WHAT IS MATHEMATICAL THINKING AND WHY IS IT IMPORTANT?

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INTRODUCTION

This paper and the accompanying presentation has a simple message, that mathematical thinking is important in three ways.

- Mathematical thinking is an important goal of schooling.
- Mathematical thinking is important as a way of learning mathematics.
- Mathematical thinking is important for teaching mathematics.

Mathematical thinking is a highly complex activity, and a great deal has been written and studied about it. Within this paper, I will give several examples of mathematical thinking, and to demonstrate two pairs of processes through which mathematical thinking very often proceeds:

- Specialising and Generalising
- Conjecturing and Convincing.

Being able to use mathematical thinking in solving problems is one of the most the fundamental goals of teaching mathematics, but it is also one of its most elusive goals. It is an ultimate goal of teaching that students will be able to conduct mathematical investigations by themselves, and that they will be able to identify where the mathematics they have learned is applicable in real world situations. In the phrase of the mathematician Paul Halmos (1980), problem solving is “the heart of mathematics”. However, whilst teachers around the world have considerable successes with achieving this goal, especially with more able students, there is always a great need for improvement, so that more students get a deeper appreciation of what it means to think mathematically and to use mathematics to help in their daily and working lives.

MATHEMATICAL THINKING IS AN IMPORTANT GOAL OF SCHOOLING

The ability to think mathematically and to use mathematical thinking to solve problems is an important goal of schooling. In this respect, mathematical thinking will support science, technology, economic life and development in an economy. Increasingly, governments are recognising that economic well-being in a country is underpinned by strong levels of what has come to be called ‘mathematical literacy’ (PISA, 2006) in the population.

Mathematical literacy is a term popularised especially by the OECD’s PISA program of international assessments of 15 year old students. Mathematical literacy is the ability to use mathematics for everyday living, and for work, and for further study, and so the PISA assessments present students with problems set in realistic contexts. The framework used by PISA shows that mathematical literacy involves many components of mathematical thinking, including reasoning, modelling and making connections between ideas. It is clear then, that mathematical thinking is important in large measure because it equips students with the ability to use mathematics, and as such is an important outcome of schooling.

At the same time as emphasising mathematics because it is useful, schooling needs to give students a taste of the intellectual adventure that mathematics can be. Whilst the highest levels of mathematical endeavour will always be reserved for just a tiny minority, it would be wonderful if many students could have just a small taste of the spirit of discovery of mathematics as described in the quote below from Andrew Wiles, the mathematician who proved Fermat's Last Theorem in 1994. This problem had been unsolved for 357 years.

> One enters the first room of the mansion and it's dark. One stumbles around bumping into furniture, but gradually you learn where each piece of furniture is. Finally, after six months of so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of, and couldn't exist without, the many months of stumbling around in the dark that precede them. (Andrew Wiles, quoted by Singh, 1997, p236, 237)

At the APEC meeting in Tokyo in January 2006, Jan de Lange spoke in detail about the use of mathematics to equip young people for life, so I will instead focus this paper on two other ways in which mathematical thinking is important.

WHAT IS MATHEMATICAL THINKING?

Since mathematical thinking is a process, it is probably best discussed through examples, but before looking at examples, I briefly examine some frameworks provided to illuminate mathematical thinking, going beyond the ideas of mathematical literacy. There are many different 'windows' through which the mathematical thinking can be viewed. The organising committee for this conference (APEC, 2006) has provided a substantial discussion on this point. Stacey (2005) gives a review of how mathematical thinking is treated in curriculum documents in Australia, Britain and USA.

One well researched framework was provided by Schoenfeld (1985), who organised his work on mathematical problem solving under four headings: the *resources* of mathematical knowledge and skills that the student brings to the task, the *heuristic* strategies that that the student can use in solving problems, the monitoring and *control* that the student exerts on the problem solving process to guide it in productive directions, and the *beliefs* that the student holds about mathematics, which enable or disable problem solving attempts. McLeod (1992) has supplemented this view by expounding on the important of affect in mathematical problem solving.

In my own work, I have found it helpful for teachers to consider that solving problems with mathematics requires a wide range of skills and abilities, including:

- Deep mathematical knowledge
- General reasoning abilities
- Knowledge of heuristic strategies
- Helpful beliefs and attitudes (e.g. an expectation that maths will be useful)
- Personal attributes such as confidence, persistence and organisation
- Skills for communicating a solution.

Of these, the first three are most closely part of mathematical thinking.

In my book with John Mason and Leone Burton (Mason, Burton and Stacey, 1982), we provided a guide to the stages through which solving a mathematical problem is likely to pass (Entry, Attack, Review) and advice on improving problem solving performance by giving experience of heuristic strategies and on monitoring and controlling the problem solving process in a meta-cognitive way.

We also identified four fundamental processes, in two pairs, and showed how thinking mathematically very often proceeds by alternating between them:

- specialising – trying special cases, looking at examples
- generalising - looking for patterns and relationships
- conjecturing – predicting relationships and results
- convincing – finding and communicating reasons why something is true.

I will illustrate these ideas in the two examples below. The first example examines the mathematical thinking of the problem solver, whilst the second examines the mathematical thinking of the teacher. The two problems are rather different – the second is within the mainstream curriculum, and the mathematical thinking is guided by the teacher in the classroom episode shown. The first problem is an open problem, selected because it is similar to open investigations that a teacher might choose to use, but I hope that its unusual presentation will let the audience feel some of the mystery and magic of investigation afresh.

MATHEMATICAL THINKING IS IMPORTANT AS A WAY OF LEARNING MATHEMATICS

In this section, I will illustrate these four processes of mathematical thinking in the context of a problem that may be used to stimulate mathematical thinking about numbers or as an introduction to algebra. If students' ability to think mathematically is an important outcome of schooling, then it is clear that mathematical thinking must feature prominently in lessons.

Number puzzles and tricks are excellent for these purposes, and in the presentation I will use a number puzzle in a format of the *Flash Mind Reader*, created by Andy

Naughton and published on the internet (HREF1). The Flash Mind Reader does not look like a number puzzle. Indeed its creator writes:

> We have been asked many times how the Mind Reader works, but will not publish that information on this website. All magicians […] do not give away how their effects work. The reason for this is that it spoils the fun for those who like to remain mystified and when you do find out how something works it's always a bit of a let-down. If you are really keen to find out how it works we suggest that you apply your brain and try to work it out on paper or search further afield. (HREF1)

As with many other number tricks, an audience member secretly chooses a number (and a symbol), a mathematical process is carried out, and the computer reveals the audience member's choice. In this case, a number is chosen, the sum of the digits is subtracted from the number and a symbol corresponding to this number is found from a table. The computer then magically shows the right symbol. The Flash Mind Reader is too difficult to use in most elementary school classes, the target of this conference, but I have selected it so that my audience of mathematics education experts can experience afresh some of the magic and mystery of numbers. As the group works towards a solution, we have many opportunities to observe mathematical thinking in action.

Through this process of shared problem solving as we investigate the Flash Mind Reader, I hope to make the following points about mathematical thinking. Firstly, when people first see the Flash Mind Reader, mathematical explanations are far from their minds. Some people propose that it really does read minds, and they may try to test their theory by not concentrating hard on the number that they choose. Others hypothesise that the program exerts some psychological power over the person's choice of number. Others suggest it is only an optical illusion, resulting from staring at the screen. This illustrates that a key component of mathematical thinking is having a disposition to looking at the world in a mathematical way, and an attitude of seeking a logical explanation.

As we seek to explain how the Flash Mind reader works, the fundamental processes of thinking mathematically will be evident. The most basic way of trying to understand a problem situation is to try the Flash Mind Reader several times, with different numbers and different types of numbers. This helps us understand the problem (in this case, what is to be explained) and to gather some information. This is a simple example of specialising, the first of the four processes of thinking mathematically processes.

As we enter more deeply into the problem, specialising changes its character. First we may look at one number, noting that if 87 is the number, then the sum of its digits is 15 and 87 – 15 is 72.

Beginning to work systematically leads to evidence of a pattern:

87	8 + 7 = 15	87 – 15 = 72
86	8 + 6 = 14	86 – 14 = 72
85	8 + 5 = 13	85 – 13 = 72
84	8 + 4 = 12	84 – 12 = 72

and a cycle of experimentation (which numbers lead to 72?, what do other numbers lead to?) and generalising follows.

Of course, at this stage it is important to note the value of working with the unclosed expressions such as 8+7 instead of the closed 15, because this reveals the general patterns and reasons so much better. Working with the unclosed expression to reveal structure is an admirable feature of Japanese elementary education.

87	87 – 7 = 80	80 – 8 = 72
86	86 – 6 = 80	80 – 8 = 72
85	85 – 5 = 80	80 – 8 = 72
84	84 – 4 = 80	80 – 8 = 72

It is also worthwhile noting at this point, that although we are working with a specific example, the aim here is to see the general in the specific.

This generalising may lead to a conjecture that the trick works because all starting numbers produce a multiple of 9 and all multiples of 9 have the same symbol. But this conjecture is not quite true and further examination of examples (more specialising) finally identifies the exceptions and leads to a convincing argument. In school, we aim for students to be able to use algebra to write a proof, but even before they have this skill, they can be produce convincing arguments. An orientation to justify and prove (at an appropriate level of formality) is important throughout school.

If students are to become good mathematical thinkers, then mathematical thinking needs to be a prominent part of their education. In addition, however, students who have an understanding of the components of mathematical thinking will be able to use these abilities independently to make sense of mathematics that they are learning. For example, if they do not understand what a question is asking, they should decide themselves to try an example (specialise) to see what happens, and if they are oriented to constructing convincing arguments, then they can learn from reasons rather than rules. Experiences like the exploration above, at an appropriate level build these dispositions.

MATHEMATICAL THINKING IS ESSENTIAL FOR TEACHING MATHEMATICS.

Mathematical thinking is not only important for solving mathematical problems and for learning mathematics. In this section, I will draw on an Australian classroom episode to discuss how mathematical thinking is essential for teaching mathematics. This episode is taken from data collected by Dr Helen Chick, of the University of Melbourne, for a research project on teachers' pedagogical content knowledge. For other examples, see Chick, 2003; Chick & Baker, 2005, Chick, Baker, Pham & Cheng, 2006a; Chick, Pham & Baker, 2006b). Providing opportunities for students to learn about mathematical thinking requires considerable mathematical thinking on the part of teachers.

The first announcement for this conference states that a teacher requires mathematical thinking for analysing subject matter (p. 4), planning lessons for a specified aim (p. 4) and anticipating students' responses (p. 5).These are indeed key places where mathematical thinking is required. However, in this section, I concentrate on the mathematical thinking that is needed on a minute by minute basis in the process of conducting a good mathematics lesson. Mathematical thinking is not just in planning lessons and curricula; it makes a difference to every minute of the lesson.

The teacher in this classroom extract is in her fifth year of teaching. She stands out in Chick's data as one of the teachers in the sample exhibiting the deepest pedagogical content knowledge (Shulman, 1986, 1987). Her pupils are aged about 11 years, and are in Grade 6. This lesson began by reviewing ideas of both area and perimeter. We will examine just the first 15 minutes.

The teacher selected an open and reversed task to encourage investigation and mathematical thinking. Students had 1cm grid paper and were all asked to draw a rectangle with an area of 20 square cm. This task is open in the sense that there are multiple correct answers, and it is 'reversed' when it is contrasted to the more common task of being given a rectangle and finding its area. The teacher reminded students that area could be measured by the number of grid squares inside a shape.

In terms of the processes of mathematical thinking, the teacher at this stage is ensuring that each student is specialising. They are each working on a special case, and coming to know it well, and this will provide an anchor for future discussions and generalisations. I make no claim that the teacher herself analyses this move in this way.

As the teacher circulated around the room assisting and monitoring students, she came to a student who asked if he could draw a square instead of a rectangle. In the dialogue which follows, the teachers' response highlighted the definition of a rectangle, and she encouraged the student to work from the definition to see that a square is indeed a rectangle.

S: Can I do a square?

T: Is a square a rectangle?

T: What's a rectangle?

T: How do you get something to be a rectangle? What's the definition of a rectangle?

S: Two parallel lines

T: Two sets of parallel lines … and …

S: Four right angles.

T: So is that [square] a rectangle?

S: Yes.

T: [Pause as teacher realises that student understands that the square is a rectangle, but there is a measurement error] But has that got an area of 20?

S: [Thinks] Er, no.

T: [Nods and winks]

Other responses to this student would have closed down the opportunity to teach him about how definitions are used in mathematics. To the question “Can I do a square?”, she may have simply replied “No, I asked you to draw a rectangle” or she might have immediately focussed on the error that led the student to ask the question. Instead she saw the opportunity to develop his use of definitions. When the teacher realised that the student had asked about the square because he had made a measurement error, she judged that this was within the student’s own capability to correct, and so she simply indicated that he should check his work.

In the next segment, a student showed his 4 x 5 rectangle on the overhead projector, and the teacher traced around it, confirmed its area is 20 square cm and showed that multiplying the length by the width can be used instead of counting the squares, which many students did. In this segment, the teacher demonstrated that reasoning is a key component of doing mathematics. She emphasised the mathematical connections between finding the number of squares covered by the rectangle by repeated addition (4 on the first row, 4 on the next, …) and by multiplication. In her classroom, the formula was not just a rule to be remembered, but it was to be understood. The development of the formula was a clear example of ‘seeing the general in a special case’. The formula was developed from the 4 x 5 rectangle in such a way that the generality of the argument was highlighted.

The teacher paid further attention to generalisation and over-generalisation at this point, when a student commented: ‘That’s how you work out area – you do the length times the width’. The teacher seized on this opportunity to address students’ tendency to over-generalise, and teased out, through a short class discussion, that LxW only works for rectangles.

S1: That’s how you work out area -- you do the length times the width.

T: When S said that’s how you find the area of a shape, is he *completely* correct?

S2: That’s what you do with a 2D shape.

T: Yes, for *this* kind of shape. What kind of shape would it *not* actually work for?

S3: Triangles.

S4: A circle.

T: [With further questioning, teases out that LxW only applies to rectangles]

In the next few minutes, the teacher highlighted the link between multiplication and area by asking students to make other rectangles with area 20 square centimetres. Previously all students had made 4 x 5 or 2 x 10, but after a few minutes, the class had found 20 x 1, 1 x 20, 10 x 2, 2 x 10, 4 x 5 and 5 x 4 and had identified all these side lengths as the factors of 20. Making links between different parts of the mathematics curriculum characterises her teaching.

Then, in another act of generalisation, the teacher begins to move beyond whole numbers:

T: Are there any other numbers that are going to give an area of 20? [Pauses, as if uncertain. There is no response from the students at first]

T: No? How do we know that there's not?

S: You could put 40 by 0.5.

T: Ah! You've gone into decimals. If we go into decimals we're going to have *heaps*, aren't we?

After these first 15 minutes of the lesson, the students found rectangles with an area of 16 square centimetres and the teacher stressed the important problem solving strategy of working systematically. Later, in order to contrast the two concepts of area and perimeter, students found many shapes of area 12 square cm (not just rectangles) and determined their perimeters.

Even the first 15 minutes of this lesson show that considerable mathematical thinking on behalf of the teacher is necessary to provide a lesson that is rich in mathematical thinking for students. We see how she draws on her mathematical concepts, deeply understood, and on her knowledge of connections among concepts and the links between concepts and procedures. She also draws on important general mathematical principles such as

- working systematically
- specialising – generalising: learning from examples by looking for the general in the particular
- convincing: the need for justification, explanation and connections
- the role of definitions in mathematics.

Chick's work analyses teaching in terms of the knowledge possessed by the teachers. She tracks how teachers reveal various categories of pedagogical content knowledge (Shulman, 1986) in the course of teaching a lesson. In the analysis above, I viewed the lesson from the point of view of the process of thinking mathematically within the lesson rather than tracking the knowledge used. To draw an analogy, in researching a students' solution to a mathematical problem, a researcher can note the mathematical content used, or the researcher can observe the process of solving the problem. Similarly, teaching can be analysed from the "knowledge' point of view, or analysed from the process point of view.

For those us who enjoy mathematical thinking, I believe it is productive to see teaching mathematics as another instance of solving problems with mathematics. This places the emphasis not on the static knowledge used in the lesson asabove but on a process account of teaching. In order to use mathematics to solve a problem in any area of application, whether it is about money or physics or sport or engineering, mathematics must be used in combination with understanding from the area of application. In the case of teaching mathematics, the solver has to bring together expertise in both mathematics and in general pedagogy, and combine these two domains of knowledge together to solve the problem, whether it be to analyse subject matter, to create a plan for a good lesson, or on a minute-by-minute basis to respond to students in a mathematically productive way. If teachers are to encourage mathematical thinking in students, then they need to engage in mathematical thinking throughout the lesson themselves.

References

APEC –Tsukuba (Organising Committee) (2006) First announcement. *International Conference on Innovative Teaching of Mathematics through Lesson Study.* CRICED, University of Tsukuba.

Chick, H. L. (2003). 'Pre-service teachers' explanations of two mathematical concepts' *Proceedings of the 2003 conference, Australian Association for Research in Education.* From: http://www.aare.edu.au/03pap/chi03413.pdf

Chick, H.L. and Baker, M. (2005) 'Teaching elementary probability: Not leaving it to chance', in P.C. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce & A. Roche (eds.) *Building Connections: Theory, Research and Practice.* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia), MERGA, Sydney, pp. 233-240.

Chick, H.L., Baker, M., Pham, T., and Cheng, H. (2006a) 'Aspects of teachers' pedagogical content knowledge for decimals', in J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (eds.), *Proc. 30th conference e International Group for the Psychology of Mathematics Education*, PME, Prague, Vol. 2, pp. 297-304.

Chick, H.L., Pham, T., and Baker, M. (2006b) 'Probing teachers' pedagogical content knowledge: Lessons from the case of the subtraction algorithm', in P. Grootenboer, R. Zevenbergen, & M. Chinnappan (eds.), *Identities, Cultures and Learning Spaces* (Proc. 29th annual conference of Mathematics Education Research Group of Australasia), MERGA, Sydney, pp. 139-146.

Halmos, P. (1980). The heart of mathematics. *American Mathematical Monthly*, 87(7), 519 – 524.

HREF1 CyberGlass Design - The Flash Mind Reader. http://www.cyberglass.biz Accessed 28 November 2006.

Mason, J. Burton, L. and Stacey, K. (1982) *Thinking Mathematically.* London: Pearson. (Also available in translation in French, German, Spanish, Chinese, Thai (2007))

McLeod, D.B. (1992) Research on affect in mathematics education: a reconceptualisation. In D.A. Grouws, Ed., *Handbook of research on mathematics teaching and learning*, (pp. 575–596).New York: MacMillan, New York.

PISA (Programme for International Student Assessment) (2006) *Assessing Scientific, Reading and Mathematical Literacy. A Framework for PISA 2006.* Paris: OECD.

Schoenfeld, A. (1985) *Mathematical Problem Solving*. Orlando: Academic Press.

Shulman, L.S. (1986) Those who understand: Knowledge growth in teaching, *Educational Researcher* 15 (2), 4-14.

Shulman, L.S. (1987) Knowledge and teaching: Foundations of the new reform, *Harvard Educational Review* 57(1), 1-22.

Singh, S. (1997) *Fermat's Enigma*, New York: Walter

Stacey, K. & Groves, S. (1985) *Strategies for Problem Solving. Lesson Plans for Developing Mathematical Thinking.* Melbourne: Objective Learning Materials.

Stacey, K. & Groves, S. (2001) *Resolver Problemas: Estrategias.* Madrid: Lisbon.

Stacey, Kaye (2005) The place of problem solving in contemporary mathematics curriculum documents. *Journal of Mathematical Behavior* 24, pp 341 – 350.