DEVELOPING MATHEMATICAL REASONING THROUGH ARGUMENTATION

Peter Gould

NSW Department of Education and Training, Australia

Engaging in argumentation in elementary classrooms involves more than taking turns when arguing. Curriculum documents describe the importance of questioning, reasoning and reflecting as contributing to Working Mathematically. The prominence and time allocated to developing this form of mathematical thinking in elementary classrooms is influenced by the teacher's confidence and familiarity with using these modes of thought. Examples of developing the classroom norms through specific mathematics lessons are needed to act as both models of and models for effective lessons involving students in reasoning through argumentation. A research lesson on the development of units of different sizes associated with measurement and fractions is proposed as a vehicle for developing mathematical reasoning through argumentation.

DESCRIBING MATHEMATICAL THINKING

Attempts to describe the essential features of mathematical thinking in Australia have been influenced by similar developments in the USA and the United Kingdom. In particular, the focus on problem solving in the 1980s (Mason, Burton, & Stacey, 1982) led to attempts to describe essential features of problem solving in various syllabus documents in Australia. The approach to problem solving in Australian syllabus documents was guided by Polya's (1957) initial description (see, plan, do and check) and elaborated by a desire to enrich the repertoire of problem solving strategies students' possessed (Ohio Department of Education, 1980).

The initial enthusiasm for developing mathematical problem solving did not by itself overcome many of the practical issues associated with transforming classrooms and improving students problem solving. Inservice programs assisted teachers in understanding the distinguishing characteristics of a mathematics problem (compared to an exercise) and described common problem solving heuristics such as *draw a diagram*, *act it out* and *work backwards*. Sometimes teachers also discussed *teaching for* problem solving, *teaching about* problem solving and *teaching through* problem solving. In elementary classrooms in New South Wales, the majority of the teaching was arguably teaching for problem solving with very little teaching through problem solving. This may have been influenced by elementary teachers' level of confidence in mathematics. That is, the attitudes that a teacher holds towards mathematics influences the teaching approach to mathematical problem solving. Teaching through problem solving requires ready access to a repertoire of rich problem tasks and the confidence to use them in teaching.

The challenges associated with developing problem solving classrooms in mathematics led to an elaboration of question types used in mathematics teaching such as open-ended questions, good questions and rich tasks. The need to have students engage in sustained problem solving also contributed to a move towards extended investigations. That is, teachers started to explore the prospect of moving from closed questions (*What is the area of a rectangle with side lengths 6 cm and 4cm?*), to open questions (*The area of a rectangle is 24 cm^2. What could its sides measure?*) and then to extended investigations (*How many rectangles have an area of 24 cm^2?*).

THE RISE OF WORKING MATHEMATICALLY

The initial flurry of interest in developing problem solving as a central component of the elementary mathematics curriculum was followed by attempts to unify efforts in Australia. The moves towards a coordinated approach culminated in a National Statement of Mathematics (Australian Education Council, 1990). With changes in governments since that time, the National Statement of Mathematics no longer has the status it did in the early 1990s. It did however, contribute to considerable curriculum collaboration between states and territories in Australia, and to the rise of common descriptions of mathematical thinking.

Even though all states and territories in Australia have their own syllabus documents, there are many common features among them. Apart from the common descriptions of content strands such as Number and Measurement, all of the syllabus documents acknowledge the role played by the process of thinking mathematically. Mathematical thinking is described in most syllabuses under a title such as Working Mathematically. The description of the processes involved in Working Mathematically in Australia has evolved from the Mathematical inquiry strand of the National Statement of Mathematics. The current description of Working Mathematically in the New South Wales' elementary mathematics syllabus (Board of Studies NSW, 2002) contains five processes:

1. questioning
2. applying strategies
3. communicating
4. reasoning
5. reflecting.

The process of *questioning* relates to students making mathematical conjectures or predictions. *Applying strategies* includes the application of known techniques and problem solving heuristics. *Communicating* encapsulates the use of appropriate language and representations to formulate and express ideas in written, oral and diagrammatic form. *Reasoning* describes the use of justification or backing underpinning mathematical modes of proof. *Reflecting* is described as a metacognitive process designed to support generalisation and making connections between different areas of learning.

The challenge of having a separate process strand of the Mathematics K-6 syllabus is addressed by having Working Mathematically described within each content strand. Syllabus pages (see http://k6.boardofstudies.nsw.edu.au/maths) are usually divided into two with the relevant knowledge and skills outlined on the left and the relevant working mathematically processes described on the right. The knowledge and skills follow the stem “Students learn about…” and the Working Mathematically processes follow the stem “Students learn to…”. Examples of each of the five processes involved in Working Mathematically are associated with the knowledge and skills described in each content strand and for each stage of schooling. A stage of schooling in NSW elementary schools corresponds to two school years with the exception of Stage 1, which includes the first year of school (Kindergarten) and is called Early Stage 1 in NSW.

The approach to explicitly teaching mathematical reasoning at the same time as specific knowledge and skills is quite new to elementary teachers in NSW. The transition to the new Mathematics K-6 syllabus was only recently made in 2005. The current advice to teachers on how to address mathematical thinking is to build it into planning of each unit of work. However, teaching to develop reflective thinking in mathematics is more often a stated aspiration than an achieved goal.

DEVELOPING THINKING THROUGH ARGUMENTATION

The intended key window for considering mathematical thinking in elementary classrooms in NSW is through communication or more specifically, argumentation. Government schools in New South Wales also use a Quality Teaching Framework (NSW Department of Education and Training, 2003b) to guide the design of lessons. The Quality Teaching Framework is made up of three domains (Intellectual quality, Quality learning environment and Significance) each with six elements. The model of pedagogy described by the Quality Teaching Framework can be applied from Kindergarten to Year 12 across all learning areas. The element of the Intellectual quality dimension of the framework of most relevance to the window of communication and argumentation is described as substantive communication.

Learning to argue about mathematical ideas is fundamental to understanding mathematics. Palincsar and Brown (1984) wrote that “ …understanding is more likely to occur when a child is required to explain, elaborate, or defend his position to others; the burden of explanation is often the push needed to make him or her evaluate, integrate and elaborate knowledge in new ways.” Argument here is taken to mean a discursive exchange among participants for the purpose of convincing others through the use of mathematical modes of thought.

Teacher led exchange

Questioning is a central component of a teacher's repertoire of instructional techniques. Look at the way that questioning is used in the following exchange.

Teacher:	Let's see if we remember how rectangles are different from other geometric shapes? How would you describe a rectangle?…Kim?
Kim:	A rectangle has straight parallel sides.
Teacher:	Hmm. Do you mean like this? (*Teacher sketches a regular hexagon on the board.*) Ryan?
Ryan:	No. That's a hexagon because it has six sides. A rectangle has four sides with all of its sides parallel.
Teacher:	(*Looks perplexed, then draws four unconnected parallel line segments.*) Four parallel lines. Is this what he means?…Sarah?
Sarah:	No, its got to be closed. A rectangle has four sides with only the opposite ones parallel.
Teacher:	(*Sketches a parallelogram with no right angles.*) Like this?
Robert:	No. For a rectangle, all four of the angles have to be right angles.
Teacher:	(*Draws a rectangle.*) So this is a rectangle. What would be a good way to describe a rectangle to someone who wasn't here?

What is interesting in this exchange is the way that the teacher intentionally creates counter-examples from the students' descriptions. Counter-examples are very important in developing an idea in mathematics. They form a "negative image" of a rectangle. The questions also prompt the need for precision of language in mathematics.

Increasing student-student exchanges

Classroom discussion is clearly important in students' coming to know the processes of mathematical thinking. The pattern of exchange above is teacher centred. Classroom questioning in many classrooms in NSW is teacher-centred. Changing the patterns of exchange to encourage more student-student responses often necessitates renegotiating the didactic contract (Brousseau, 1984). Teachers establish implicit contracts with students. If the teacher does most of the work within the classroom, including the role of sole questioner, then this becomes part of the implicit contract of the classroom.

Developing classroom norms to establish ways of engaging in constructive argumentation in mathematics requires attention to the role of critical listening (Wood, 1999). Learning to argue mathematically contributes to developing shared understandings of mathematics. The ways in which students seek to justify claims, convince their classmates and teacher, and participate in the collective development of publicly accepted mathematical knowledge contribute to mathematical argument. Establishing the network of mutual expectations for participating in mathematical argument in class, including participating in disagreeing is explicitly negotiated by the teacher. Once students begin to freely agree and disagree, the teacher can extend his or her expectations for how children should interact and talk with one another during a mathematical argument. This can include the types of questions that might be appropriate or even how to mount a challenge to someone’s work or justify your reasoning. In a culture that expects student understanding, teaching mathematics is more than merely telling or showing students; teachers must enable students to create meanings through their own thinking and reasoning.

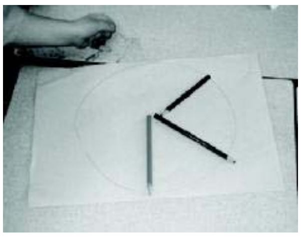
Maintaining students’ participation in substantive communication involving mathematical thinking is a complex process and is greatly in need of exemplars. In particular, classroom argumentation needs opportunities to move from authority-based arguments (because the teacher says so or the text states this) to reasoning with mathematical backing.

Arguing about fractional units

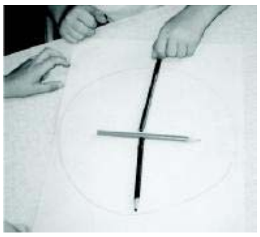
Fractions are a particularly troublesome area of the elementary mathematics curriculum in NSW. The language associated with fractions in English contributes to a number of misconceptions for students. Unlike most Asian languages, English uses the same terms for naming ordinals and fractions (e.g. third, sixth, ninth). It is also easy for students to not hear the soft sounds at the end of fraction names, which can lead to confusion between whole numbers (e.g. six) and fractions (e.g. sixth). Thus, although six sixes are thirty-six, six sixths are one. Interpreting fractions as relational numbers provides a good context for students reasoning, particularly with reference to the equal whole that allows comparisons of the size of fractions.

It is also quite common to have students of different ages in the one classroom in NSW. Consequently, argumentation about fractional units needs to enable participation across different year groups. Curriculum support material (lesson ideas) such as *Fractions: Pikelets and lamingtons (NSW Department of Education and Training, 2003a)* emphasises the use of sharing contexts in developing children’s fraction concepts. In particular, students’ evoked fraction concepts suggest that equality of area is not the feature abstracted from regional models used in teaching fractions (Gould, 2005). The following sequence of images shows a pair of Year 3 students engaged discussing how to create thirds of a circle.

1.



2.



3.

4.

Figures 1–4. Partitioning a round model into equal three parts

Justifying fair shares and the equivalence of fractional parts becomes a topic for argument in elementary schools in NSW. That is, “how do you know?” is the key question. At the upper elementary stage, the mathematical thinking focus of the research lesson is reversibility. Having students reconstruct the whole from a fractional part before creating a different fraction draws on what Tzur (2004) described as the *reversible fraction conception.* For example, if this piece is three-quarters of a strip of paper, how long would half of the original strip of paper be? The expectation is that students arrive at consensus through reasoned argument, reconciling different approaches through demonstration using a common model.

References

Australian Education Council. (1990). *A national statement on mathematics for Australian schools*. Melbourne, Victoria: Curriculum Corporation.

Board of Studies NSW. (2002). *Mathematics K-6 syllabus*. Sydney: Board of Studies NSW.

Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H. G. Steiner (Ed.), *Theory of mathematics education* (pp. 110-119). Bielefeld, Germany: Institut für Didaktik der Mathematik.

Gould, P. (2005). *Year 6 students' methods of comparing the size of fractions.* Paper presented at the Building connections: Theory, research and practice. 28th Annual conference of the Mathematics Education Research Group of Australasia, RMIT Melbourne.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. Workingham, England: Addison-Wesley.

NSW Department of Education and Training. (2003a). *Fractions: Pikelets and lamingtons*. Sydney: NSW Department of Education and Training.

NSW Department of Education and Training. (2003b). *Quality Teaching in NSW public schools*. Sydney, Australia: NSW Department of Education and Training.

Ohio Department of Education. (1980). *Problem solving…a basic mathematics goal.* Columbus OH.

Palincsar, A. S., & Brown, A. L. (1984). Reciprocal teaching of comprehension-fostering and comprehension-monitoring activities. *Cognition and Instruction, 1*(2), 117-175.

Polya, G. (1957). *How to solve it, 2nd Edition*. Princeton, NJ: Princeton University Press.

Tzur, R. (2004). Teacher and students' joint production of a reversible fraction conception. *Mathematical Behavior, 23,* 93-114.

Wood, T. (1999). Creating a context for argument in mathematics class. *Journal for Research in Mathematics Education, 30*(2), 171-191.