ENCOURAGING MATHEMATICAL THINKING THAT HAS BOTH POWER AND SIMPLICITY

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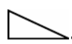
There is a world-wide desire to improve the teaching of mathematics, yet while teachers strive to improve performance on tests, there is a growing realization that practicing procedures to be able to perform them fluently is not sufficient to develop powerful mathematical thinking. The brain works by focusing on important information and suppressing inessential detail. Sometimes the detail that worked before may later prove to be inappropriate and cause difficulties. There are thus two important issues to address: taking account of ideas that students have met before that affect their current learning, and helping them to focus on essential ideas that become the basis of more subtle thinking. I will use the notion of 'compression of knowledge' (Thurston, 1990) to refer to the shift in focus from a process occurring in time (such as addition) to a concept that can be thought about as a mental entity. This dual use of a symbol for process and concept is called a procept (Gray & Tall, 1994).

It is my intention to build on a theoretical framework for the long-term development of mathematical thinking from new-born child to adult which requires powerful ideas to be compressed into thinkable concepts that apply in new situations. This suggests that teachers need to act as mentors to rationalize the use of ideas that students have met before and to encourage them to compress knowledge into powerful ideas that can be linked together in coherent ways. I will illustrate this by considering specific mathematical ideas that occur in school mathematics and refer to recent research findings from studies around the world. A curriculum based only on practicing procedures becomes increasingly complicated unless the student's knowledge is compressed into thinkable concepts that make mathematical thinking not only powerful, but essentially more simple.

LONG-TERM LEARNING OF MATHEMATICAL CONCEPTS

How do we learn about mathematical concepts? How do we grow over the years to learn to think mathematically in sophisticated ways? Let us begin with two mathematical concepts:

(a) What is a 'triangle'? (b) What is '5'?

A 'triangle' evokes descriptions like 'a three-sided figure', 'a figure made of three straight lines', or a picture like this  or this  or this . It is a physical or mental object that can be 'seen' or imagined in a thought experiment. A triangle is a *prototype* representing a whole category of figures, which can look very different, yet have the same essential properties as a three-sided polygon. To 'see' a figure as a triangle requires a focus of attention on the significant properties (the number of straight sides) ignoring inessential properties (e.g. lengths, angle size and orientation).

The number '5', on the other hand can be described as 'the number after four' or its properties might be evoked such as '5 and 5 make 10' or pictured as five objects. It is related not to the particular objects counted, but to a *procedure*: the procedure of counting elements in a set using the number names 'one, two, three, four, five'.

Piaget distinguished two fundamental modes of abstraction of properties from physical objects: *empirical abstraction* through teasing out the properties of the object itself, and *pseudo-empirical abstraction* through focusing on the actions on the objects, for instance, counting the number of objects in a collection. Later he speaks of *reflective abstraction* focusing on operations on mental objects where the operation themselves become a focus of attention to form new concepts.

Initially, therefore, we distinguish two ways of building a concept:

- The first is from the exploration of a particular object whose properties we focus on and use first as a description – 'a triangle has three sides' – and then as a definition – 'a triangle is a figure consisting of three straight line segments joined end to end'. (The latter definition already assumes knowledge of meanings such as 'figure' and 'straight line segment'.)
- The second arises from a focus on a sequence of actions and on organizing the sequence of actions as a mathematical procedure such as counting, addition, subtraction, multiplication, evaluation of an algebraic expression, computation of a function, differentiation, integration, and so on, with the compression into corresponding thinkable concepts such as number, sum, difference, product, expression, function, derivative, integral.

The first way gives a long-term cognitive development which, in geometry, has been

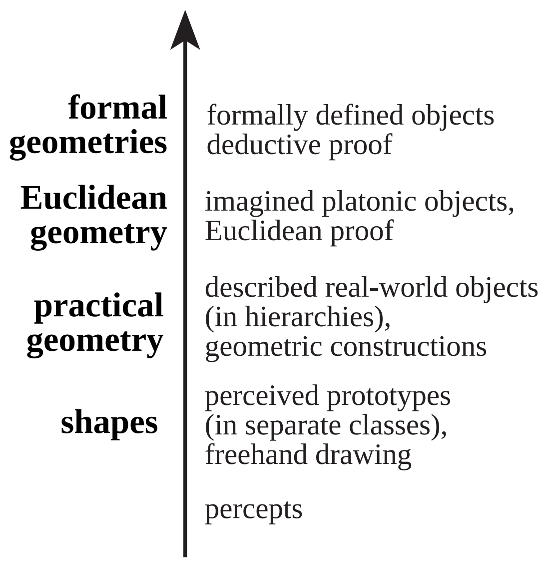


Figure 1: cognitive development of geometrical concepts
(Tall, et al 2001, after van Hiele, 1986)

formulated by van Hiele, building from perception of shapes, to description of their properties, practical constructions, definitions of figures that can be used for deductions, building to a coherent theory of Euclidean geometry (figure 1).

In general, the building of concepts from perception of, and actions on, physical objects and the growing sophistication towards definitions, deductions and formal theory is called the conceptual-embodied world of mathematical development.

The focus on actions, such as counting, has a quite different form of development, in which symbols are used to represent desired actions that are then also used for the outputs of those actions. This use of symbolism to shift from process to concept is termed a *procept*. Procepts occur widely in symbolic mathematics (Table 1).

symbol	***process***	***concept***
4	counting	number
3+2	addition	sum
–3	subtract 3 (3 steps left)	negative 3
3/4	sharing/division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\left.\begin{array}{c}\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)\\ \sum_{n=1}^{\infty} 1/n^2\end{array}\right\}$	tending to limit	value of limit
$(x_1, x_2, \ldots, x_n)$	vector shift	point in n-space
$\sigma \in S_n$	permuting $\{1,2,\ldots,n\}$	element of S_n

Table 1: Symbols as process and concept

This gives two different forms of mathematical development that interact at all levels:

- the **conceptual-embodied** (based on perception of and reflection on properties of objects);
- the **proceptual-symbolic** that grows out of the embodied world through action (such as counting) and symbolization into thinkable concepts such as number, developing symbols that function both as processes to do and concepts to think about (called procepts).

These two developments focus increasingly on the properties of the concepts involved and a switch to focus on properties expressed in set-theoretic terms leads to

- the **axiomatic-formal** (based on formal definitions and proof) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions.

The whole system can be represented in a single diagram of overlapping categories of cognitive development (Figure 2).

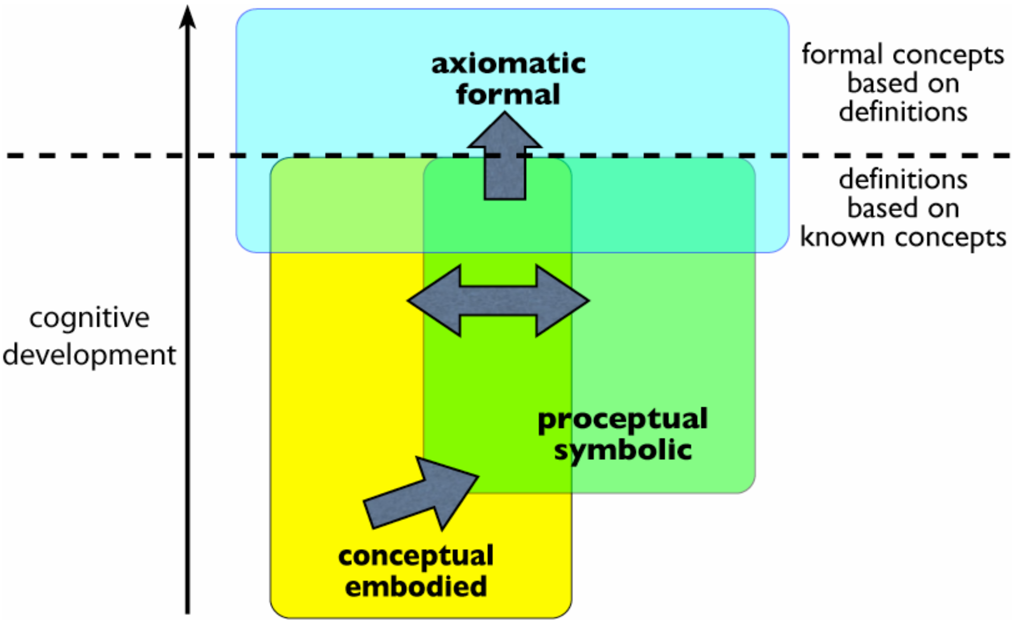


Figure 2: The cognitive growth of three mental worlds of mathematics

We use two words to describe each world of mathematics because the component terms, such as ‘embodied’, ‘symbolic’ and ‘formal’ are used in different ways in the literature. For instance, Lakoff (1987) says that *all* thought is ‘embodied’, Peirce (1932) and Saussure (1916) use the term ‘symbolic’ in a wider sense than this, Hilbert (1900) and Piaget (Piaget & Inhelder, 1958) use the term ‘formal’ in different ways—Hilbert in terms of formal mathematical theory, Piaget in terms of the ‘formal’ operational stage when teenagers begin to think in logical ways about situations which are not physically present.

Here the term ‘conceptual-embodied’ refers to the embodiment of abstract concepts as familiar images (as in ‘Mother Theresa is the embodiment of Christian charity’), ‘proceptual-symbolic’ refers to the particular symbols that are dually processes (such as counting, or evaluation) and concepts (such as number and algebraic expression), ‘axiomatic-formal’ refers to Hilbert’s notion of formal axiomatic systems. When these terms are used in a context where their meaning is clear, they will be shortened to *embodied*, *symbolic* and *formal*.

At first the child coordinates perception and action, allowing it to use its perceptions and actions to build early conceptual-embodied conceptions of the world, and to increasing sophistication of geometric development through descriptions, constructions, definitions, deductions and on to Euclidean and non-Euclidean geometries. In parallel, a focus on actions and symbolism leads to the proceptual symbolism of counting, arithmetic, algebra, symbolic trigonometry, functions, symbolic calculus. The two distinct worlds of conceptual embodiment and computations and manipulations with symbols as procepts have many links.

By compressing the dual names conceptual-embodied, proceptual-symbolic and axiomatic-formal to *embodied*, *symbolic* and *formal*, (with the shorter terms carrying the meaning of the originals), it is possible to consider them in combination:

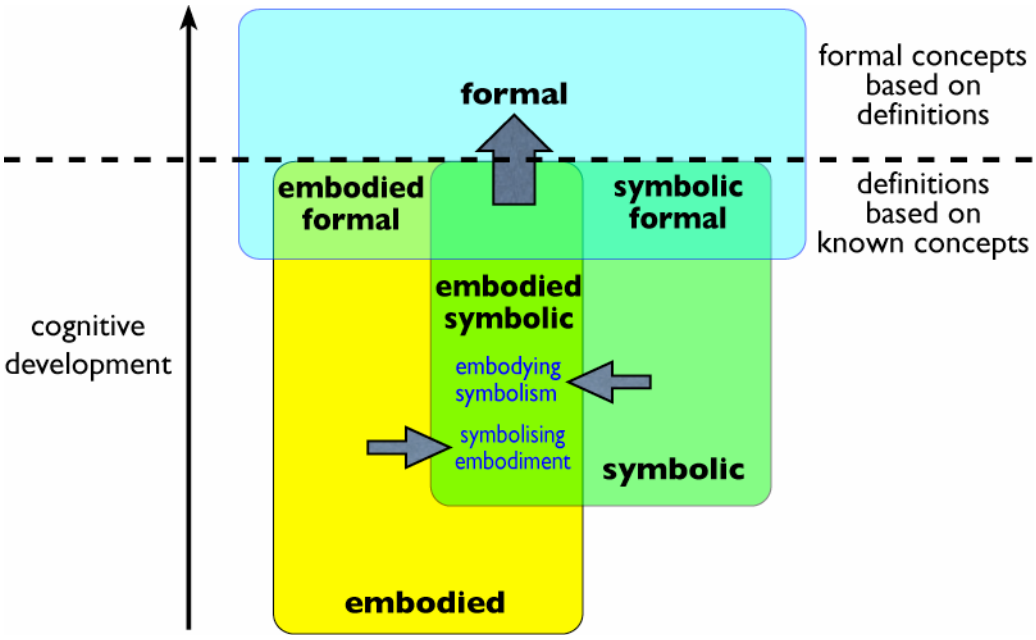


Figure 3: Interrelationships between combinations of worlds of mathematics

This gives a variety of routes to concepts as cognitive development grows. For instance, having built the natural numbers by compressing the procedures of counting to the concept of number, the concept of fraction may be introduced either by a new embodiment sharing sets or objects into a number of equal parts, or as new operations with number symbols; integers may be seen as pairs of numbers as credits (positive) and debts (negative) or as operations shifting the number line to the right or left.

Is there a general principle that suggests that a particular route is likely to be more appropriate (say building symbolism from embodiment or giving embodiment to symbolism, or a balanced combination of the two)?

Embodiment evidently gives human meaning, for instance, the following picture of 2 rows of 3 objects can also be seen as 3 columns of 2 objects, so the total number of objects, 2¥ 3 is the same as 3¥ 2.

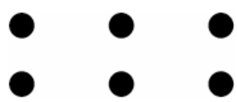


Figure 4: 2¥ 3 is the same as 3¥ 2.

In the embodied world, we categorize concepts together that satisfy certain properties. Figure 4 could just as easily represent any other rectangle with a whole number of rows and columns, say 4¥ 3 or 28759¥ 953246 or even *m*¥ *n* where *m* and *n* are any whole numbers. Indeed, if we look at a picture with a larger number of rows and columns, human perception can no longer instantly see how many rows or columns there are, but is well able to see that the array has equal sized rows and columns, and so verify the commutative law perceptually.

The embodied representation of the product as a rectangle of objects gives insight into the order irrelevance of multiplication of whole numbers, but more effort is needed to represent the commutative law for fractions and for mixed positive and negative numbers. On the other hand, if one needed to *calculate* 28759×953246 and 953246×28759 to check that they are equal, then the complication of the arithmetic might cause young learners concern. This suggests a principle that embodiment can give a real human insight to the simpler forms of mathematical structure but that more subtle forms may require a different approach.

In an algebraic approach, the commutative law $a \times b = b \times a$ is assumed as a fundamental law based on experiences in arithmetic. In the formal world in various axiomatic systems, it is an *axiom*.

Algebraic identities such as $a^2 - b^2 = (a - b)(a + b)$ can be given an embodied meaning as in the following picture (figure 5).

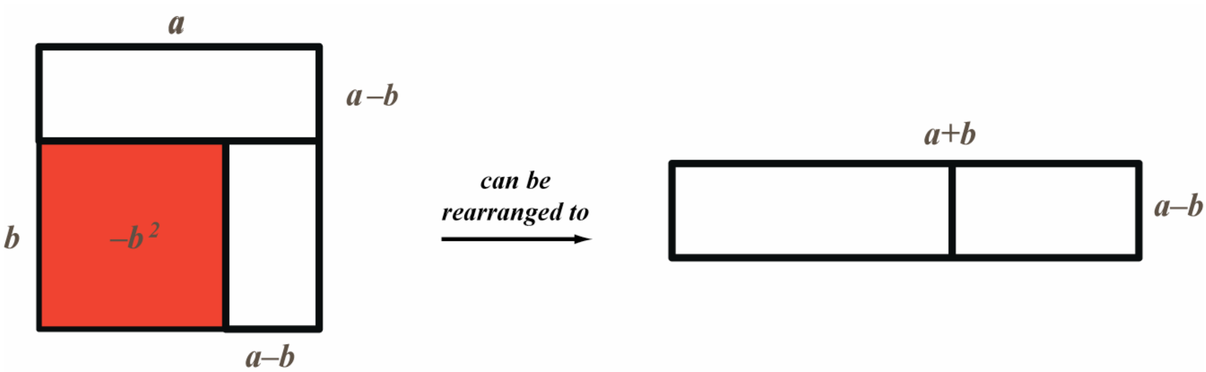


Figure 5: $a^2 - b^2 = (a - b)(a + b)$

The problem here is that the embodied representation is more complicated when $b > a$ or the values a and b may be positive or negative. It requires meaning being given to

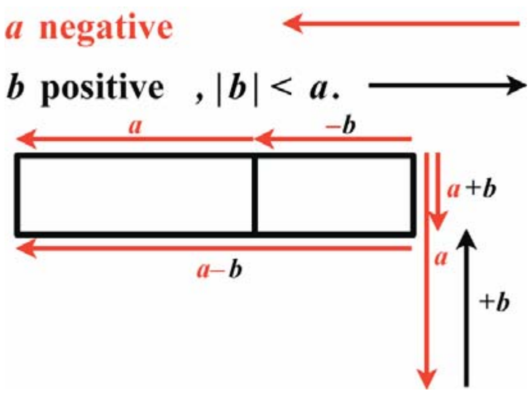


Figure 6: $a^2 - b^2 = (a - b)(a + b)$ for $a < 0$, $b > 0$, $b < |a|$

negative lengths (by reversal) and negative areas (by turning over). Can you 'see' this in figure 6?

Matters become more complicated with the formula

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Can you 'see' it in three dimensions as in figure 7? It is easier to do with physical manipulatives for positive *a* and *b*.

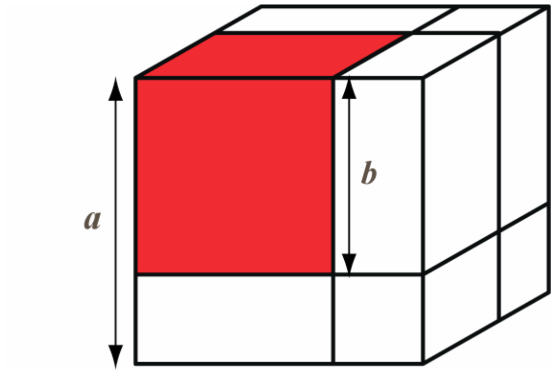


Figure 7: The difference between two cubes $a^3 - b^3$

The visualization is more complicated for positive and negative values of *a* and *b*, and as we move to $a^4 - b^4$ in four dimensions, we are no longer in familiar territory. The case of $a^n - b^n$ in general is impossible to visualize for ordinary mortals, certainly for values of *n* greater than 3.

Meanwhile, the meaning via symbolic manipulation is routine. The case $n = 4$ even consists of two applications of the case $n = 2$:

$$\begin{aligned} a^4 - b^4 &= (a^2 - b^2)(a^2 + b^2) \\ &= (a - b)(a + b)(a^2 + b^2) \end{aligned}$$

We see therefore that there is a genuine need to switch from embodiment to symbolism as the mathematics becomes more complex.

COMPRESSION OF KNOWLEDGE FROM PROCEDURE TO THINKABLE CONCEPT

Mathematics requires more than the ability to carry out procedures to *do* mathematics, it requires the construction of thinkable concepts to manipulate in the mind and as symbols on paper.

The symbols $a^2 - b^2$ and $(a - b)(a + b)$ represent quite different sequences of evaluation. The first squares the values of *a* and of *b* and then subtracts the latter square from the former. The second subtracts *b* from *a*, then adds *a* and *b* and then multiplies them together. So the expressions $a^2 - b^2$ and $(a - b)(a + b)$ represent different procedures of evaluation but always give the same result.

The functions $f(x) = x^2 - 4$ and $g(x) = (x - 2)(x + 2)$ are likewise different procedures of evaluation, but are considered as giving the *same* function, because for given input, they always give the same output.

Various theories of process-object encapsulation (eg Dubinsky, 1991; Sfard, 1991, Gray & Tall, 1994) suggest that the conceptions begin as step-by-step actions (or procedures) and then are re-conceptualised as overall processes focusing on the relationships between input and output. On closer inspection, there is a whole spectrum of compression (Figure 8):

- pre-procedure (before the full step-by-step procedure is constructed);
- a single step-by-step procedure,
- more than one procedure, giving the possibility of selecting the most efficient in a given context;
- seeing the process as a whole,
- conceiving the process as a thinkable concept (a procept) that may be manipulated.

The learning of procedures is part of mathematics, as is the practice for increasing speed, and the development of shorter procedures for increasing efficiency. But

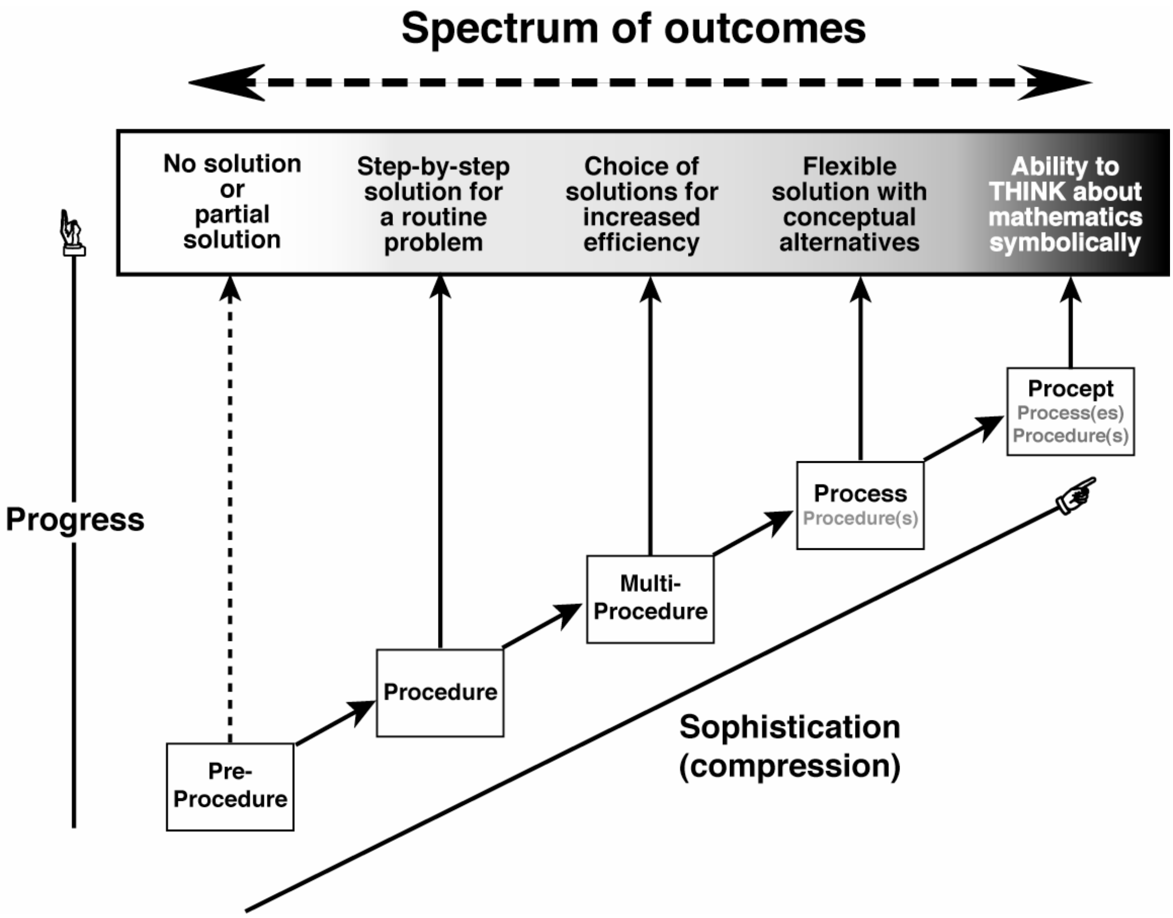


Figure 8: Spectrum of outcomes from increasing compression of symbolism (expanded from Gray, Pitta, Pinto & Tall, 1999, p.121).

without compression to thinkable concepts, the building of connections between ideas and the transition to the next level of development is likely to prove more difficult. Procedures occur *in time* and while it may be easy to practice them until they become fluent, it is less easy to think about them as thinkable concepts. For instance, a child whose conception of arithmetic is mainly in terms of counting procedures without a flexible understanding of known facts and the ability to derive new number facts from old is likely to find the transition to fractions, negatives and later to algebra, increasingly challenging. Students may be able to solve routine problems and pass examinations but they may then be compromised in their learning at the next stage as they lack the simple thinkable concepts to put together to make sense of the new situations.

It may happen that two students correctly solve the same problem, but one may have a very different conception of the solution. As an example, consider the following:

Circle the expressions that give the same result. Write another expression that is the same: $(x+3)(x-2)$, x^2+5x-6, x^2+x-6.

Student 'John' correctly circled the first and the third, $(x+3)(x-2)$, x^2+x-6. He then correctly wrote the expression $(x-2)(x+3)$. On the surface this is just the original expression with the terms in brackets reversed. But when asked what he did, John obtained the final expression $(x-2)(x+3)$ by factorising x^2+x-6 and he had not realised that the result was the same as the first expression.

Does this really matter? On the surface, John gets the routine questions correct. His problem is, however, that he lacks flexibility to move on and he had increasing difficulties coping with the procedures of calculus, which increasingly he tried to commit to memory in a way that proved to be fallible and began to fall apart.

THINKABLE CONCEPTS AND MET-BEFORES

Long-term learning becomes increasingly sophisticated. The complex multi-processing brain needs to focus on essentials and suppress irrelevant detail to be able to make decisions (Crick, 1994). Language enables us to focus on any phenomenon of interest, be it an object, an action, a property, an emotion, or whatever, to be able to name it and talk about it to make its meaning more precise. The notion of 'procept' is a typical example. Eddie Gray and I (Gray & Tall, 1994) realised that the same symbol such as 3+2 was being used by some children as a cue to carry out a counting procedure, while for others it was a thinkable concept, the sum '5'. By naming this phenomenon as a 'procept', it gave the facility to talk about it, to realise that procepts occurred throughout the symbolism of arithmetic, algebra, trigonometry, calculus, functions, transformations, and so on. It allowed us to see that the same symbol was interpreted very differently by different individuals and caused greater cognitive complications for some than for others who had compressed the symbol into a thinkable concept. It then allowed the idea to be refined and analysed, seeing ‘operational’ procepts in arithmetic, always producing a result, ‘potential’ procepts in algebra as expressions such as 3*x*+2 can only be evaluated when *x* is known, ‘potentially infinite procepts’ in the form of limits, and so on.

The formation of thinkable concepts is essential in the increasing sophistication of long-term development. The learner builds new conceptions on experiences that they have met before. Technically, I define a *met-before* (Tall, 2004) to be a current conceptual structure in the mind that is linked to a previous experience. This allows us to reflect on what children are doing and to talk to each child about some previous experiences that are helpful in a new context and others that may now be causing them difficulties. For example, arithmetic always produces ‘answers’, but this met-before can cause great difficulties when algebraic expressions do not give ‘answers’. On the other hand, the embodied shifting of objects around is a convenient met-before that may enable the learner to move the symbols around in an expression $3x + 4y + 2x$ to ‘move like terms together’ and combine 3*x* and 2*x* into 5*x* to transform the expression into $5x + 2y$. Such a shifting of terms may work with expressions, but it causes difficulty with equations such as $3x + 4 = 2x + 5$ where moving the 2*x* next to the 3*x* requires not only ‘change sides’ but also ‘change signs’.

Of course we would wish to teach algebra with meaning, and there are various ways that we might embody an equation to give meaning, for instance by considering it as a ‘balance’, so that $3x + 4 = 2x + 5$ is a balance with three ‘*x*’s and 4 on one sided and two ‘*x*’s and 5 on the other. Removing two ‘*x*’s from both sides leaves the sides in balance and gives $x + 4 = 5$, then removing 4 from each side leaves the solution $x = 1$. As we saw earlier, however, an embodiment that works in a simple case may not work in a more sophisticated context; this embodiment becomes more complicated if there are negative terms or negative values involved. (Vlassis, 2002). The learner must now use the technique when the embodiment may be meaningless.

The alternative is to make meaning not in the embodiment, but in the symbolism. Here the expression $3x + 4$ needs to be given meaning as an expression will depend on *x* and which can be manipulated as a generalised arithmetic operation: a difficult conception for many learners, especially those who see an arithmetic expression 3+2 as a procedure to be carried out, rather than a concept that can be manipulated in itself. Such an analysis would imply that students who lack proceptual flexibility with arithmetic will find algebra difficult to comprehend and be forced into procedural learning of the operations required to manipulate the symbols without meaning.

LONG-TERM LEARNING

This discussion leads us to a long-term view of learning, building on the genetic capabilities of the learner and the successive learning experiences over a life-time:

- The child is born with generic capabilities *set-before* in the genetic structure;
- Current cognitive development builds on experiences that were *met-before*;

- This occurs through long-term potentiation of neuronal connections which strengthens successful links and suppresses others;
- Actions are coordinated as (procedural) *action-schemas*;
- Ideas are compressed into *thinkable concepts* using language & symbolism;
- Thinkable concepts are built into wider (conceptual) *knowledge schemas*;
- Mathematical thinking builds cognitively through *embodiment*, *symbolism* and, later, *formal proof*, each developing in sophistication over time;
- Success in mathematical thinking depends on the effect of met-befores, the compression to rich thinkable concepts, and the building of successive levels of sophistication that is both powerful and simple.

In particular, it suggests that procedures that are not compressed into thinkable concepts may give short-term success in passing tests, but if those procedures are not given a suitable meaning as thinkable concepts (in this case, procepts), then they may make future learning increasingly difficult.

Various studies carried out by doctoral students at Warwick University in countries around the world reveal a widespread goal of 'raising standards' in mathematics learning, which are tested by tests that *could* promote conceptual long-term learning, but in practice, often produce short-term procedural learning that is may be less successful in developing long-term flexibility in understanding and solving non-routine problems.

Procedural conceptions of fraction

A study by Md Ali (2006) of the teaching of fraction in Malaysia focused on the methods used to raise the standards of all children learning mathematics by a curriculum that is intended to develop conceptual learning. Children are taught fractions in a caring and helpful way that includes the flexibility of seeing that a product, such as 'two-fifths of twenty-five' can be performed in two distinct ways: the first works out a fifth of twenty five, which is five, then multiplies by two, to get ten. The second multiplies two times twenty-five to get fifty and divides by five, also to get ten. However, the process is done by getting the children to recite the procedure, with the teacher saying successive parts of it and inviting the children to fill in the required words. For instance, the teacher might say, 'How do we work out two-fifths of twenty-five?' and draw three circles on the board one above the other for numerator and denominator of the fraction, the other for the whole number. 'What do we put in the top circle? The nu…', to which the class gleefully says 'the numerator!'. 'What do we put in the bottom circle? The de…', the class replies 'denominator!'. 'Of means mul…', 'multiply', and so the lesson continues, building up the ritual of the procedure of multiplication by a fraction.

The children's achievement in fractions tests is improved, but it is achieved by focusing upon persistent routine exercises. While this focuses on increasing *efficiency* of calculation using two different methods, it does not focus on the flexibility of fraction as a thinkable concept. The general consensus of teachers interviewed was that they faced a dilemma: on the one hand the curriculum recommended conceptual teaching and learning but on the other they had to succumb to the demand to achieve the school target in the examination. The teachers focused extensively on mastery of techniques through their lesson structure, their emphasis on the content and the way in which they presented it at the expense of the children's understanding of the fraction concept and the ability to perform creatively to solve even mildly different problems.

'Magic' embodiments in algebra

Working with a group of committed teachers in Brazil, Rosana Nogueira de Lima (de Lima & Tall, 2006) found that the teachers' concern to help students pass their algebra examinations led to focusing on the required techniques. Their experience of algebra included the manipulation of expressions such as reordering $3a+3b+2a$ to give $3a+2a+3b$ and then to simplify to $5a+3b$. This involves 'moving' the $2a$ 'next' to the $3a$ and then adding them together to get $5a$. In our mind's eye, we might sense this as 'picking up' the '$2a$' and moving it over the '3b' to get 'like terms' together.

The students were taught to solve linear equations by using the principle of 'doing the same thing to both sides, but many of them focused not on the general principle, but on the specific actions required to get the solution. The solution of $3x+2=8$ is then achieved by moving the numbers to the same side. Unlike the met-before of moving like terms together in an expression, shifting the 2 to the other side requires the 'magic' of 'change sign' to get $3x=8-2$ and simplifying to get $3x=6$. This may be solved by 'moving the 3 over the other side', this time 'putting it underneath' to get

$$x=\frac{6}{3}.$$

Such an activity seems to give a kind of *procedural embodiment*, remembering a sequence of actions to perform, rather than a *conceptual embodiment*, which involves giving coherent meaning to the underlying concepts. It works for a few able students who are able to carry out the procedure accurately, but without meaning, the procedure is fragile and many students make mistakes, such as changing $3x=6$ with the additional magic of 'changing signs' to get

$$x=\frac{6}{-3}.$$

Once such an error occurs and is marked as wrong by the teacher, the student tries to 'correct' mistakes, which can produce a new range of mixtures of errors.

In solving quadratics, the situation became worse as the teachers, knowing the difficulties with linear equations, focused on teaching the formula, which they know will solve all quadratic equations. However, in order to be able to *use* the formula, it may be necessary to first manipulate the symbols in the equation and here problems arose when the students were asked to show that the equation $(x-2)(x-3)=0$ had roots 2, 3. Many could not begin and, of those that could, *none* saw that on substituting the values the equation was satisfied; instead they attempted to multiply out the brackets and solve the equation using the formula. Few succeeded.

Complications in the function concept

As we move through into the secondary curriculum we come to concepts like the notion of function, which the NCTM standards see as being an essential underpinning of a wide range of mathematics. In some countries, such as Turkey, the function concept is taught from its set-theoretic definition and seen as a fundamental foundational idea. It is quite simple. You have two sets *A* and *B* and for each element *x* in *A*, there is precisely one corresponding element *y* in *B* which is called $f(x)$ (eff of eks). That's it!

However, this is used in the curriculum to weave a huge web of knowledge: linear functions, quadratic functions, trigonometric functions, exponentials and logarithms, formulae, graphs, sets of ordered pairs, set diagrams, and so on. How does one help the student make sense of this complicated mass of ideas?

If the purpose is to score marks on the examination, one strategy is evidently to give them all the details of all the techniques they require to pass the examination. But perhaps there is another way, more suitable to enable the student to focus on relevant detail and to build thinkable concepts that naturally link one to another.

Bayazit (2006) reports a study of the teaching of two teachers with very different approaches.

Teacher Ahmet saw his duty to mentor the students and help them make sense of the notion of function. So whenever he studied functions, he emphasised the simple property that a function $f: A \to B$ mapped each element of the domain *A* into the domain *B*. For example, in considering when a graph could be a function, he looked at the definition and related it to the fact that each *x* corresponded to only one *y*, and related this to the 'vertical line' test. When he considered the constant function, he went back to the definition and revealed the constant function $f(x) = c$ as the *simplest* of functions which mapped *every* value of *x* in *A* onto the single element *c* in *B*. Nothing could be simpler! Likewise, when he studied inverse functions, such as the square root, the inverse trigonometric functions and the relationship between logarithm and exponential, he patiently referred everything back to the definition of a function and encouraged his students to focus on the essential simplicity of the ideas. With piece-wise functions, which were new to the students, he again went back to the definition and confirmed how these too satisfied the simple requirement that for every *x* there was a unique.

The other teacher, Burak, was well aware of his students' potential difficulties and misconception with the functions, and was also aware of the sources of such obstacles. However, he focused on what was necessary for the students to pass the exam. He taught the 'vertical line test' as a specific test for functions, practising examples to get it right. He considered that students rejected the constant function because of their inability to conceive an ‘all-to-one’ transformation as a possible interpretation of the function definition, explicitly addressing the absence of x in the formula as a particular source of misconception. He interpreted the students’ difficulties with the inverse function as an indicator of their inability to move back and forth between the elements of domain and co-domain without losing sight of the ‘one-to-one and onto’ condition. He considered that the students had problems with visible discontinuities of the graph of piecewise-defined functions, predicting that the student would join the points by broken-lines or curves.

However, Burak made no effort to eliminate these obstacles during his classroom teaching. His range of knowledge of his students is a complicated collection of different problems. So he gave the students all the detail they needed to answer the examination questions. He constrained his teaching of inverse function to working out the formula, so that to find the inverse of $y = 2x + 3$, one would express x in terms of y by subtracting 3 from both sides and dividing by 2 to get $x = \frac{y-3}{2}$ then interchange x and y to get the inverse function as $y = \frac{x-3}{2}$. He explained that a constant function does not involve x and that its graph is a horizontal line parallel to the x axis, When teaching the piecewise function, he avoided the difficulties and did not give any illustration to encourage his students to reflect on what happened when the graph of the function was made up of disjoint parts or isolated points.

He would often indicate to students that an examination or test required a particular way of learning:

> If you want to succeed in those exams you have to learn how to cope.
>
> Do not forget simplification. It is crucial, especially [in] a multiple-choice test.

It would appear that his desire for success over-rode his deeper conceptions of thinking so that he provide an action-oriented teaching practice in which his students’ difficulties and misconceptions were peripheral to the rules and procedures that would lead to success in the types of problem asked in examinations.

REFLECTIONS

Looking at the total picture of long-term learning, what emerges is the absolute necessity of the teacher helping the student to construct thinkable concepts that not only enable students to solve current problems, but also to move on to greater sophistication. In a given situation, the learning of efficient procedures to *do* mathematics is an important part of learning, but in the long-term, it is essential to compress knowledge into thinkable concepts that will work in more sophisticated ways. This can be done by building on embodied experiences that can give insightful meanings suitable for initial learning but may include met-befores that can hinder future sophistication. Here it is essential to focus on the development of flexible thinking with the symbolism that compresses processes that can be used to solve mathematical problems into procepts that can be used to *think* about mathematics.

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