DESIGNING MATHEMATICS CONJECTURING ACTIVITIES TO FOSTER THINKING AND CONSTRUCTING ACTIVELY

Fou-Lai Lin

Department of Mathematics, National Taiwan Normal University

Based on the perspective that "a good lesson must provide opportunities for learners to think and construct actively". This paper will focus on (1) presenting a framework of designing conjecturing (FDC) with examples; (2) showing the supporting role of conjecturing on each phase of mathematics learning activity – conceptualizing, procedural operating, problem solving and proving; and (3) concluding that conjecturing is to encourage thinking and constructing actively, hence to drive innovation.

INTRODUCTION

Based on the data of TIMSS 2003, a dichotomy between students' achievement and self-confidence of mathematics among the APEC member economies is resulted. Comparing with the international average, students in high achievement countries, such as Singapore, Korea, Hong Kong, Japan, and Taiwan, have high percentage of grade 8 students with low self-confidence, while in those relatively low achievement countries, such as Malaysia, Australia, U.S., Indonesia, Chile, and Philippine, students have higher percentage of grade 8 students with high self-confidence than the above high achievement countries (ref. Table 1&2).

Table 1. Students' Self-Confidence in Learning Math-Grade 8 (TIMSS 2003)

Countries	High SCM		Medium SCM		Low SCM	
	% of students	Avg. Achievement	% of students	Avg. Achievement	% of students	Avg. Achievement
Singapore	39	639	34	594	27	571
Korea	30	650	36	592	34	534
Hong Kong	30	627	38	581	33	556
Taiwan	26	661	30	593	44	534
Japan	17	634	38	580	45	538
International Avg.	40	504	38	453	22	433

Table 2. Students' Self-Confidence in Learning Math-Grade 8 (TIMSS 2003)

Countries	High SCM		Medium SCM		Low SCM	
	% of students	Avg. Achievement	% of students	Avg. Achievement	% of students	Avg. Achievement
Malaysia	39	546	45	490	16	471
Australia	50	542	31	483	19	451
U.S.	51	534	29	483	20	461
Indonesia	27	420	59	408	15	416
Chile	35	427	42	369	23	361
Philippines	29	405	59	369	12	366
International Avg.	40	504	38	453	22	433

Within an education system, promoting both students' achievement and self-confidence in mathematics has seemed to be a dilemma. Why is high achievement often coupled with low self-confidence in mathematics? In the study by Lin and Tsao (1999) on the phenomenon of learning and teaching mathematics in Taiwan secondary schools, they concludes that competitive examination system in Taiwan drives passive and rote learning. In terms of education reform, a rationale, such as a good lesson must provide opportunities for learners to think and construct actively, should be put into practice.

VIEWS OF THINKING

Scientific thinking, mathematical thinking, arithmetic thinking, geometric thinking, algebraic thinking, statistic thinking, thinking in problem solving, high order thinking, and advanced mathematical thinking are all meaningful in mathematics education community. Those terminologies indicate that structure (components or mechanism) of thinking is not only subject-oriented but also hierarchical. A collection of extracts from experiential/phenomenological point of view, behavioral point of view, concept-development point of view, geometrical point of view, mathematics problem solving point of view, and meta-cognition on components of thinking is quoted below as an example.

1. Experiential/phenomenological point of view

Wertheimer (1961), a gestalt psychologist, has described the component of thinking from phenomenological point of view.

> Thinking consists in
>
> envisaging, realizing structural features and structural requirements; proceeding in accordance with, and determined by, these requirements; and thereby changing the situation in the direction of structural improvements, which involves:

that gaps, trouble-regions, disturbances, superficialities, etc., be viewed and dealt with structurally;

that inner structural relations – fitting or not fitting – be sought among such disturbances and the given situation as a whole and among its various parts;

that there be operations of structural grouping and segregation, of centering, etc.;

that operations be viewed and treated in their structural place, role, dynamic meaning, including realization of the changes which thus involves.

To make sense of the thinking components above, one may imagine oneself working in an archaeological field and trying to rebuild the living phenomena from several pieces of broken objects (furnace).

2. Behavioral point of view

Wertheimer (1961) also suggests that thinking process can be noticed in terms of the behaviour: comparison and discrimination (identification of similarities and differences); analysis (looking at parts); induction (generalisation, both empirical and structural); experience (gathering facts or vividly grasping structure); experimentation (seeking to decide between possible hypotheses); expressing 'one variable is a function of another variable'; associating (items together and recognising structural relationships); repeating; trial and error; learning on the basis of success (with or without appreciating structural significance).

3. Concept-development point of view

In Vygotsky's (1986) theory, scientific concepts formation consists of thinking in complex, generalizing and abstracting. Thinking in complex has four variations: associative complex, collections complex, chain complex, and diffuse complex. Generalizing and abstracting are functioning interactively during concepts formation.

4. Geometrical point of view

In van Hiele's model of geometric thinking, there are five hierarchical levels: level 0: visualization; level 1: analysis; level 2: informal deduction; level 3: deduction; level 4: rigor (Crowley, 1987).

5. Mathematics problem solving point of view

Thinking in mathematics problem solving has been structured as specializing, generalizing, conjecturing, and convincing (Polya, 1962; Mason, Burton, & Stacey, 1985).

6. Meta-cognition

Wilson (2001) has identified that the components of meta-cognition consists of awareness, evaluation, and regulation.

Participating in conjecturing activities, what would be the structure of thinking is still an open issue for research.

EXAMPLES AND A FRAMEWORK FOR DESIGNING CONJECTURING ACTIVITY

1. Three entries of Conjecturing

A conjecturing activity may start with one of the three entries: a false statement, a true statement, and a conjecture of learners.

1-1 False statement as starting point

Using students' misconception as starting point is an example, such as

(1) Multiplication makes bigger; division makes smaller.
(2) 4/9＞2/3 (if a＞c and b＞d, then b/a＞d/c)
(3) a multiple must be an integer or a half
(4) the additive strategy on ratio task
(5) a quadrilateral with one pair of opposite right angle is a rectangle
(6) the sum of a multiple of 3 and 6 is a multiple of 9
(7) the square of a given number is even
…

A *proceduralized refutation model*(PRM) (Lin & Wu, 2005) can be applied to design a conjecturing activity by substituting each students' misconception into the first item in the worksheet which follows student's activities step by step in the model (ref. Fig. 1).

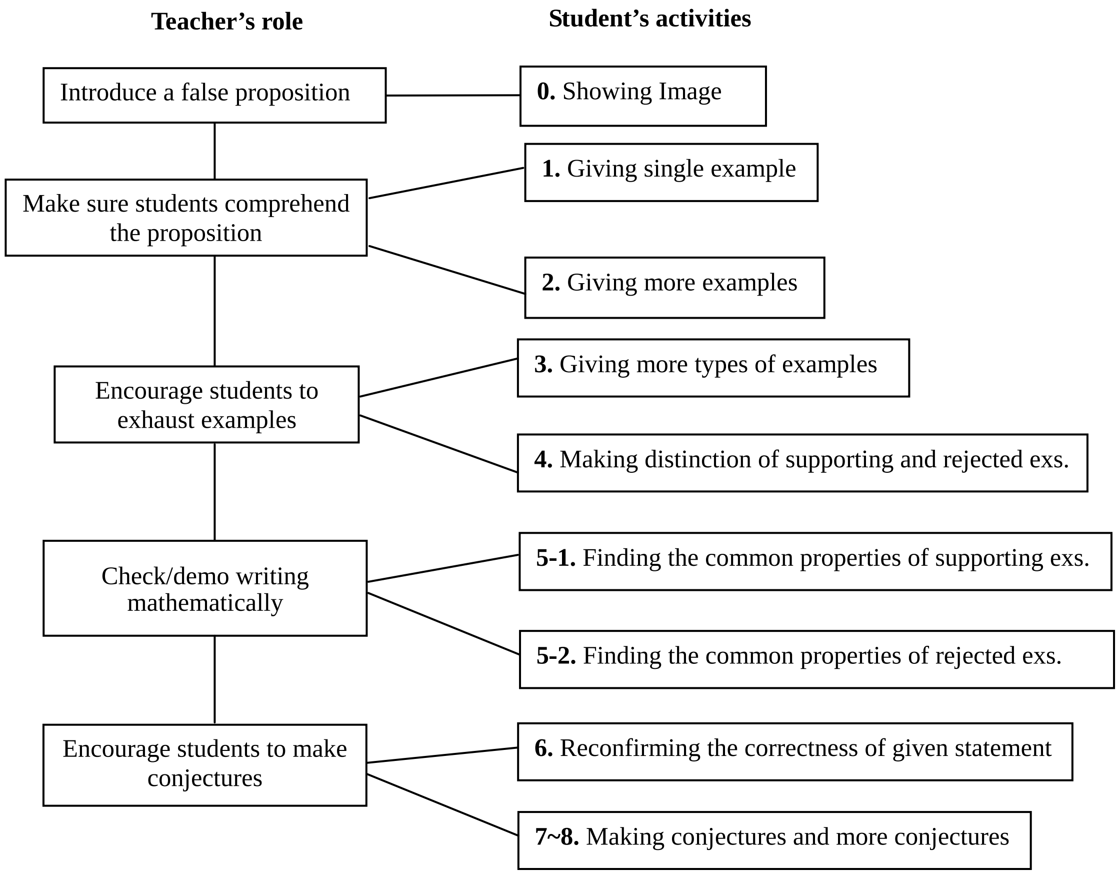


Figure 1. Model of Proceduralized Refuting and Making Conjectures

1-2 True statement as starting point

For example, Heron's formula is a good starting point: A=$\sqrt{s(s-a)(s-b)(s-c)}$, where s=$1/2\,(a+b+c)$

If a, b, c are the three sides of a given triangle, then A is its area.

The conjecturing activity regarding this formula can be designed as the following items (i)~(vi).

(i) Making your own sense of the formula:

Convincing yourself that A do represent the area of a triangle with three sides a, b, and c.

Observing it's beauty.

(ii) A model of conjecturing: A triad of mathematics thinking

Many teaching experiments show that high school students are able to notice the beauty of formula A which is *symmetry*, with respect to a, b, c, and the *degree of expressionin in A is two*, it stands for area. Students are also convinced by applying the area formula with some *special/extreme cases* of triangles.

Thinking in symmetry, degree of the expression and special/extreme cases composes a triad of mathematics thinking which can be generalized to make conjectures for formulae of geometry quantities.

(iii) Application of the Triad

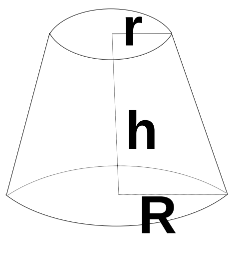
e.g. What can you say about the formula B?

B=$\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where s=$1/2\,(a+b+c+d)$

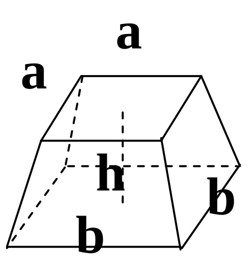
(iv) Your conjecture about B will be…

(v) Convincing yourself and peers about your conjecture.

(vi) Conjecturing the volume of the following two solids respectively.



V=?



V=?

1-3 Starting with students' own conjecturing

(1) Defining, by its nature, is a good conjecturing activity

e.g. Swimming Pool (Lin & Yang, 2002)

Conan is going to move to a new home, he has a rectangular swimming pool built in the backyard. When he checked the pool, he said, "Is it really a rectangular swimming pool?" If you were <u>Conan</u>, what places and what properties would you ask the workers to measure so that you can be sure it is rectangular?(It costs NT$1000 to check each item.)

Be sure, the payment is the less the better.

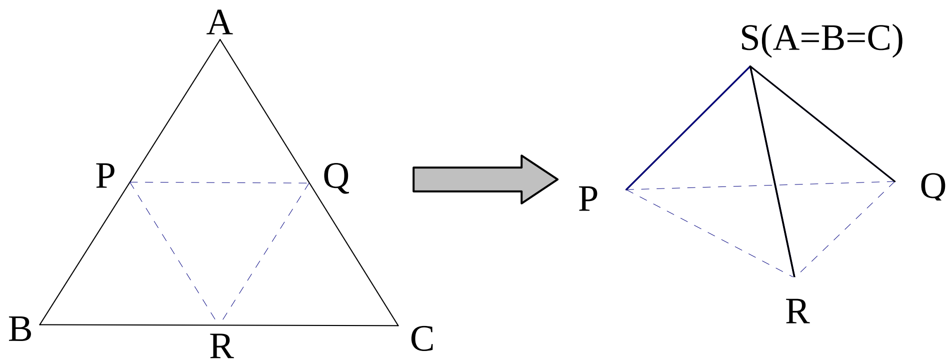
During the defining activity, teacher very often can collect many cognitively meaningful statements from students for students-centered teaching resources. For instance, in the swimming pool task, a student insisted that a quadrilateral with one pair of opposite right angle is a rectangle (Lin & Yang, 2002).

(2) Perceiving from an exploration

Taking 'triangle and tetrahedron' as an example, conjecturing activities can be designed as the following items (i)~(viii).

(i) Demo:

Folding out a tetrahedron from a given regular triangle:



(ii) Could you folding out a tetrahedron from a given isosceles triangle?

(iii) Would some kind of isosceles triangles work?

(iv) Could an isosceles right triangle work?

(v) How would you classify the triangles?

(vi) According to your classification, which kind of triangle would work?

(vii) Making your conjectures

(viii) Un-folding a tetrahedron, which kind of polygon you can obtain?

(3) Constructing premise/conclusion

Asking students to complete a conditional statement if p then q, given either p or q (not both) is prevalently used in Taiwanese mathematics classrooms. These constructing premise/conclusion activity is also encouraging students to make their own conjectures.

ex. If…, then the sum (product) of two numbers is even

If the sum (product) of two numbers is even, then …

If…, then their product is bigger than each of them

If their product is bigger than each of them, then …

If…, then the line L bisects the area of the quadrilateral

If a is an intersection point of two diagonal lines of a quadrilateral, then …

2. A frame for designing conjecturing (FDC)

Examples used to interpret the three entries of conjecturing in the above section also have shown the meaning of the following frame for designing conjecturing (FDC).

Table 3. A frame for designing conjecturing

Starting	Learning Strategy/Process
False Statement	➢ Proceduralized refutation learning model
True Statement	➢ A thinking triad ➢ "What if not" strategy to improve problem posing (Brown & Walter, 1983) ➢ Specialization/Generalization (Polya, 1962; Mason, Burton, &Stacey, 1985) ➢ Analogous ➢ Re-modification: modify-remodify till one makes sense of it
Conjecture	➢ Defining ➢ Exploration ➢ Constructing Premise/Conclusion

This frame is not only a theoretical frame but also an operational frame for carrying out teaching exploration to validate its effectiveness.

3. Conjecturing supports all phases of mathematizing

Mathematizing is an organizing and structuring activity according to which acquired knowledge and skills are used to discover unknown regularities, relations and structures (de Lange, 1987). Kilpatrick, Swafford, and Findell (2001) call such activity "mathematics proficiency," which consists five components: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

3-1 Conjecturing to enhance conceptual understanding

ex.(1) Using students'misconceptions as the starting statement in PRM.

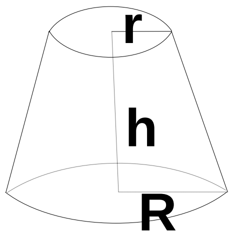
ex.(2) Inviting students to make conjecture of fraction addition after they have learned the meaning of fractions. Using the error pattern a/b + c/d = (a+c)/(b+d) as the starting statement in PRM.

Taking those examples as evidences, we might be convinced that conjecturing can enhance conceptual understanding both in prospective learning and in retrospective learning (Freudenthal, 1991). Certainly, we would like to invite teachers to carry out their teaching exploration with similar designing as the above examples.

3-2 Conjecturing to facilitate procedural operating

ex.(1) Using "the sum of a multiple of 3 and a multiple of 6 is a multiple of 9" as the starting statement in PRM.

ex.(2) Focusing on the Thinking Triad to make conjecture of the volume of a conical shape:



Those conjecturing activities provide good opportunities in computational operation for students in primary or junior high school on ex.(1) and in senior high school or university on ex.(2).

3-3 Conjecturing to develop competency of proving

Conjecturing and proving very often are discontinuous. In order to merge those two learning activities, learning strategy such as "constructing premise/conclusion" and "defining" are proved to be effective.

3-4 Conjecturing is a necessary process of problem solving

Based on Polya's (1962) thoughts, Mason, Burton, and Stacey (1985) have argued that specializing, generalizing, conjecturing and convincing are the components of thinking in problem solving.

CONJECTURING APPROACH DRIVES INNOVATION

From the examples of conjecturing activity elaborated in previous sections, we are convinced that participating in a conjecturing activity designed with FDC in which everyone is encouraged,

1) to construct extreme and paradigmatic examples,
2) to construct and test with different kind of examples,
3) to organize and classify all kinds of examples,
4) to realize structural features of supporting examples

5) to find counter-examples when realizing a falsehood,
6) to experiment
7) to self-regulate conceptually
8) to evaluate one’s own doing-thinking
9) to formalize a mathematical statement
10) to image/extrapolate/explore a statement
11) to grasp fundamental principles of mathematics

involves learners in *thinking and constructing actively*.

Since conjecturing encourages learners to think and to construct actively, and thinking and constructing actively is the foundation of innovation, conjecturing is indeed an adequate learning strategy for innovation.

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