensei Kinen Ronbunshu Henshu-iinkai. *Gakko Sugaku no Kaizen.* Toyokan Publishing Co., Ltd., 1995

Mondai Kaiketsu no Shido. Shogakko Sansu Jissen Shido Zenshu 11 Kan, Nobuhiko Noda (Ed). *Mondai Kaiketsu no Noryoku wo Sodateru Shidou.* Nihon Kyouiku Tosho Center, 1995

Kimiharu Sato. *Neriai wo Toshite Takameru Shingakuryoku.* Kyouiku Kagaku, *Sansuu Kyouiku* September 1995 issue

In this book, feeling has been put into the use of some words in Japanese. For example, the term "developmental discussion" has been used to describe the aim of restructuring the meanings and procedures that children have through dialectical conversations with them. Furthermore, from the standpoints of "if there is nothing extraordinary, then the idea cannot be truly tried or structured" and "extending the concept cannot be done without the risk of over generalization," we replaced the word "error (*Ayamari* in Japanese)" with "over-generalized idea. (*Kari* but read *Ayamari* in Japanese)" This is in line with the meaning of misconception and at the same time is used in the background of alternative framework on the theory of constructivism.

The Secrets of the 5 Times Multiplication Table (6th **lesson of 13 lessons)**

(Each student developed at least one secret card of each property about 5 times multiplication in last lesson)

Teacher (First question of the beginning of the lesson): "Are you just going to hold onto your secret cards?"<1>

Students (begin to place cards on the blackboard). <1>

Teacher (walks to desks of students lagging behind to provide individual guidance. Pretends not to see the work on the blackboard. More and more students finish putting up the cards and return to their desks).

Teacher: "Is that what you want? Putting them all over the place like that?" (speaks in a stern voice, moves to the blackboard from the backside of classroom).

Taro (walking over to the teacher to make his point): "Like, for example, you could use the chalk to draw lines on the blackboard, and then write the secrets." (Photo 1)

Teacher (stops for a moment to think). <2>

Teacher: "You still want to scatter them around like that? Can you understand anything from this?" <2>

Teacher (after a while): "Taro's going to give us some ideas for putting up the cards. Let's listen to him." <2>

Taro (abbreviated): "Like, using the chalk to write, 'all of the cards have 5s' or something like that."

Taro (abbreviated): "Like, using the chalk to write, 'all of the cards have 5s' or something like that."

A Student: "People think the same thing?"

Teacher: "Taro's trying to say that if you write down your idea, and then attach the card with your own secret under that, it will be easier to organize the ideas. You see now? (slight pause) Yesterday, the comment given most often was, 'the answers go up 5 at a time.' (writes this on the whiteboard off to the side of the classroom) So write like this, and then stick the cards to the blackboard. <3>

Students (all begin to rearrange the cards on the blackboard).

Teacher (listens to the students talking, and then draws lines to categorize the cards. (Photo 2) <3>

Photo 2

Explanation of Norm and Evaluation

<1>Norm: In this classroom, the classroom has already a norm for the method of comparing cards with classmates, putting up your own cards with classmates' cards judged to be of the same contents and then working together to think about and arrange them.

<1>Teacher's goal: Arranging a situation in which the individual students will need to consciously compare the ideas of others with their own, thereby fostering their own thinking powers.

Photo 1

<2>Evaluation: The teacher decides to intervene after looking at how the cards have all been attached in scattered fashion and judging that leaving everything up to the students won't work out. An instant conclusion is reached on how to get the students to rearrange the cards on their own, with the next step planned. To begin, the students are encouraged to reorganize the cards on their own. If that doesn't go well, the proposal made by Taro will also be used.

<3>Evaluation: Taro's explanation is insufficient, and a student recognized that he is talking about students who wrote down "all of them have 5s attached." and gathering it. The teacher sees that the majority of students fail to understand what he proposed, and uses the example of "the answers go up 5 at a time." He doesn't write this on the blackboard, which has no workspace left, and instead uses the whiteboard off to the side of the classroom which he already prepared it before the classroom.

<2>, <3> Forming Norm with regard to "rulemaking": The teacher's response to Taro illustrates that, for the rulemaking in this classroom, the ideas of students are reflected in responses to the teacher's needs.

Photo 2. Evaluation: As seen in the comparative examination session to follow, while the teacher realizes that the students will be unable to systematically classify their own thoughts, he intentionally places ideas in the same brackets according to how the thoughts were explained. It is here that the decision by the teacher to offer guidance clearly appears.

At the Discussion of Whole Classroom (6th lesson)

Teacher: "Someone put up cards that say, 'The numbers increase, or get bigger.' Will one of you explain that?" <4>

Kumi (raising her hand): "With the 5 times multiplication we did this time, the numbers increase like it says on the blackboard. For example, with 5 x 1 the next equation is 5 x 2. So the numbers change from 1 to 2." (Photo 3)

Photo 3

Teacher (as Kumi explains to teacher, he turns his back to her and moves from the middle of the group off to the side). <5>

Yuki (raising her hand): "The answers steadily increase by the first number in each equation."

Teacher: "Can you tell that there is actually a slight difference between what Kumi and Yuki said?"

Secrets of 2 Times Multiplication Table (8th lesson of 13 lessons)

(Leaning ways of 5 times multiplication table were used for the 2 times multiplication table. At the 7th lesson, the teacher was absent and students developed 2 times table by themselves with their experience until 6th lesson. Secrets were discovered, then written on the cards. At 8th lesson, students displayed cards)

Students (begin to place cards on the blackboard).

Teacher (breaking into a conversation in some students): "Are these the same? So if someone says they're the same, that's all right?" <4>

Teacher (breaking into a conversation in other students): "Is the counting method of jumping 2 at a time and increasing by 2 each time the same thing?" (writes on the blackboard, then stares at it).

Students (begin to talk among themselves again).

Teacher: "So, I'll let you go ahead on your own, OK?" (Kumi is behind the teacher in Photo 4) <5>

Kumi (joining with another student who has come up to put on cards, staring at the blackboard together): "I don't think these are the same." (Photo 5).

Explanation of Norm and Evaluation

<4>Evaluation/rulemaking: Two different ideas are placed in the same category. The students mutually realize that their examination at the blackboard was insufficient, compare to see if their ideas are the same or different with those of their classmates, then put their heads together to see if that can come up with some rules. This approach is used in an attempt to foster students capable of conducting comparative examinations on their own volition.

<5> Forming Norm with regard to "rulemaking": The teacher intentionally leaves the center of the group, to help the students cultivate the skills to work together in conducting comparative examinations.

Photo 4

Photo 5

Photo 5. Kumi's progress: At the Discussion of 6th lesson, three classes ago, Kumi was unable to determine whether the ideas were the same or different. She recognize it in front of all people. At this session, after being told by the teacher that the students would be responsible for figuring things out by themselves, she begins to closely examine the idea of classmates who bring up more cards to attach to the blackboard. Clearly surfacing here is the attitude to conduct her own comparisons, while considering whether the ideas expressed by others are the same as hers. It can be recognized here that the teacher's goal has borne fruit as rules in Kumi's approach.