MEXICAN PERSPECTIVE ON MATHEMATICAL THINKING

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QUESTION 1: HOW MATHEMATICAL THINKING IS DEFINED IN YOUR CURRICULUM DOCUMENTS

Mathematical thinking is not explicitly mentioned as such in the Mexican Curriculum for grades 7-9. Nevertheless, it can be read from the curriculum documents that mathematical thinking is approached as a series of intellectual skills consisting on:

- Using abilities to mathematically express and represent situations taken from different socio-cultural environments.
- Using procedures to pose and solve mathematical problems.
- Promoting positive attitudes towards the study of mathematics.
- Promoting collaboration and critical thinking.

In order to achieve these intellectual skills the school should built an environment that favors students' autonomous and flexible mathematical activity. An environment where students put forward and validate conjectures, develop their own procedures and acquire the tools and mathematical procedures socially established.

A positive attitude towards mathematics consists in motivate and develop students' curiosity and interest for investigate and solve problem situations. Besides, it means creativity to put forward conjectures, flexibility to modify their own view, and intellectual autonomy to confront unknown situations and gain self confidence on their capability to learn.

Collaborative and critical thinking will be supported by organizing group activities in which students are required to formulate, communicate and provide arguments to validate mathematical statements by properly using mathematical rules for debating that led them to appropriate decisions in each situation.

QUESTION 2: WHAT IS YOUR KEY WINDOW FOR CONSIDERING MATHEMATICAL THINKING?

According to the Mexican Curriculum for grades 7-9 (12-15 year olds) the key windows for considering mathematical thinking can be summarized as follows:

- *Posing and solving problems*. This implies that students are able to identify, pose and solve different kinds of problems or problem situations. For example, problems that have only one solution, problems with mores than one solution or possibly with no solution; problems where they have more o less data than required to solve the problem. Students should also be capable to solve a problem using more than one procedure and find out which one is the most efficient. As well, it is required that students are able to generalize procedures to solve problems.

- The curriculum documents acknowledge the need to define lines of progress to follow the development of students thinking. In the case of problem solving it is considered as a line of progress "from students' solving a problem being assisted to autonomous problem solving."
- *Argumentation*. Once the teacher has achieved that their students took the responsibility to find out at least one way to solve a problem, the teacher must create favorable condition for students to see the need to put forward solid mathematical arguments to sustain the procedure or solution they have obtained. These arguments can be seen from three levels of complexity and meet three different aims: (i) explain, (ii) show and justify informally and (iii) mathematical proof.

  The line of progress for this window acknowledged by the curriculum documents goes from "the empirical justification to the axiomatic justification".

- *Communication*. Consists of the capability to express represent and interpret mathematical information associated to a given event. This requires understanding and using different representations for qualitative and quantitative information related to a given event; establishing relations amongst these representations; clearly presenting the mathematical ideas; deducing the information drawn from the different representations and inferring characteristics, properties and tendencies related to the given event.
- *Procedural skills*. This window focuses on efficiently using procedures and representations when performing mathematical operations. The efficient use of procedures makes the difference between those who successfully solve a problem and those who do not. This skill does not only refers to mechanically apply a procedure, it is required that students are able to find out the most adequate procedure to solve a problem situation and evaluate the correctness of results. It is recommended that for students to achieve a good command on a procedure it is necessary that they test its validity by using it in as many different problem situations as possible.

QUESTION 3: HOW CAN WE DEVELOP MATHEMATICAL THINKING THROUGH THE LESSON?

According to the results from a research on a program for professional teachers development (Cedillo, 2006), the authors of the present draft propose a teaching approach guided by the principle of *framing the classroom events according to the students' ways of reasoning instead of students following their teacher's ways of reasoning*. By adopting such a principle it is assumed that learning is an active construction process that is socially shared by the learners and the teacher. This requires the teacher to play a different role. In order to briefly describe how it is conceived that teacher’s role, we used the metaphor of conceiving the activity in the classroom as a chess game; in such game the teacher is the expert player who simultaneously plays against 30 other players who can communicate and discuss amongst themselves before making a move. The expert makes the first move and has to be prepared to receive up to 30 different challenging responses; upon the second move on the chess board the expert player (teacher) has to give specific and challenging responses to each player, and so on. A major difference between the conventional chess game and the version we use in the metaphor is that in our chess game the teacher must manage the game in such ways that, eventually, the students legally win.

This metaphor encapsulates the stance that mathematics learning is an active confrontation between learners and mathematical challenges. It encloses the position of seeing mathematics learning as an active confrontation of the learners with challenging mathematics. Krainer (2004) suggests that to fulfill that position it is necessary to consider both the prior students’ and prior teacher’s knowledge: “It is unproductive to ignore students’ recent understanding and fresh ideas, and it is equally unproductive to ignore the knowledge produced by generations of mathematicians. Thus, teaching mathematics is a continuous dilemma situation for teachers: on the one hand, they need to start where the students are, and on the other they aim at supporting students in developing an understanding of the mathematical concepts that are part of a socio-historically constructed body of mathematical knowledge” (Krainer, 2004, p. 87).

Next we include three examples designed by the authors of this draft paper attempting to show how the curriculum recommendations might be put in a lesson format. The examples show the basic ideas for a teacher to follow in order to plan a lesson.

EXAMPLES

In the case of arithmetic a central theme in the Mexican Curriculum for Grade 2 of Middle School is divisibility (13-14 year olds). The examples correspond to the following topics: (i) Multiples and Factors; (ii) Maximum Common Factor, and (iii) Minimum Common Multiple.

To approach these topics we use a set of questions aimed at encouraging students to engage in mathematical inquiry and to uncover relationships between these concepts. We expect that this approach may lead students to put forward generalizations by observing numerical regularities and finally to express and justify these generalizations by using algebraic code. The students are allowed to work cooperatively in small groups if they wanted to do so. Next we will describe the problem situations we used for each topic.

Multiples and factors

The theme of multiples and factors is treated on the basis of the students' responses to the following questions:

- Can you find numbers that have exactly two divisors? In the next four minutes list as many of these numbers as you can. The challenge is that your list cannot include any number not fulfilling the given condition.
- Can you find numbers that have exactly three divisors? Can you show a rule that allows us to construct many numbers having exactly three divisors? Is there more than one rule that allows us to do that?
- Can you find numbers that have exactly four divisors? Can you show a rule that allows us to construct many numbers having exactly four divisors? Is there more than one rule that allows us to do that?
- Can you find numbers that have exactly n divisors? Can you show a rule that allows us to construct many numbers having exactly n divisors? Is there more than one rule that allows us to do that?
- Can you find a natural number different from 1 that you cannot factor using exclusively prime numbers as factors?

Maximum common factor

The topic of maximum common factor is approached using a known problem that involves three liquid containers, none of which is graduated, and whose capacity is known. The first and the second containers have capacities that may be different. The third container is larger that the other two. The problem consists in deducing a general rule that allows one to know the number of liters that can be obtained given the capacity of the first two containers. The solution relies on the concept of maximum common divisor. We use this problem in order to provide an opportunity for students to recreate their notions about the concept of maximum common divisor and to encourage them to find numerical regularities that eventually lead them to put forward general solutions. The problem can be extended to empirically approach Diophantine equations. The specific questions and sequence in which the questions were posed to the students are described below:

- You have three jars: one has a capacity of 3 liters; the second a capacity of 5 liters. The third jar is used to hold a certain amount of liquid larger than 8 liters. None of the jars is graduated. Can you get 4 liters passing liquid from one jar to the other? Can you find a way to record the sequence of movements you made from one jar to the other so that you convince us that your answer is correct?
- Consider the same conditions as the problem before, but now you have one 2-liter jar and one 4-liter jar. Can you get 1 liter passing liquid from one jar to the other? Can you find a way to record the sequence of movements you made from one jar to the other so that you convince us that your answer is correct?

- Consider the same conditions as the problem before, but now you have one 6-liter jar and one 9-liter jar. Can you get 1 liter passing liquid from one jar to the other? Can you find a way to record the sequence of movements you made from one jar to the other so that you convince us that your answer is correct? Can you get any integer amount of liters passing liquid from the 6-liter jar to the 9-liter jar? Can you make a list with the different amount of liters you can get passing liquid from the 6-liter jar to the 9-litter jar?
- Now you have one 7-liter jar and one 10-liter jar. Can you get 1 liter passing liquid from one jar to the other? Can you find a way to record the sequence of movements you made from one jar to the other so that you convince us that your answer is correct? Can you get any given amount of liquid passing liquid from the 7-liter jar to the 10-liter jar? Can you make a list with the different amounts of liters you can get passing liquid from the 7-liter jar to the 10-litter jar?
- Look carefully at the lists you made with the different amounts of liters you can get passing liquid from one jar to another. Do you notice some regularity fulfilled by these numbers? Can you find a general strategy that allows you to know whether you can or cannot get a given amount of liquid just knowing the capacity of each jar?

Minimum Common Multiple

The topic of minimum common multiple is approached using the gears problem. The version of the problem we posed to students involves two gears as shown in the figure below; the number of “teeth” in each gear can be different. Students were asked to find how many turns the gears have to make so that they will coincide again at the point the turning commenced. This problem was extended to include the case of more than two gears and to include word problems like “a friend of mine bought apples and oranges. For each apple she paid 5 pesos and for each orange 3 pesos. She paid the same for the apples as for the oranges. How many of each did she buy?

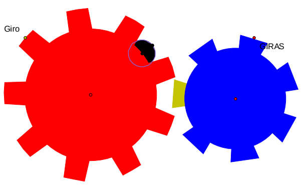

Fig. 1. The gears problem

References

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