DEVELOPING MATHEMATICAL THINKING THROUGH PROBLEM-BASED LESSONS

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The Philippine curriculum for elementary and secondary education envisions to produce thinking Filipino learners. This is by providing them the opportunities to be equipped with the knowledge, skills, and values that would enable them to learn throughout life. Although it does not explicitly mention "mathematical thinking," it emphasizes the use of teaching strategies and students' activities that elicit various thinking skills and dispositions, in the mathematics teaching and learning process. This report relates the importance that the curriculum places on the development of learning how-to-learn skills that are also associated with mathematical thinking, how problem solving provides rich opportunities for the development of mathematical thinking, and how lessons can be crafted to develop mathematical thinking.

THE CURRICULUM'S VISION: THINKING FILIPINO LEARNERS

The present curriculum was a reform that started implementation in 2002. The reform is viewed as "one more step towards developing thinking learners, learning organizations, and people who will create their own future and not the future created for them." It aims to make Filipinos learn how-to-learn skills to empower them to competently engage in lifelong learning in an ever-changing world. To achieve this, among others, it enjoins teachers to make conscious efforts to develop students' creative and critical thinking skills and to use interactive activities that will make students take greater responsibility for their own learning (Department of Education 2002). It does not have any direct reference to mathematical thinking. However the mathematics curriculum, envisions Filipino learners to be competent at thinking skills such as: exploring, analysing and interpreting data, predicting, looking for patterns, establishing relationships among quantities, generalizing, visualizing abstract mathematical ideas, presenting alternative solutions to problems, and applying mathematics to daily life. Notably, in the competencies for each topic, solving problems is repeatedly mentioned (Department of Education Bureau of Elementary Education 2003, Department of Education Bureau of Secondary Education 2002). The thinking skills and problem solving cited in the curriculum are also related to mathematical thinking (NCTM 2000). Furthermore, the curriculum also stresses the development of values such as honesty, flexibility, perseverance, cooperation, open-mindedness, respect of others' ideas, and the like.

To attain said vision, the mathematics curriculum recommends using a variety of teaching strategies to include the more learner-centred ones: problem solving, investigations, practical work, and discussion, besides the more common ones: exposition and practice and consolidation. It further recommends organising students into cooperative learning groups whenever appropriate.

An Example of a Lesson Activity that Promotes Thinking Skills

How is the curriculum's vision being attempted to be realized in the classroom? In a grade 5- lesson to introduce the concept of prime and composite numbers, there are clear indications of developing the thinking skills earlier mentioned (Department of Education Bureau of Elementary Education 2003). The pupils are made to work in groups and each group is given 20 pieces of square strips. They are asked to arrange a specific number of squares starting from 2 squares up to 20 squares, to form different rectangles. Supposedly based on the activity, the pupils should be able to identify a relationship between the number of different rectangles formed from a given specific number of squares and the number of factors of the number of squares. Using this relationship, the teacher can introduce the concept of prime and composite numbers. So in a way, the lesson is good because the pupils are actively involved in the development of the concept. However, the approach is still structured because the lesson prescribes how the pupils should explore. It also presents to them a table that they are required to fill with the prescribed information. It asks them to answer questions that will lead them to generalize the desired relationship upon which the concept of prime and composite numbers can be introduced.

Even with the lesson's limitations, it nevertheless provides an opportunity for the pupils to use thinking skills such as exploring systematically, observing, comparing, classifying, identifying patterns, and making generalizations. But experiencing these thinking skills can be made more meaningful to the pupils if the approach is more open. If so, then they can be left to decide how they will proceed with their exploration and devise a way so as not to miss any case; and how they will record and organize their results and identify what information they need to show so that they can observe patterns that can be the basis of determining a relationship that they can generalize. Hence, they do not need to rely on some external focus questions to be able to analyze relationships.

If the approach is more open, the pupils will have more opportunities to learn the how-to-learn skills. The teacher can just go around the groups and extend help only when needed, by asking appropriate questions. Compared with the traditional approach where the definitions are given first and then numbers are classified as prime or composite, the given sample lesson shows the good attempts to develop pupils' thinking skills while they learn mathematics. But such lessons can still be improved so that the responsibility of learning depends more on the pupils.

Examples of Critical and Creative Thinking

The curriculum does not explain what it means by critical and creative thinking. So in this paper, the more common notion of these types of thinking cited by Marzano (1988) will be adopted. Unlike problem solving and decision making which are considered thinking processes that involve several thinking skills, creative and critical thinking refer more to quality of thinking. Critical thinking is evaluative. It judges reasonableness, validity, accuracy, efficiency, truth, and the like. Creative thinking is generative. It results to an output that is new, original, different, unusual, and the like. Critical and ceative thinking are complementary and a particular thinking may be described as being of a greater degree in one than in the other.

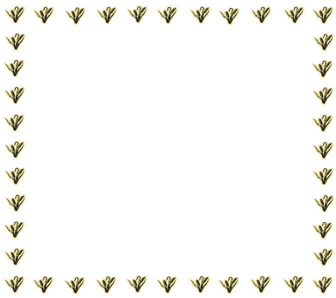
To judge that the most efficient method in solving the problem presented in the next section is Method 4 because it uses the least number of operations among all the methods, is an evidence of critical thinking. Meanwhile, presenting Method 5 as a possible solution to said problem is an example of creative thinking because it is a method different from all the rest.

PROBLEM SOLVING: A KEY WINDOW FOR CONSIDERING MATHEMATICAL THINKING

Solving non-routine problems uses thinking skills that are included in the processes involved in mathematical thinking (Romberg 1994; Mason, Burton & Stacey 1990). Examples of these skills are interpreting, visualizing, representing, and identifying patterns. So non-routine problem solving is important in considering mathematical thinking. To illustrate this, consider the problem below. Supposedly, it is to be used in grade 6 as an example of solving multi-step problems involving the use of different operations. The situation is a familiar one for pupils. But they will have to invent different ways of finding the answer to the question posed. These ways are not taught to them.

Problem: Mr. Ben has a 10 m x 10 m garden. He planted seedlings 1 m apart around the land. How many seedlings did he plant in all? Show different ways of doing this. Write the number sentence for each method. Be able to explain your methods.

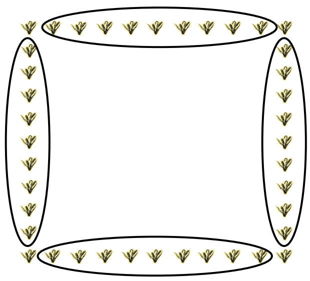
This problem was actually given to some elementary school mathematics teachers in a seminar that emphasized the importance of exploring multiple solutions to a problem. From the given information that the garden is 10 m x 10 m, the teachers concluded that it must be a square one. Some of them suggested that they should assume that the seedlings are very thin to be consistent with the information that the seedlings are really 1 m apart and that they are planted on each side of the garden that has a length of 10 m only. There were those who thought that since the length of a side is 10 m, there must be 10 seedlings on each side. However, many drew how they interpreted the problem. Their drawing is similar to the one shown below.



And so they argued that there are 11 seedlings, not 10, on each side of the garden.

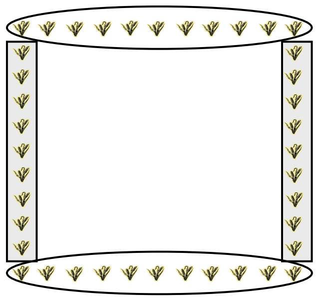
The different methods that the teachers gave are visually represented by the following drawings with their corresponding number sentences.

Method 1



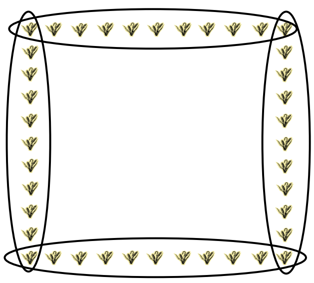
4 x (11- 2) + (4 x 1) = 40

Method 2



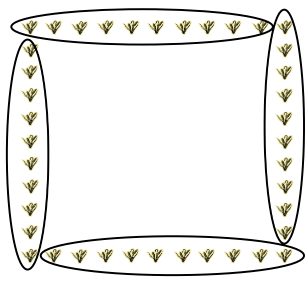
(2 x 11) + 2 x (11 - 2) = 40

Method 3



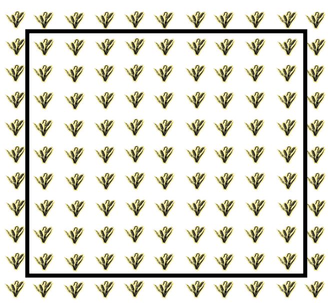
(4 x 11) - (4 x 1) = 40

Method 4



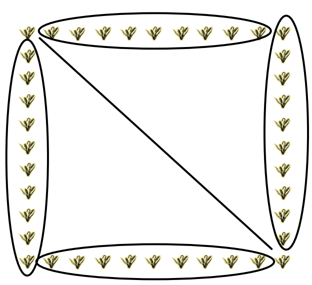
4 x (11 - 1) = 40

Method 5



(11 x 11) - (11 - 2) x (11 - 2)

Method 6



2[(11-1) + (11-2)] + 2 = 40

The explanations given for the different methods are given below. In their explanations, the teachers already referred to the sides and vertices of a square instead of the sides and corners of the garden. This implies that they first extracted the mathematical aspects of the real life situation in order to solve the problem, and this is a part of the mathematisation process (OECD 2003, Romberg 1994).

Method 1: Count only the seedlings found on the sides but not on the vertices. Then add those that are on the vertices. For each side, there are 11 seedlings but 2 of them are on the vertices included on each side. So there are 4 x (11- 2) seedlings on the sides excluding those on the vertices. And there is one seedling on each of the 4 vertices. So there are (4 x 1) seedlings on the vertices. Hence, there are (4 x 9) + 4 = 40 seedlings planted in all.

Method 2: Count only those seedlings on the horizontal sides. Then count those seedlings on the vertical sides that have not been counted yet. So the total number of seedlings is (2 x 11) + 2 x (11-2) = 40.

Method 3: Count those seedlings on each side. Since the ones found at the vertex are counted twice, these duplications should be subtracted from the total. So this is (4 x 11) – (4 x 1) = 40.

Method 4: On each side there are two vertices included. Count only those seedlings that are on a side excluding that on one vertex. So there are 4 x (11 – 1) = 40 seedlings in all.

Method 5: Think as if the entire garden had been planted with seedlings. Each seedling is 1 m apart from its neighboring seedlings vertically and horizontally. There is a total of 11 x 11 seedlings in all. Since only the seedlings on the sides are to be considered, the number of seedlings inside need to be subtracted from this total. There are (11- 2) x (11– 2) seedlings inside. Thus, the total number of seedlings left on the sides is (11 x 11) - (9 x 9) = 40.

Method 6: Cut the square diagonally. The numbers of seedlings on the horizontal and vertical sides of the square on the left and right side of the diagonal excluding the seedlings along the diagonal are the same. There are 2 seedlings on the diagonal. So the total number of seedlings is 2 x [(11 – 1) + (11 – 2)] + 2 = 40.

To summarize, the teachers tried to understand the problem by drawing. Then by marking their drawing, they visualized a systematic way of counting. This representation shows the underlying reasoning for their method. They communicated this reasoning both in writing and verbally. They used the relevant mathematics that they know to write the number sentences. Specifically, in writing the number sentences, they used (1) the properties of a square that is, of having 4 sides of equal lengths and 4 vertices and, (2) the relationships of operations with numbers, particularly in expressing repeated addition as multiplication.

In effect, the teachers represented their ideas in different forms, that is, by using drawing, words, and symbols. They also identified the relationships that existed among these representations. For example, for each method, they were able to relate every number and every operation in their number sentence to their corresponding interpretations in their drawing and explanation of their reasoning.

When the teachers were asked to predict the number of seedlings for square gardens with longer sides, they first added 1 to the number representing the length, to get the number of seedlings on each side. Then most of them chose Method 4, the method that has the least number of operations, and used it. They were also able to make a conjecture on how to get the number of seedlings using any method because in the process of predicting for other cases for varying lengths of a side of the garden, they observed in their number sentences what numbers changed and what numbers remained the same. For Method 4 for example, their conjecture was that the number of seedlings equals 4 x (number of seedlings on a side – 1). And they generalized this relationship even for cases that they have not considered because the pattern that they observed in their method as shown on the drawing, clearly continues.

When the teachers were asked to modify the original problem, the change they made involved other regular polygonal shapes although the context was not a garden anymore. They conjectured that all the methods would still work with some corresponding changes.

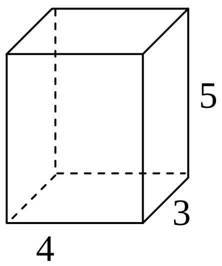
Solving a non-routine problem such as the one discussed here used processes. It involved mathematisation. Moreover, initially, it dealt with only the specific case presented. Then based on other results generated by predicting for other cases, conjectures on possible relationships were formulated. The representations and the reasoning underlying the different methods were then used to argue convincingly that the patterns associated with the methods would continue. So the conjectures were considered generalizations. Mathematical thinking also uses the same processes such as specializing or considering specific cases, generalizing, conjecturing, and convincing (Mason, Burton, & Stacey 1990). Moreover, it also uses specific thinking skills such as interpreting, representing, and identifying patterns. So non-routine problem solving can be a rich context for developing mathematical thinking.

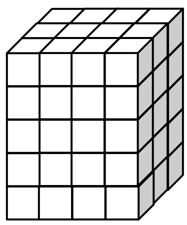
HOW TO DEVELOP MATHEMATICAL THINKING IN LESSONS

Students' mathematical thinking can be developed if they experience thinking mathematically (Polya 1957; Romberg 1994; Mason, Burton, & Stacey 1990; NCTM 2000). Solving non-routine problems can provide this experience as the example earlier has shown. It can be the source of a difficulty, gap, contradiction, or surprise that often triggers mathematical thinking (Mason, Burton, & Stacey 1990). This implies that lessons in mathematics should be based on non-routine problem solving.

The following discussion broadly relates how a problem solving approach can be used to introduce the concept of the volume of a rectangular prism. Usually the formula for getting the volume of a rectangular prism is given and the pupils practice on computing this for different rectangular prisms with specified dimensions. An alternative way can be to show several empty transparent rectangular containers and some unit cubes such that an exact number of the unit cubes can fit into each container. For example, one rectangular container may be 4 units by 3 units by 5 units. Group the pupils then let each group estimate the greatest number of unit cubes that can be exactly placed into the container. It may be good later to ask them about the basis for their estimate.

Afterwards, let them put as many cubes as needed to completely fill each container and to record their results. Ask which they think is the container that occupies the most space. They are expected to respond that the one which used the greatest number of cubes. After they have done this, show them a big empty transparent rectangular container. Then ask how much space it occupies. Since this container is the biggest among all the containers considered so far, there will not be enough cubes to use and even if there were, it will take a long time to fill it. So the pupils are confronted with a problem. Hopefully, based on what they have experienced earlier some would be able to think of how they would do it for smaller containers, say one that is 4 units by 3 units by 5 units.

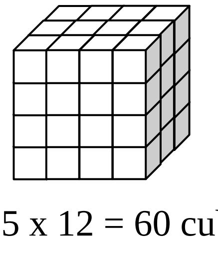




Most likely, when filling the smaller containers, the pupils did so by layers, from bottom to top. So for the 4 units by 3 units by 5 units container, they can reason that there are 12 cubes per layer and since there are 5 layers, then the total number of cubes is 5 x 12 = 60. Other ways that they may think of in getting the answer are described below.

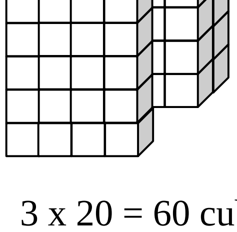
Pupils would have noticed that the cubes are aligned in rows and columns. One reasoning is that the container can be "sliced off" on top (Method 1). This method is actually similar to the one just described. Since there are 12 cubes on each layer and there are 5 layers, then there are 5 x 12 = 60 cubes. Another reasoning is to "slice off" the container vertically at the front (Method 2). Since there are 20 cubes on this vertical layer and there are 3 such layers, there are 3 x 20 = 60 cubes. Still another reasoning can be to "slice off" a vertical layer at the left part of the container (Method 3). Since there are 15 cubes per layer and there are 4 layers, there are 4 x 15 = 60 cubes.

Method 1



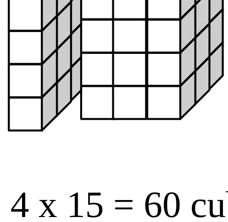
5 x 12 = 60 cubes

Method 2



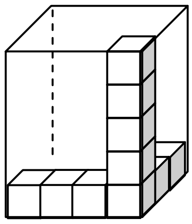
3 x 20 = 60 cubes

Method 3



4 x 15 = 60 cubes

Later they can likewise figure out that the number of cubes on each layer can also be determined by multiplying (1) the number of cubes along the length by the number of cubes along the width in the case of Method 1, (2) the number of cubes along the length by the number of cubes along the height in the case of Method 2, and (3) the number of cubes along the height by the number of cubes along the width in the case of Method 3. They have encountered quite a similar process where they used square tiles rather than cubes when they got the area of a rectangle. Then they can conclude that they do not have to fill the entire container with cubes to know how much space it occupies in terms of unit cubes. They only have to determine the number of cubes along the length, width, and height of the rectangular container and multiply these numbers. So in this case, it is 4 x 3 x 5 = 60. And this amount of space occupied by an object expressed in number of unit cubes can then be introduced as volume.



Eventually, they will also realize that they can just multiply the length, width, and height of the rectangular prism to get its volume.

In this lesson, pupils work together in groups using actual objects and discuss what they need to do to solve the problem. As a result of this collaborative work, they discover a relationship that enables them to get the volume of rectangular prisms. As it is, the said relationship is still a conjecture. It works for a few particular cases and they may consider some more cases to verify if it is correct. Then with their reasoning they should be able to convince others that their conjecture can be made into a generalization.

The pupils can also extend their understanding of the new knowledge that they acquired by posing problems like: How do you get the volume of a triangular prism or a cube? What would be the volume of the biggest cube that can be completely contained in a rectangular prism so that the cube does not touch any face of the prism except the bottom? Suppose there is a rectangular box that is 5 units x 3 units x 7 units and another rectangular box that is 6 units x 10 units x 21 units. What is the greatest number of the smaller boxes that can be contained exactly in the bigger box?

Mathematical thinking needs a supportive environment where it can grow. It is one where the teacher believes that pupils are capable of thinking on their own and so the teacher gives them enough time and space for it (Mason, Burton, & Stacey 1990). It is where questions that require thinking are asked at the correct timing (Polya 1957). It is where the ideas, responses, and questions of pupils are welcomed and valued (Romberg 1994). So lessons should be planned such that pupils can make a substantial contribution to the development of their own mathematical understanding at their level.

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