DEVELOPING MATHEMATICAL THINKING THROUGH LESSON STUDY: OVERCOMING BARRIERS TO EFFECTIVE IMPLEMENTATION

Patsy Wang-Iverson

Gabriella and Paul Rosenbaum Foundation, USA

Implementing jugyo kenkyu (lesson study) effectively to develop students' mathematics thinking in the United States is proving to be quite challenging for a variety of reasons, including (1) teachers' mathematics content knowledge, (2) the plethora of textbooks that do not follow a coherent and concise scope and sequence, (3) mathematics standards that vary from state to state, (4) different conflicting perspectives on what is effective mathematics instruction, and (5) a policy mandate requiring schools to focus on student test scores rather than student learning.

BACKGROUND

Interest in lesson study in the United States has been attributed (see Lewis & Perry, 2006; Takahashi, 2006) to the following sequence of events: release of the assessment and survey data from the Third International Mathematics and Science Study beginning in late 1996 (TIMSS; http://nces.ed.gov/timss; http://www.timss.org) followed in 1999 by the release of a video study of representative eighth grade mathematics classrooms from Germany, Japan, and the United States (USED, February, 1999) and the publication of *The Teaching Gap* (Stigler & Hiebert, 1999).

As part of the video study, U.S. teachers were asked, in advance of being videotaped, if they were familiar with the NCTM *Professional standards for teaching mathematics* (released in 1991), and whether they believed they were implementing these standards in their classroom (USED, February, 1999). The answer was "yes" to both questions. However, analysis of the videotaped classes revealed that it was the Japanese classrooms, and not the U.S. classrooms, which better represented what was called for in the NCTM standards (USED, February, 1999, p. 122). The videos captured the attention of educators, who sought to learn how Japanese teachers acquired their teaching skills. A possible answer was suggested in *The Teaching Gap*, which included a chapter on lesson study, excerpted from Makoto Yoshida's doctoral dissertation, which later was published as a book (Fernandez & Yoshida, 2002).

Although not explicitly named as such, the components of lesson study were described in Harold Stevenson's earlier detailed research examining mathematics classrooms in Japan, China, and the United States (Stevenson, 1987; Stevenson, 1993; Stevenson & Stigler, 1992; Stigler & Stevenson, 1991). These earlier findings were further corroborated in the Case Study Project of education in Germany, Japan and Germany directed by Stevenson and Lee as part of TIMSS 1995 (USED, June, 1998; USED, March, 1999; USED, 1999; Kinney, 1997-98).

Other researchers also have contributed to our understanding of lesson study through work either not directly connected with TIMSS (Lewis & Tsuchida, 1997; Lewis & Tsuchida, 1998; Lewis, 2002; Fernandez & Yoshida, 2004), or which have built upon TIMSS (Askey & Wang-Iverson, 2005; Wang-Iverson & Yoshida, 2005). Still others have identified characteristics of effective instruction which are similar to work connected with lesson study (e.g. Hirsch, 1996). U.S. implementation of lesson study is still in its infancy compared to its history in Japan, which dates back to 1873 and was informed by Western ideas (Isoda, 2006).

MISUNDERSTANDING OF LESSON STUDY?

Although the 1995 TIMSS videos inspired many educators to learn about lesson study, they also contributed to a misunderstanding that lesson study would help teachers learn how to teach like the Japanese (Fernandez, in Wang-Iverson & Yoshida, 2005, p. 98) and that there was one right way of instructing students. These 1995 videos of representative Japanese lessons all presented a problem-based approach to learning mathematics, which led some U.S. educators to think that all Japanese lessons were structured in this format. However, examination of the TIMSS data revealed that Japanese teachers used this format in less than half of the videotaped research lessons (44%); they devoted a comparable amount of time to having students practice routine procedures (USED, February, 1999, p. 102). Japanese teachers have been engaged in lesson study since 1873, and in that time, teaching approaches have changed, suggesting that *lesson study does not lead to any one particular way of teaching.* Rather, it serves as a vehicle for teachers to make improvements collaboratively in teaching and learning through better understanding of student learning, thinking, and misunderstanding and observing each other's classes.

For some people, engaging in lesson study seems to be equated with developing lessons that not only are one problem-based, but also "student-centered" and "constructivist" in nature, with students given the freedom to "explore" and "discover" (Inprasitha, 2006; Lim, 2006; Takahashi, 2006). When confronted with the term "student-centered", many people conjure up an image of a classroom where students are sitting in groups and working cooperatively. Might this imply that classrooms with teachers in the front doing most of the talking are bad (Leung, 2006; Lim, 2006)? Such mental images of effective mathematics instruction have created a tension between a teacher-centered versus a student-centered classroom (Lim, 2006). Does this debate distract teachers from focusing on student learning of mathematics?

In TIMSS 1999, the video study component included seven countries: Australia, Czech Republic, Hong Kong, Japan (reanalysis of 1995 videos), Netherlands, Switzerland, and the U.S. The six other countries were selected based upon their students' higher performance in mathematics than the U.S. One of the questions posed was whether these five other higher achieving countries exhibited teaching practices similar to Japan. The answer was a resounding “no.” The Hong Kong eighth grade mathematics classes in the TIMSS 1999 video study were coded as the most “traditional”, with students uttering few words, while the teacher stood in the front of the class, doing most of the talking (USED, March, 2003; Leung, 2006). Ironically, these lessons were rated the most highly by a team of mathematicians and mathematics educators who analyzed the lesson transcripts with respect to lesson coherence, level of mathematics, and degree of student engagement. Lim reported similar findings; better results apparently were obtained with teachers at the front of the class (Lim, 2006). Secondary analysis of the TIMSS 1999 public release mathematics lessons by teams of mathematicians and pre-service and in-service mathematics educators confirmed the coherence of the Hong Kong lessons (Askey & Wang-Iverson, 2005). The issue of effective mathematics learning, however, lies not in the format of a classroom, but in what mathematics students have opportunities to learn.

IMPROVING MATHEMATICAL THINKING THROUGH LESSON STUDY

In recent years, educational research has focused on trying to better understand how students learn (NRC, 2001; NRC, 2005). The theme of the December 2006 APEC Conference is on mathematical thinking, a topic of personal interest. I have observed many U.S. mathematics classrooms in different states; in too many cases, I could detect no opportunity for student thinking, mathematical or otherwise. What passes for student thinking frequently consists of students trying to guess the answer the teacher is seeking. When students parrot an answer the teacher wants, how can the teacher truly assess student thinking and understanding?

What needs to occur in a classroom to stimulate students’ mathematical thinking, and how might lesson study improve teachers’ abilities in this area? Can evidence of student thinking be observed in a class where the teacher is in front of the room? On the other hand can students be busily engaged in activities without displaying any mathematical thinking?

How is mathematical thinking defined in U.S. curriculum documents?

Curriculum documents, or textbooks, vary by school, by district, and by state in the U.S (Schmidt, Houang, & Cogan 2002, Fig. 2: p. 5, Fig. 3: p. 11). They are informed by state standards, which in turn may or may not be informed by the NCTM *Principles and Standards in School Mathematics* (2001). For these reasons, it is difficult to provide a succinct definition of mathematical thinking in this country. The NCTM standards are delineated by grade bands (K-2, 3-5, 6-8, 9-12) and do not provide grade level teachers with explicit guidelines for what they should teach and what mathematical thinking entails. In an effort to explicate their recommendations in more concrete and specific language, the National Council of Teachers of Mathematics recently published *Curriculum focal points for prekindergarten through grade 8 mathematics* (2006), with three focal points per grade. This document, in conjunction with earlier publications that elucidate ways in which students can be helped to develop their ability to think mathematically (NRC, 2001), aim to move U.S. mathematics away from its earlier incoherent states (Fig. 2 and 3) described by Schmidt, Houang & Cogan (2002) and contrasted with the scope and sequence of the A+ countries (Schmidt, Houang, & Cogan, 2002, Fig. 1: p. 4), which is reproduced below:

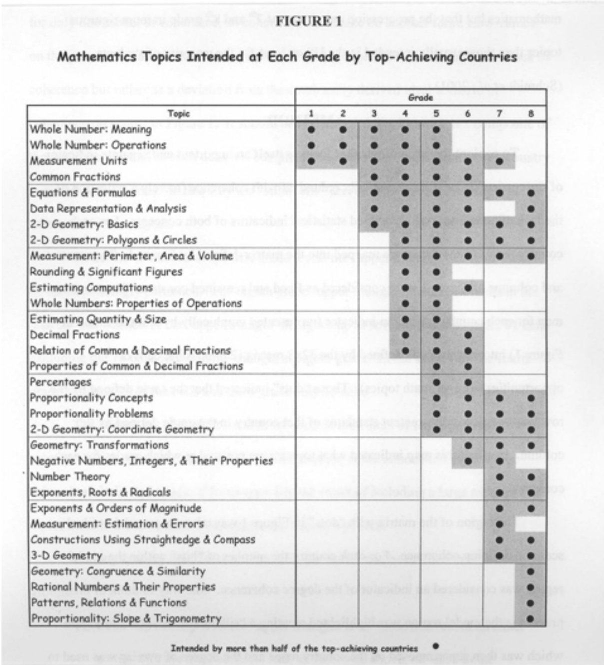
FIGURE 1

Mathematics Topics Intended at Each Grade by Top-Achieving Countries

Topic	Grade 1	2	3	4	5	6	7	8
Whole Number: Meaning	●	●	●	●	●			
Whole Number: Operations	●	●	●	●	●			
Measurement Units	●	●	●	●	●	●	●	
Common Fractions			●	●	●	●		
Equations & Formulas			●	●	●	●	●	●
Data Representation & Analysis			●	●	●	●		●
2-D Geometry: Basics			●	●	●	●	●	●
2-D Geometry: Polygons & Circles				●	●	●	●	●
Measurement: Perimeter, Area & Volume				●	●	●	●	●
Rounding & Significant Figures				●	●			
Estimating Computations				●	●	●		
Whole Numbers: Properties of Operations				●	●			
Estimating Quantity & Size				●	●			
Decimal Fractions				●	●	●		
Relation of Common & Decimal Fractions				●	●	●		
Properties of Common & Decimal Fractions					●	●		
Percentages					●	●		
Proportionality Concepts					●	●	●	●
Proportionality Problems					●	●	●	●
2-D Geometry: Coordinate Geometry					●	●	●	●
Geometry: Transformations						●	●	●
Negative Numbers, Integers, & Their Properties						●	●	
Number Theory							●	●
Exponents, Roots & Radicals							●	●
Exponents & Orders of Magnitude							●	●
Measurement: Estimation & Errors							●	
Constructions Using Straightedge & Compass							●	●
3-D Geometry							●	●
Geometry: Congruence & Similarity								●
Rational Numbers & Their Properties								●
Patterns, Relations & Functions								●
Proportionality: Slope & Trigonometry								●

Intended by more than half of the top-achieving countries ●

The A+ countries identified by Schmidt et al. were the top achievers in TIMSS 1995: Belgium (Flemish), Czech Republic, Hong Kong, Japan, Korea, and Singapore (Beaton et al., 1996; Mullis et al., 1997).

How is mathematical thinking defined (or not defined) in two lessons?

To prepare students to think mathematically, it is important for teachers to understand students' current state of thinking and know how to move them to the next level. I will use two lessons from two different schools as illustration:

Lesson I. A lesson plan was developed using a problem adapted from a grade 6 NSF-funded curriculum and taught out of sequence to a mathematics class not belonging to the teacher who taught the lesson:

> Paulo and Paula are tending the brownie booth at the school fair. The brownies are baked in rectangular pans, and they can be sold as fractional parts of a pan. A full pan of brownies costs $24. The cost of any portion of a pan is that fraction of $24.
>
> One pan of brownies was 2/3 full. Mr. Sims bought ½ of what was in the pan.
>
> 1. What fraction of a full pan did Mr. Sims buy?
>
> 2. How much did he pay?
>
> Use pictures, words, and/or a number sentence to explain your thinking.

The teacher asked students to read the problem out loud and then asked for explanations of what the problem was asking them to do. Various students selected different parts of the problem to repeat. The teacher followed up by asking them to explain in their own words. To further assess their understanding, the teacher asked students to identify the operation(s) they would use. Students (S) volunteered the following answers:

S1: multiplication

S2: division

S3: subtraction

When a fourth student suggested addition as a possibility, other students responded that it would not work, indicating that they thought the first three operations were all possible. The teacher then asked students to begin working in pairs to solve the problem, without commenting on which, if any, of the three proposed operations was correct or incorrect.

A number of students wrote down their answers without showing their work:

#1 – ½

#2 - $12

Their answers implied that they misunderstood the wording of the problem and simplified it to their level of understanding or overlooked the 2/3 in the problem statement. Another pair of students set up the problem as a division problem in the following manner:

$\frac{2}{3}\overline{\left)\frac{1}{2}\right.}$

and then could not progress beyond this point.

Other students' work can be seen in Fig. 1-4. In Fig. 1, observers not familiar with the class initially could not understand why the students had changed 24 into 24/100. During the post-lesson discussion, it was explained that in the previous lesson the students had learned about percent. This example shows how students try to integrate what they previously learned into a new problem, even though there may not be a connection between the two situations. Additionally, in this case the students attempted to incorporate all the numbers from the problem in their computation even though what they were doing made no sense to them.

In Figures 2-4, in response to the first suggestion offered by the teacher to "use pictures", students drew pictures to represent the situation. In Figure 2, the students identified the operation as division of $24 by 3 rather than multiplication of $24 by 1/3. It could not be ascertained if they understood clearly that multiplying $24 by 1/3 was equivalent to dividing $24 by 3. They wrote down 3 divided by 24, although what they meant was the reverse. In Figure 3, the students showed their understanding by recording their thinking in a logical sequence, although $24 of 1/3 is not mathematically correct. In Figure 4, one can only see the beginning of an attempt to write a mathematical expression.

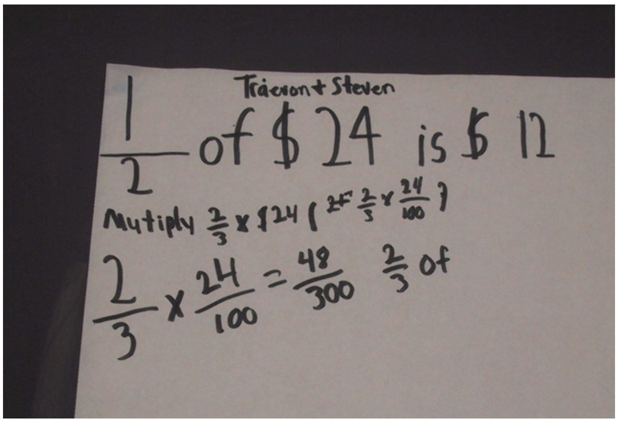

Figure 1

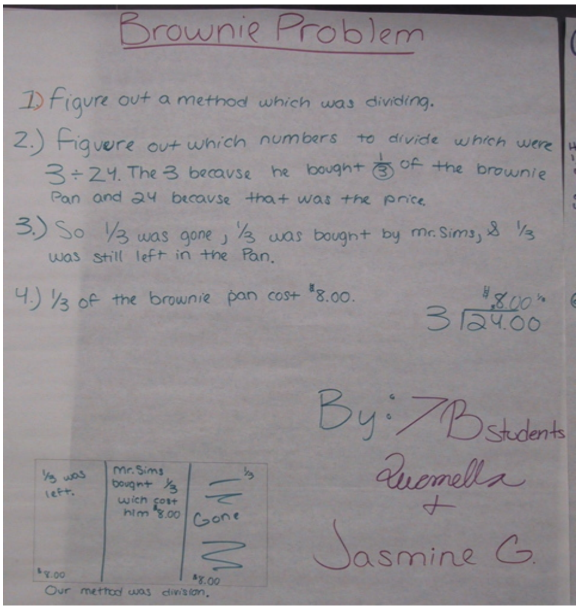

Figure 2

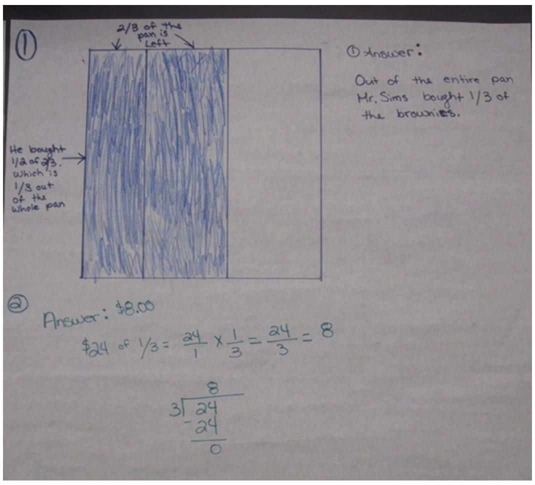

Figure 3

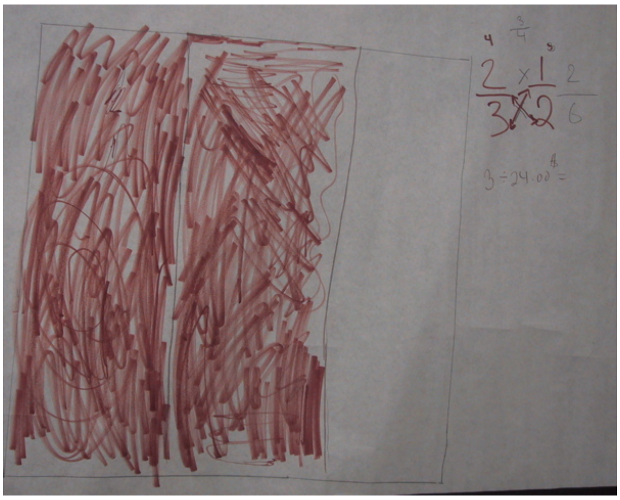

Figure 4

Lesson II. In this fifth grade lesson on angles and figures, the teacher first reminds the students of what they had learned previously and then puts up many different angles and asks students to go to the board to try to form a triangle using their choice of a combination of three angles (Fig. 5).

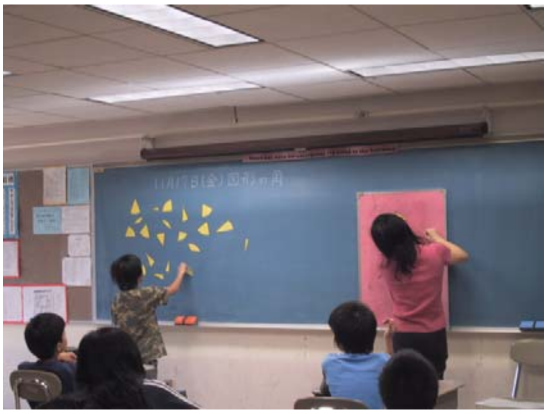

Figure 5

A number of students who first went to the board seemed to be selecting angles at random, without engaging in any mathematical thinking. Seated students approved of or disagreed with angles selected. Eventually, through collaborative effort, the class came up with five sets of angle combinations to form triangles (Fig. 6). The teacher asked what they were thinking as they selected certain angle combinations. As students described the angles they selected, such as thin, the teacher asked what they meant by thin, which prompted the student to respond “measure of small angle.”

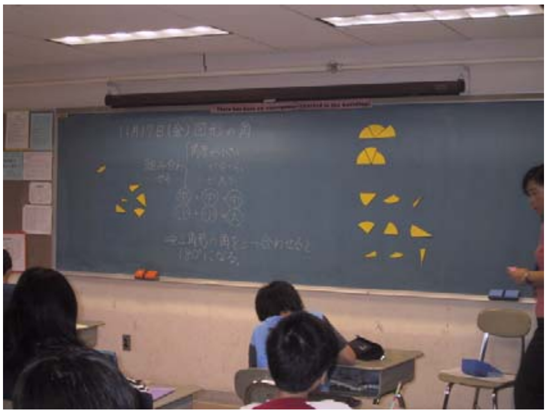

Figure 6

To stimulate further student thinking, the teacher said, “To make a triangle, there may be a pattern in which angle measures are combined.” Students responded with “medium, medium, medium; small, small, large.” The teacher asked if these ideas could be used to find other combinations that would work. Students came up with an additional combination of small, medium, and large. The teacher then demonstrated on the board how the three angles could be combined to form a straight line (Fig. 6; right side). She asked if it was really true that the sum of three angles of any triangle is 180^o, or were students simply accepting this fact? To allow them to determine for themselves whether the sum of the angles is, indeed, 180^o, she then gave them triangles to cut and investigate individually. A page from one student’s workbook can be seen in Figure 7.

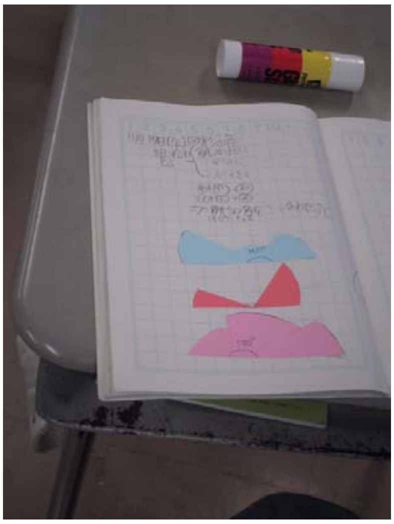

Figure 7

Key window for considering mathematical thinking

A key window for considering mathematical thinking is a combination of teacher guidance of student thinking, student communication, teacher understanding of students’ communication, and teacher reflection on needed intervention. The first lesson illustrated a great deal of student confusion and inability to think logically and sequentially. But the students provided the teacher with the necessary information for what was needed the next day.

How can we develop mathematical thinking through the lessons?

The student works illustrated in Fig. 1-4 are from a seventh grade mathematics class. Although the students are making an effort to think about what they are doing, a large number of them do not have the tools for thinking coherently, and mathematics appears to makes no sense to them in ways that would allow them to make the necessary judgments Katagari describes in Mathematical Thinking and How to Teach It (2006):

> before one calculates on paper or with a calculator, one must be able to make the judgment "what numbers need to be calculated, what are the operations that need to be performed on those numbers, and in what order should these operations be performed?" (p. 5)

In lieu of "conserving cogitative energy" (Katagari, 2006), the students invested more "cogitative energy" in not arriving at a solution. The class ended before the teacher could summarize the lesson. Many students in this classroom were not prepared for the lesson they were taught. Rather than having their suggestions to multiply, divide and subtract accepted without comment, could the students have been asked to explain why they proposed these operations and did they make sense? Why or why not? In classrooms with large numbers of students who have a weak grasp of mathematics, it may be necessary to conduct the lessons as a whole class undertaking in order to maximize student learning.

Richard Askey (retired mathematician, University of Wisconsin-Madison), one of the observers of the grade 5 lesson (lesson II), commented that the lesson illustrated what he would call pre-mathematics, which is a necessary prerequisite for mathematical thinking. However, as I have observed, many U.S. lessons never progress beyond pre-mathematics.

Askey suggested to the teacher after the class that for the next lesson the students could prove informally that the sum of the angles of a triangle is 180^o if they know that a rectangle is made of up four right angles, and the sum of the four angles is 360^o. By dividing the rectangle into two equal triangles (see Figure a), they can conclude the sum of the angles of a right triangle is 180^o:

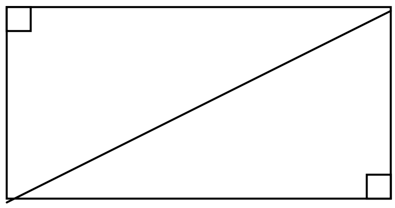

Figure a

Having proved that the sum of a right triangle is 180^o, students then can apply this knowledge to the proof of a general triangle.

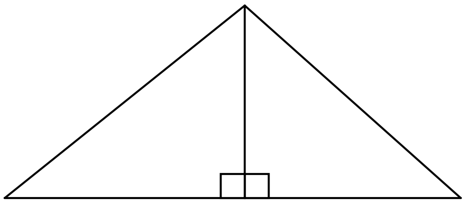

Figure b

Draw a general triangle (Fig. b) and drop a perpendicular from a vertex to the opposite side to form two right triangles. The sum of the angles of each right triangle is 180^o, resulting in a total of 360^o. Subtracting the two interior right angles, leads to a sum of 180^o for any general triangle.

When teachers provide a solid and concrete pre-mathematical experience for students, it paves the way more easily to mathematical thinking about the situation.

CONCLUSIONS

In the abstract I listed five barriers to effective implementation of lesson study in the U.S. The first barrier was teacher content knowledge, including teachers' own ability to engage in mathematical thinking (Lim, 2006). Lesson study, defined as a collaborative and collegial approach to understanding students' current state of mathematical knowledge and figuring out how to move them along the continuum to mathematical thinking, is necessary for improving instruction and learning. However, lesson study in the U.S. is not sufficient by itself due to many teachers' own limited mathematics content knowledge. Teachers of mathematics now are being asked to teach in ways they themselves did not learn mathematics; many of them do not know mathematics deeply enough to develop students' mathematics thinking in the ways outlined in the APEC Dec. 2-7, 2006 conference announcement (http://www.criced.tsukuba.ac.jp/math/apec/apec2007/). For many teachers, their knowledge of division by fractions is limited to "ours is not to wonder why; just invert and multiply" (Ma, 1999). In recognition of teachers' limited knowledge of mathematics, the Conference Board of the Mathematical Sciences recommended a minimum number of mathematics courses pre-service teachers should take in preparation to teach at elementary, middle and secondary levels, respectively (2001). Over the past five years, some colleges and universities have developed mathematics courses to bridge the gap in teachers' knowledge, but these courses have not been developed according to a common set of standards.

Like lesson study, mathematics content courses by themselves are necessary but not sufficient; limited data are available on student performance in classes of teachers who have taken these content courses. A possible effective combination is to have teams of teachers engaging in lesson study with some of them having taken university content courses from which they can draw upon in the development of lesson plans and deepening of their own knowledge. Furthermore, as illustrated by the example of Askey's role in the second lesson, a knowledgeable other can provide crucial help to teachers as they attempt to develop students who are able to think mathematically (Watanabe & Wang-Iverson, 2005).

The second barrier is the plethora of textbooks that run to many hundreds of pages without a coherent scope and sequence, as they attempt to meet the different standards across the states (Schmidt, Houang & Cogan, 2002). Development of textbooks has become attempts to include the many state standards (to maximize sales) rather than on improving the content. However, U.S. educators now have access not only to the Singapore *Primary Mathematics* textbooks (http://www.singaporemath.com), but to English translations of two sets of Japanese elementary mathematics textbooks published by Tokyo Shoseki (http://www.globaledresources.com) and Gakkoh Tosho (http://www.gakuto.co.jp/20050131e/index.html), which can be used as resources and supplements. The common scope and sequence of these texts are outlined in Fig. 1 in Schmidt, Houang & Cogan (2002).

With the recent publication of NCTM's *Focal Points* (2006), one can only hope that the third barrier, the standards that vary by state, will converge by consensus among states to examine this new document more closely for adoption in each state. If, by mapping *Focal Points (2006*) against the scope and sequence of the A+ countries, one sees a similar upper triangle matrix in the introduction and retention of topics, then schools will have justification for choosing to use the Singapore or Japanese mathematics textbooks in mathematics classrooms. The last barrier, the use of standardized testing to measure student progress, can actually become an asset in schools' decision to use curricula that are more coherent and concise (Garelick, 2006).

This cross-country effort spearheaded by APEC to arrive at a better understanding of a clear definition of mathematical thinking and how to achieve it in both teachers and students provides a concrete goal. It circumvents the insoluble and trivial debate surrounding teaching strategies that has plagued the mathematics education community.

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