Cultivating Mathematical Thinking through Representation
－Utilizing the Representational System－

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Thank you for your introduction. My name is Tadao Nakahara. The secondary theme of this international conference is “Cultivating the Thinking of Children through Activities Involving Exchanges and Increased Depth”. I have conducted various types of research into “representations in mathematics education”. Mathematical thinking is conveyed and deepened through these representations. Since the research I have been engaged in is closely related to this conference’s theme, I have undertaken the task of delivering the keynote speech.

I will be speaking for approximately 40 minutes on the theme:

Cultivating Mathematical Thinking through Representation
－Utilizing Representational Systems－

1. Introduction

Representation plays extremely important roles in mathematics education (this also includes “arithmetic education”, here and throughout this speech). These important roles can be categorized as follows:

a) Thinking through what is represented (as a method of thinking)
b) Recording what was thought through representations (as a method of recording)
c) An important method for communication

I will first classify the various types of representation used in mathematics education, then consider the characteristics of each classification, and present utilization principles for each representation method based on these characteristics. Before we begin, however, I would like to explain two points.

The first point is that with respect to mathematics education, “representation” is broadly defined. When it comes to representation in mathematics education, people first think of things such as symbols and expressions. For the purpose of this discussion, we will not limit ourselves to these things, but rather include figures and graphs (of course), as well as Japanese-language expressions, teaching aids, physical objects, and so on in the definition of “representation”. Note, however, that we will not include representations through voice. Therefore, in general, we will be focusing on “visual representations related to mathematics”.

The second point is that, as a rule, I will be using these three terms in the following manner:

a) Representational method:

A representational methods is each concrete expressions such as “3+2” or “add 3 and 2”

b) Representational mode:

A representational mode is a set of concrete representational methods that is categorized from a certain perspective, such as “representation through symbols” and “representation through figures”

c) Representational system:

A representational system systematically organizes mutual relationships between representational modes

2. Previous Research into Representational Systems in Mathematics Education

A great deal of research has been conducted into representations as related to mathematics education, both internally and externally. I would like to introduce two areas of research here that form the foundation of my research.

(1) Bruner's EIS principle

When the American cognitive psychologist Bruner focused on the cognition of children, as well as representative thinking, he pointed out that it is possible to divide representation into the following three classifications, which describe the sequential development stages of representation:

(E) Enactive representation

(I) Iconic representation

(S) Symbolic representation

His system is referred to as the "EIS principle", based on an abbreviation of the three stages of representation. This principle has been utilized for early instruction in mathematical concepts, as well as for remedial instruction aimed at students who are stumbling. Above all, this principle has played a leading role in the so-called "modernization movement" in mathematics education. Since this principle was not intentionally presented for the sake of mathematics education, however, I believe that there is room for improvement when one considers the representational systems applicable to mathematics education.

(2) Lesh's representational system

Next, I would like to introduce the research into representational systems conducted by the American Lesh, who organized the Rational Number Project and did work in fractions instruction. The members of this project positioned representational systems as one of the important central points of this research, and presented the representational systems shown in Figure 1:

(Lesh's Representational Systems)

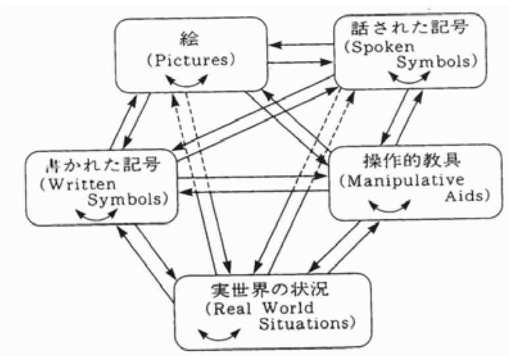

Figure 1 Lesh's Representational Systems

These representational systems can be said to have been improved in the following three ways over the aforementioned representational research conducted by Bruner, as pertains to mathematics education:

a) Bruner's enactive representation is divided into two categories, "real world situations," and "manipulative aids." These two categories have important differences in levels of concreteness and abstractness, and should be differentiated.

b) Rather than thinking linearly, interrelationships and mutual transformations between representational modes are considered.

When one considers the utilization of representation in class, and the utilization of representation in solving problems, mutual transformations are important.

c) The conversion of representations within the same representational mode are considered, as well.

This type of conversion is used a great deal in mathematics education as well.

On the other hand, Lesh's representational systems include auditory representations such as "spoken symbols," and positions representations of mathematically different natures in "written symbols" as well. For this reason, there is still room for improvement.

3. Nakahara's research into representational systems

(1) Categories of representational modes and basic characteristics

I have analyzed and examined a great deal of earlier studies, starting with the research of Bruner and Lesh. Based on this, I propose the broad organization of representational modes in mathematics education into the following five categories:

S2. Symbolic representation

Representations used in mathematical notation, such as numbers, letters, and symbols

S1. Linguistic representation

Representations that use everyday languages, such as Japanese or English

I. Illustrative representation

Representations that use illustrations, figures, graphs, and so on

E2. Manipulative representation

Representations such as teaching aids that work by adding the dynamic operation of objects that have been artificially fabricated or modeled

E1. Realistic representation

Representations based on actual states and objects

I would like to introduce actual examples of each of these categories here:

S2. 3+2=5

S1. Adding 3 and 2 gives you 5.

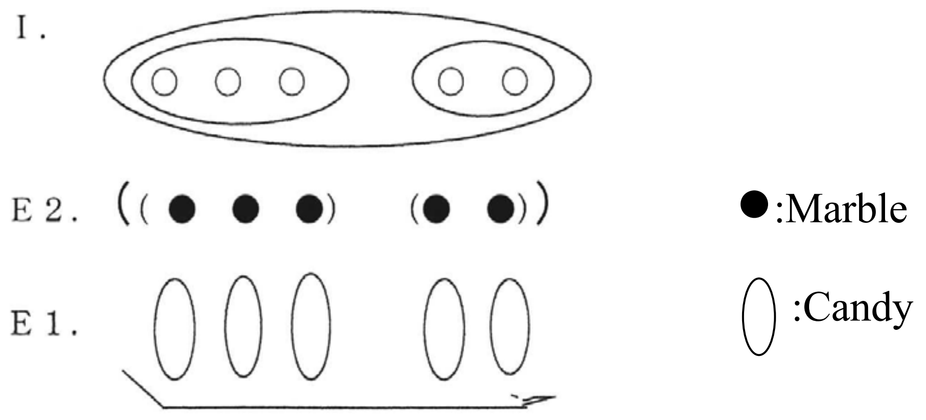

Figure2

When one compares these categories with Bruner's EIS, enactive representation is divided into the two categories realistic representation and manipulative representation, and symbolic representation is divided into the two categories linguistic representation and symbolic representation. Also, when compared with Lesh's representational mode, "spoken symbols" are removed, and "written symbols" are split into "linguistic representation" and "symbolic representation". I believe this classification method does a better job of taking advantage of the characteristics of representation in mathematics education, as it possesses characteristics that are particular to the representational modes of mathematics education, as follows:

Symbolic representation: This type of representation, which is governed by rules, is extremely succinct and unambiguous.

Linguistic representation: This type of representation is also governed by convention, but lacks succinctness. On the other hand, this representation is descriptive and can give a sense of familiarity.

Illustrative representation: Since the relationship with the referent is analogical, this type of representation is both visually and intuitively rich.

Manipulative representation: This type of representation is dynamic, somewhat concrete, and artificial.

Realistic representation: This type of representation is dynamic and extremely concrete and natural.

(2) Nakahara's Representational systems

Next, I would like to examine the mutual relationships between the five aforementioned representational modes, and organize them as follows.

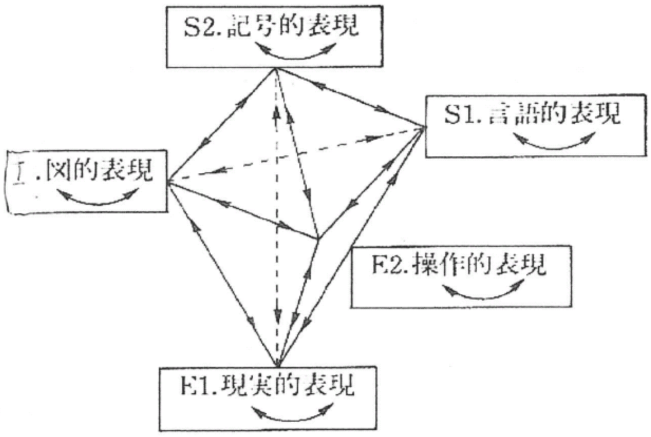

Figure 3 Representational systems in mathematics education

This is based on the following types of mutual relationships between representational modes:

a) The basic order is E-I-S, from bottom to top. This represents a transition from representations that are concrete to highly abstract representations.

b) The mutual conversions between representational modes and within representational modes are indicated with arrows.

c) In particular, conversions between representational modes are referred to as "translations". This activity of translation gives birth to mathematical thinking, deepens understanding, promotes problem-solving.

4. The characteristics and utilization of representational modes

Next, I would like to examine the characteristics and functions of two representational modes that play important roles in the cultivation of mathematical thinking, the understanding of mathematical concepts, and problem-solving. I will also derive the principles for utilizing these representational modes.

(1) The characteristics and utilization of manipulative representation

The following examples show situations where manipulative representation is used:

Example A: Addition with borrowing (13-8)

The tiles shown to the right can be used for this problem.

First, subtract 8 from 10 to get 2. Add 3 to get the answer, which is 5.

Figure 4

Example B: The factorization of x^2+5x+4

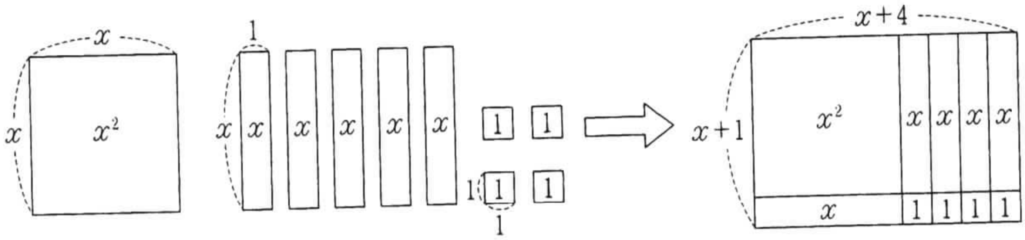

Figure 5

As shown above, tiles are used to factorize as follows:

$x^2+5x+4 = (x+4)\quad(x+1)$

As shown in the above examples, manipulative representation have the following characteristics:

a) Dynamic operations and problem-solving: Manipulative representation uses dynamic operations to solve problems.

b) Intermediateness and mediation: Manipulative representation is positioned in the middle between realistic representation and symbolic representation, and mediates between the two.

The following points are important in the utilization of these characteristics:

a) The children themselves carry out operational activities.

b) To attempt generalization concepts or methods through dynamic operations and to depart from manipulative representation.

(2) The characteristics and utilization of illustrative representation

The various different types of illustrative representation used in arithmetic and mathematics classes are organized as follows:

I-a) Situation diagram: A diagram of a realistic situation or state that has some relationship with arithmetic or mathematics

I-b) Scene diagram: A diagram that shows a scene based on arithmetic or mathematics

I-c) Procedural diagram: A diagram that shows procedures for calculations, operations, and so on

I-d) Structure diagram: A diagram that shows the structure of issues in arithmetic or mathematics

I-e) Conceptual diagram: A diagram that shows concepts of arithmetic or mathematics

I-f) Principle or relationship diagram: A diagram that shows the principles and relationships of arithmetic or mathematics

I-g) Graph: A diagram using any of a variety of different types of graphs

I-h) Figure: A diagram using any of a variety of different types of figures

Several examples of these different types are shown below.

[An example of illustrative representation]

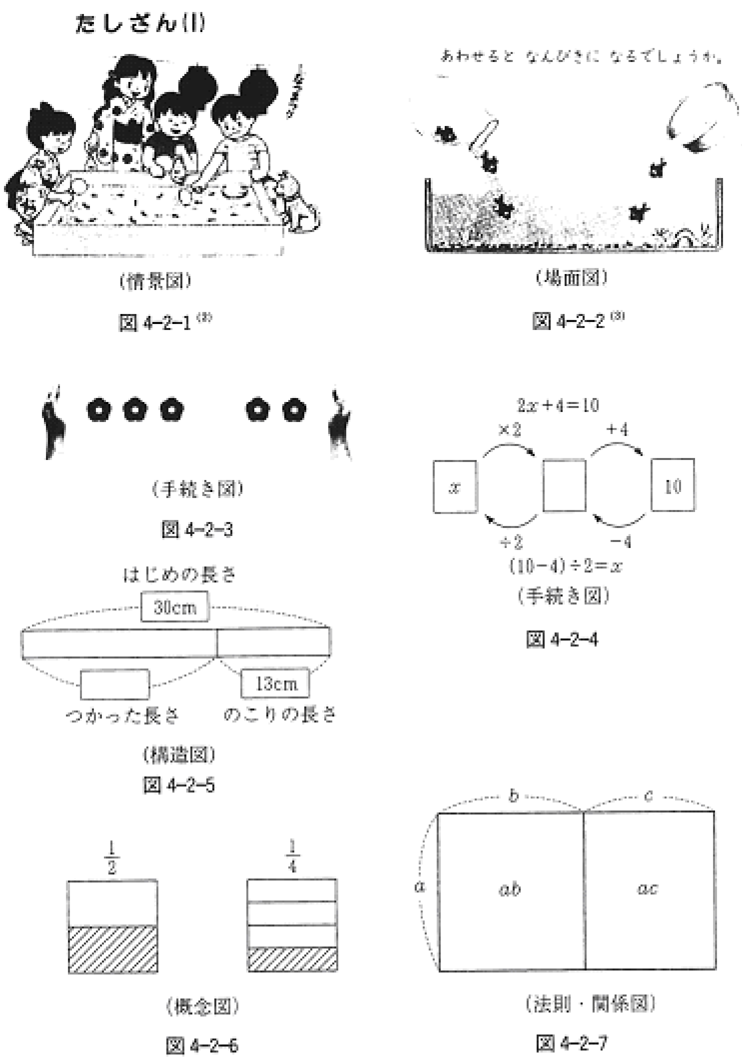

Figure6

As previously shown, “an analogical relationship with the referent, and both visual and intuitive richness” are important shared characteristics of these various types of illustrative representation. Illustrative representation uses these characteristics, which greatly contribute to the formation of images, to evoke images. Furthermore, illustrative representation is also characterized by the fact that it shows the entirety of the scene, relationships, and structure. These characteristics are referred to as “imaging” and “totality”. It is important to utilize illustrative representation in a way that takes advantage of these analogical, visual, intuitive, imaging, and totality characteristics.

Note also that the illustrative representation types described from I-a to I-h above are suitable for different roles and applications, as follows:

A. Situation and scene diagrams: Used to link realistic representations with lesson content.

B. Procedural and component diagrams: Used to show clues or methods for solving problems.

C. Conceptual, principle, and relationship diagrams: Used to form images of lesson content.

D. Graph and figure diagrams: Used to show graphs or figures.

Each different type of illustrative representation needs to be utilized in a manner that is fully based on these roles.

5. The utilization of representational systems

Finally, based on this discussion, I would like to consider the utilization of the aforementioned representational systems.

Many of the concepts and methods taught in arithmetic and mathematics classes arise from the solving of realistic problems. Therefore, it is advisable to start from here in the classroom. When a class is constructed, it is first necessary to utilize “realistic representation”. In the final scene of the class, the concepts or methods learned are usually represented with mathematical symbols. “Symbolic representation” is used at this point These two types of representation are connected with “manipulative representation,” “illustrative representation,” and “linguistic representation.” Therefore, when considered from the perspective of representational systems, the creation of a class starts with the application of realistic representations, and then apply manipulative representation, illustrative representation, and linguistic representation as appropriate based on the aforementioned principles. Finally, the class ends with the application and understanding of symbolic representation.

As an example of this, let us consider the lesson of “3+2” as shown in section 2. This class basically starts from the realistic representation shown there, and then uses manipulative representation to solve this, while showing the process with both illustrative representation and linguistic representation. Finally, the expression “3+2” is shown, and the class ends once this can be both understood and applied.

In this case, rather than simply proceeding in a straight line from bottom to top, it is also important to inject activities that express “3+4” with manipulative representation and realistic representation. This means the inclusion of the previously mentioned mutual translation between representational modes. These activities both deepen understanding and cultivate the ability to apply what is learned.

Children utilize a variety of different types of representation in these classes, while thinking for themselves and presenting the results. The utilization of representation draws out the children’s ideas and gives them a way to convey them to others. This process gives birth to mathematical thinking on the part of the children, while deepening understanding and cultivating the ability to solve problems.

Finally, I would like to introduce one result of this kind of class. The following chart organizes the various ideas of children during a class covering the question “which is larger, 3/4 or 4/5?”. Symbolic representation and illustrative representation were utilized in this class to draw out the children’s diverse ways of thinking.

[Bibliography omitted]

[The diverse thinking of children with respect to the question “which is larger, 3/4 or 4/5?”]

[The diverse thinking of children]

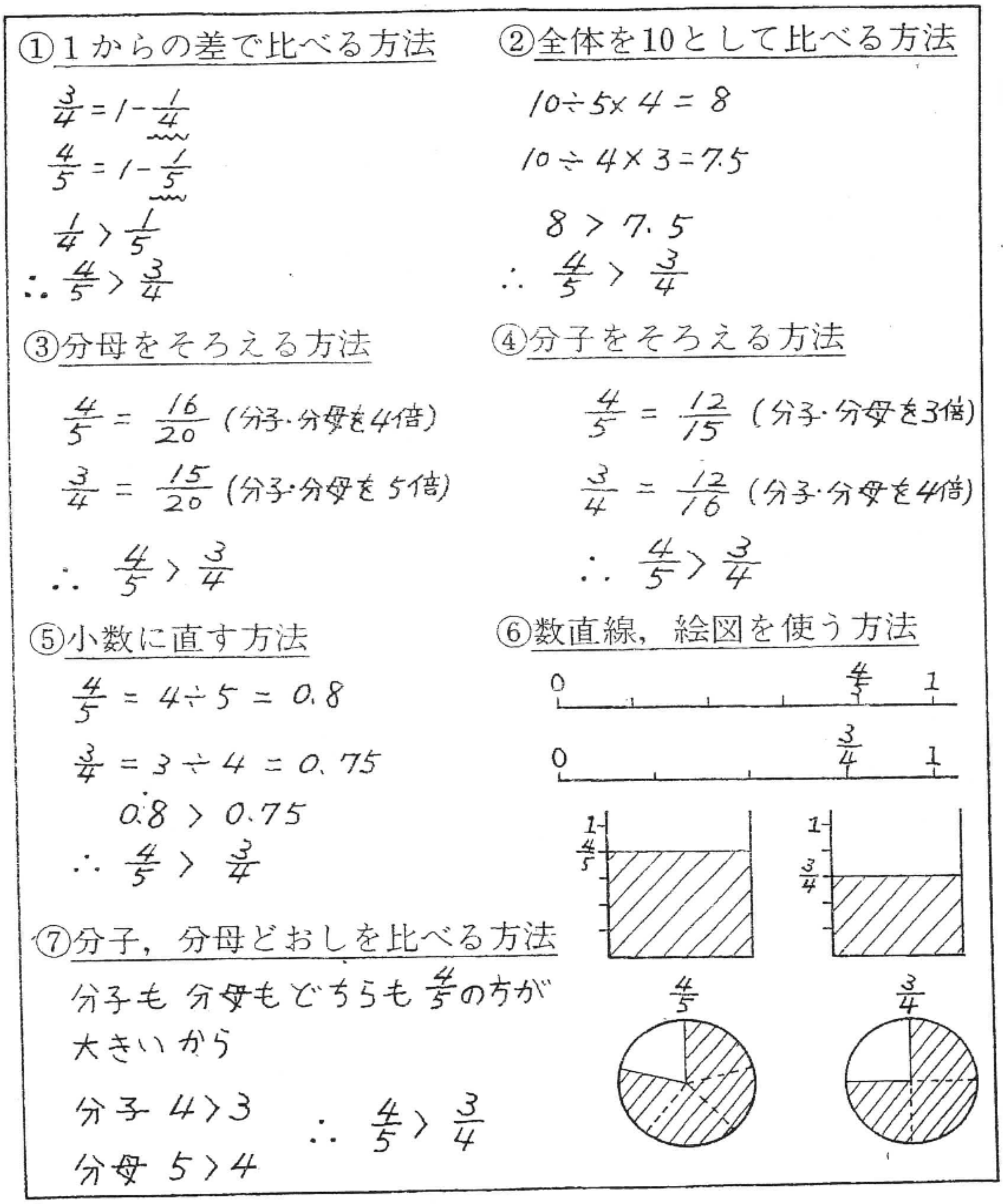

Figure 7