Emergent modeling and iterative processes of design and improvement in mathematics education

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Introduction: design and improvement

The Japanese lesson studies have become internationally famous. The core idea is that teachers design, try out, observe, analyze, and improve innovative lessons collectively. In this manner, lesson studies offer an alternative for the top-down approaches to innovations in education that proved to be problematic. Apart form the fact that teachers are the key agents in this approach, and therefore develop a strong sense of ownership, it is the iterative character of the lesson studies that make them potentially powerful. In respect to the latter, we may observe a striking similarity with the mathematical teaching cycle of Simon (1995). He developed this idea of a teaching cycle in conjunction with the notion of a hypothetical learning trajectory. Point of departure for him was the question of how to reconcile the constructivist stance that students construct their own knowledge with the obligation of formal education to strive for pre-given educational goals. Simon's solution to this problem is that teachers try to anticipate what mental activities the students will engage in when they participate in the envisioned instructional activities, and consider how those mental activities relate to the end goals one is aiming for. This is what he calls a hypothetical learning trajectory. The envisioned learning trajectory is hypothetical in that the actual learning trajectory may differ. The teacher, therefore, has to investigate whether the actual learning of the students corresponds with what was anticipated. This will lead to new understandings of the students' conceptions. These new insights, and the experience with the instructional activities as such will form the basis for the constitution of a modified hypothetical learning trajectory for the subsequent instructional activities. The alternation of anticipation, enactment, evaluation and revision creates a cyclic, iterative, process that shows a strong resemblance with the lesson study approach, in which lesson experiments are not seen as tests of preconceived designs, but function as learning situations for teachers, which may open up unanticipated avenues.

The hypothetical learning trajectory offers a basis for an elaboration of the lesson study approach from a socio-constructivist perspective. From this perspective, the task of the teacher is to help students to build on their own thinking while constructing more sophisticated mathematics. In this respect, one speaks of a transition from 'instruction' to 'construction'. What makes the latter difficult, is that the teacher can only indirectly influence what the students construct. This is exactly the problem that Simon tries to tackle with the hypothetical learning trajectory. By construing tasks or problem situations in which the students are expected to reason in a certain manner, the teacher tries to steer the constructing activity of the students. Designing long-term learning processes, however, is much more challenging than designing relatively short term learning trajectories. To reach given educational goals, one has to think of a more long term learning process. This is a very complicated task that surpasses the scope of what may be

expected of teachers. We may argue therefore that teachers should be offered a more general framework that enables them to design hypothetical learning trajectories on a day-to-day basis. Such a framework of reference may be offered by a so-called local instruction theory, which consists of a theory about a possible learning process, and the means of supporting that process. Such a theory is called local in that it is tailored to a given topic, such as addition of fractions, multiplication of decimals, or data analysis. Local instruction theories may be developed in design-research teaching experiments (Gravemeijer & Cobb, 2006), which are characterized by iterative trials and improvements similar to both lesson studies and Simon's (1995) approach.
The idea is that a local instruction theory offers a framework of reference for the teacher who is construing a hypothetical learning trajectory for his or her classroom at a given moment in time. To support teachers further, we may develop a set of exemplary instructional activities that the teacher may adopt and adapt. Mark, that the objective here is not to construe textbook like ready-made scripts, but to offer additional resources for teachers—which they can choose from, and to which they can make their own adaptations. That is to say, although the teacher may rely on theory and tasks developed by others, there is still room—and a need—for the teacher to construe his/her own hypothetical learning trajectories. For, tasks will have to be trimmed to the specific situation of this teacher with his/her own goals, with these students, at this moment in time. To make such decisions, the teacher has to construe hypothetical learning trajectories, but when doing so, the local instruction theory can be used as a framework of reference. The theory can be seen as a travel plan, while the teacher and the students make the journey (to borrow Simon's (1995) journey metaphor). Like a journey, a long-term teaching-learning process can be planned in advance, and in a similar manner, the actual teaching-learning process has to be constituted in interaction with the conditions and developments one encounters. In this sense, an externally developed local instruction theory can function as a 'travel plan' for the teacher, or better, informs the teacher's travel plan. Like a traveler, the teacher will have to adjust this plan continuously by construing hypothetical learning trajectories that fit his/her interpretations of, and choices in, the actual situation.

In this paper I want to elucidate the importance of the design of local instruction theories that aim at supporting students in constructing, or reinventing, mathematics on their own accord. In addition, I want to elaborate one of the key instructional design heuristics that may help designers/researchers to develop such local instruction theories. That is the 'emergent modeling' design heuristic that is part of the domain-specific instruction theory for realistic mathematics education (RME). Finally, I will address the relation between the design of local instruction theories and lesson studies, and I will argue that lesson studies can be cast as an extension of the design research that aims at developing local instruction theories. I will start by discussing the importance of supporting students in constructing, or reinventing, mathematics on their own accord. I will do so by first asking the question, what makes mathematics so difficult?

What makes mathematics so difficult?

In answering the question, 'What makes mathematics so difficult?' we may start by asking ourselves, 'How do people learn mathematics?' Here we may consider various sophisticated learning theories. But, if we limit ourselves to the general practice of learning mathematics in schools, it may be more useful to take our point of departure in popular notions of teaching and learning. In practice, learning is usually thought of as making connections between what is already known, and that what has to be learned. In the case of mathematics, the latter concerns an abstract, formal body of knowledge. I believe that it is this popular notion of *learning mathematics as making connections with an external body of knowledge* makes learning mathematics so difficult. In the following, I will first substantiate this claim; next I will describe an alternative that makes mathematics more accessible.

The learning paradox

If we think of learning as making connections with the external body knowledge that has to be acquired, the task for mathematics educators is to shape mathematics instruction that helps students in bridging the gap between the student's personal knowledge on the one hand, and the abstract formal mathematical knowledge on the other hand. However, it seems as if the gap between the formal mathematical knowledge and the personal knowledge of the students is too big.

To accommodate to this problem, curriculum developers try to devise tactile or visual models (often called 'manipulatives') that represent mathematical relationships and concepts to students in a readily apprehensible form. The underlying idea is that these external representations will facilitate the process of making connections with the represented mathematical relationships and concepts. In this respect the word 'transparent' is used, which suggests that students can look through the models and see the mathematics. This will enable the students to construct internal mental representations that mirror those embodied in the external representations.

The most well known manipulatives that fit this representational view are the Dienes blocks, or MAB materials, that are meant to concretize the decimal system (see fig. 1).

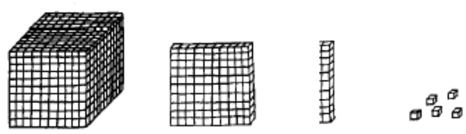


Figure 1. Dienes blocks.

Students are expected to see one big block as consisting of one thousand tiny cubes, a flat slice as one hundred, and a 'rod' as ten tiny blocks. Practice, however, shows that this interpretation is not self-evident (Labinowics, 1985).

Cobb, Yackel and Wood (1992) claim that the problematic character of the Dienes blocks is inherent to the assumption that instructional representations are the primary source of the students' mathematical knowledge. For us it is self-evident what these instructional representations signify, but this is not the case for most students. The plausibility of the usefulness of representations resides in the fact that we, as adult mathematics educators, experience mathematical constructs, such as, 'tens', 'ones', and 'hundreds' as object-like entities that can be pointed to and spoken about. An feeling that is not only due to our

individual mathematical sophistication, but also to our experience of being able to talk and reason about these ‘objects’ unproblematicly while interacting with others.
As a result of such experiences, one may start to entertain the notion of mathematics as an independent, objective body of knowledge. From a constructivist point of view, however, the assumption that objective knowledge exists independently of the act of knowing is highly problematic. Likewise, the idea that that objective knowledge can be accessed directly via external representations is equally problematic.

Teachers and instructional designers are experts who already comprehend the abstract mathematical knowledge that the students still have to acquire. From their perspective, it makes perfect sense to try to develop ‘transparent’ models that make the abstract mathematical knowledge apprehensible for students. They see their knowledge of the decimal system reflected in the blocks. For students, however, the Dienes blocks are nothing but just wooden blocks. We cannot expect the students to see more sophisticated mathematics in the blocks than the mathematics they already have acquired. This raises the question of *how* students are to learn abstract mathematics from concrete external representations. This problem is known as the *‘learning paradox’* (Bereiter, 1985), which Cobb et al. describe as:

> (T)he assumption that students will inevitably construct the correct internal representation from the materials presented implies that their learning is triggered by the mathematical relationships they are to construct before they have constructed them. (…)
> How then, if students can only make sense of their worlds in terms of their internal representations, is it possible for them to recognize mathematical relationships that are developmentally more advanced than their internal representations? (Cobb et al., 1992, p. 5)

The consequence is that when students do not see what there is to be seen, the teacher does not have many options, other than to spell out the correspondences between the blocks and the algorithm in detail. The result of that policy, however, will sooner be rote learning than understanding. This is exactly what happens with a conventional ‘mapping instruction’, in which actions with the blocks are mapped upon steps in execution of the written algorithm on paper, and vise versa. As might be expected from the lack of transparency of the blocks, mapping instruction does not lead to understanding or proficiency (Resnick & Omanson, 1987). Moreover, students develop al sorts of ‘buggy algorithms’ in trying to get a handle on procedures they do not understand (Brown & Van Lenn, 1982). Another effect of teaching about a body of knowledge that is not accessible for the students, is that they start to treat school mathematics and everyday-life reality as two disjunct worlds.

School math and student reality, two different worlds

We may illustrate this with an interview with a first-grader conducted by Cobb (1989). First, the student, Auburn (Grade 1), is presented with some addition tasks that are presented as numerical expressions:

16 + 9   =
28 + 13 =
37 + 24 =
39 + 53 =

In this part of the session, Auburn solves ‘16 + 9’ by counting on, and she arrives at the answer, ‘16 + 9   = 25’. Later, Auburn has to fill out a worksheet that contains the same task, now written In a column format (see fig. 2).

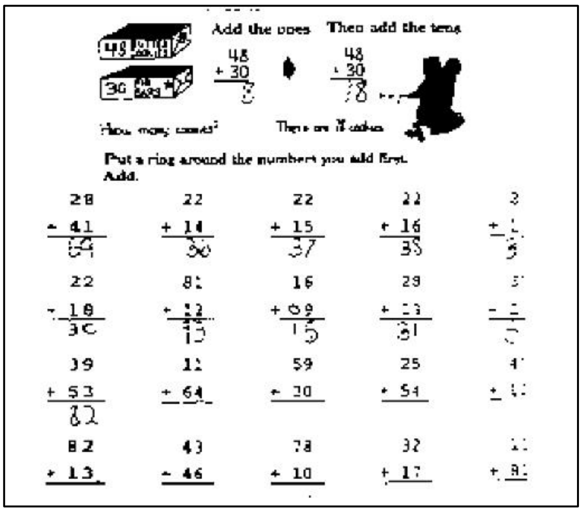


Figure 2. Aburn’s worksheet.

Auburn solves this problem in the following manner:

16
 9 +
15

This then constitutes the starting point for the following exchange between the interviewer (I), and Auburn (A).

I : Is that correct that there are two answers?
A : ?
I : Which do you think is the best?
A : 25
I : Why?
A : I don’t know.
I : If we had 16 cookies and another 9 added, would we have 15 altogether?
A : No.
I : Why not?

A : If you count them altogether you would get 25.
I : But this (15) is sometimes correct?
Or is it always wrong?
A : It is always correct.For us this answer may be highly surprising, but for Auburn, the mathematics of the worksheets seems belong to a different world, a world that appears to be disconnected from the world of everyday-life experience. One of the consequences is that Auburn will not be inclined to use everyday-life knowledge to make sense of 'school-math' problems. For her mathematics has its own set of arbitrary rules that you just have to accept on the authority of teachers and textbooks.

In the beginning of this paper, I mentioned that the difficulty of learning mathematics could lay in the formal, abstract, character of mathematics. We may conclude from the above, however, that the actual problem seems to be in level of sophistication of the mathematical knowledge of teachers (and textbook authors). The large difference between the abstract knowledge of the teachers and the experiential knowledge of the students causes a mismatch. Teachers and textbook authors (mis)take their own more abstract mathematical knowledge for an objective body of knowledge with which the students can make connections. However, the gap between the knowledge of the teachers and the knowledge of the students is too big to make this work. Instructional representations cannot bridge this gap, because, what those materials signify is in the eye of the beholder. Only the experts who know the mathematics can see the mathematics.

Different frameworks of reference

In this respect, we may quote Van Hiele (1973), who observed that teachers and students have different frameworks of reference, and as a consequence, may use the same words but with a different meaning.
Van Hiele takes the word 'rhombus' in geometry as an example. Some junior-high-school students will claim that a square is not a rhombus, unless maybe if it is tilted (see fig. 3).

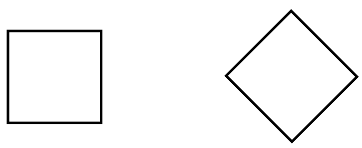


Figure 3. Is a square a rhombus?

For students who reason this way, the word rhombus signifies a figural shape. For the teacher, however, the word rhombus signifies a set of geometrical relations:

- the sides are two by two parallel,
- all sides have equal lengths,
- the diagonals intersect orthogonal,
- the facing angles are equal.

The teacher will accept a sloppy drawing of a quadrangle as a rhombus if he or she is told that all sides have equal lengths. On the basis of this, the teacher will conclude that the

diagonals intersect orthogonal, that the facing angles are equal, and that the sides are two by two parallel. For the students who connect the word rhombus with the figural image, however, such a line of reasoning does not make any sense.
This example illustrates Van Hiele's claim that teachers and students often speak different languages—without being aware of it. In fact, the teacher is talking about a mathematical reality that does not exist for students. The teacher thinks of a rhombus as a mathematical object that derives its meaning from a set of geometrical relations. But for students, who have not construed the necessary network of geometrical relations, there is no mathematical object to reason about.

This phenomenon may be illustrated with another example.
For us as adults, '1+1=2', shows common sense, but this may be very different for young children. At a certain age, young children do not understand the question: 'How much is 4+4?' Even though they may very well understand, that 4 apples and 4 apples equals 8 apples. The explanation for this phenomenon is that, for them, number is still tied to countable objects, like in 'four apples.'
At a higher level: 4 will be associated with various number relations, such as:

$4 = 2 + 2 = 3 + 1 = 5 - 1 = 8 : 2$, etc.

At this higher level, numbers have become mathematical objects that derive their meaning form a network of number relations (c.f. Van Hiele, 1973). When an elementary-school teacher is talking about numbers, he or she may very well be talking about mathematical objects that do not exist for students. So here again our everyday-life notion of teaching as helping students in making connections with new knowledge proves to be inadequate. How can students, for whom a number is a sort of adjective, make connections with numbers as mathematical objects?
I might add, that telling students that 2+2=4, etcetera, will not help if the students do not know what '2+2' means.

Learning as making connections, the source of the problem

In sum, we may conclude that the problem does lay in the fact that the common conception of learning, as making connections between the internal knowledge of the student and some external knowledge that has to be acquired, does not fit mathematics education. It causes teachers to try to force students to make connections with external knowledge that does not exist for them. In relation to this the observation of Davis and Hersh (1986) comes to mind, who describe of mathematicians who speak of esoteric mathematical constructs as if they are real objects, which are completely unimaginable for non-mathematicians. Apparently teachers and students live in two worlds, the world of the mathematics of the teacher, and the world of everyday life of the students.

Instruction in mathematics that is based on the popular notion of 'learning as making connections' apparently asks from students to make connections with a body of knowledge, which they cannot grasp. My conclusion, therefore, is that it is the tradition of trying to teach along these lines, which makes mathematics so difficult to learn. One might, of course, counter that reality shows that (at least some) people appear to have

learned mathematics in spite of this form of instruction. We may reason, however, that their actual learning process will have been different from making connections. We may conjecture that what those mathematics learners really did was construct their personal theories about the alien body of knowledge that was presented to them. Theories they revised and adjusted on the basis of experiences and feedback.
This kind of learning has serious drawbacks, however. In the first place, it is very difficult. The process generates many misconceptions that one has overcome. The second drawback is the inherent uncertainty, the learner is always guessing about what is really meant by the teacher or the textbook. Knowledge and understanding is always preliminary, until the next contradiction that will show that one's latest conjecture of what the body of knowledge entails is still off. A very likely consequence is math anxiety. Moreover, this lack of certainty, and always being dependent on the authority of teachers and textbooks, is in contradiction with the very nature of mathematics. Even if one develops some proficiency in this manner, we may ask ourselves if it is mathematics one has learned.

We may conclude that the popular notion of learning as making connections between what the learner already knows, and that what has to be learned, does not fit mathematics education. We may summarize the problems:

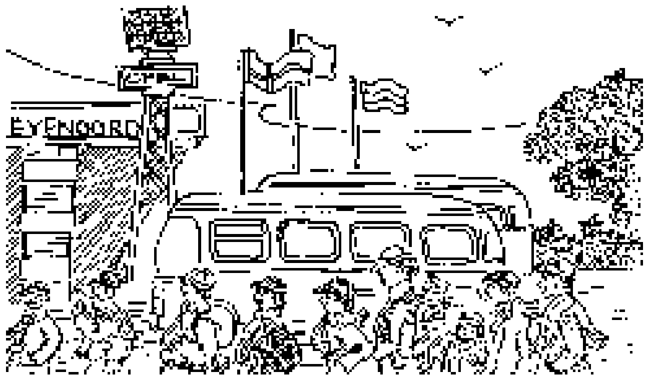
- First, there is the problematic character of body of knowledge, with which students have to make connections. For them, this body of knowledge does not exist, this knowledge only exist in the minds of teachers and textbook authors.
- Second, trying to represent objective, scientific, knowledge with 'transparent' instructional materials results in a learning paradox—how can students learn if they cannot see the mathematics, they do yet not know, in the materials?

Realistic mathematics education as an alternative
A different way of critiquing the instructional approaches discussed above, is by observing that the end product of the mathematical activity of many outstanding mathematicians is taken as a starting point for the instruction of young students. Freudenthal (1973, 1991) calls this an anti-didactical inversion. The alternative, he goes on to say, is to create opportunities for students reinvent mathematics. In relation to this, he speaks of 'mathematics as a human activity'. Just like the activity of mathematicians resulted in mathematics as we know it now, the activity of students can result in the construction of mathematics. This approach, therefore, offers an alternative for teaching students mathematics as a ready-made product.

Let me elaborate Freudenthal's point further. For him—as a mathematician—mathematics is primarily an activity. An activity that he denotes 'mathematizing' or organizing. In relation to this, he speaks of the activity of organizing subject matter to make it more mathematical. This may concern, both organizing matter from reality to make it accessible for mathematical means, and organizing mathematical matter to make it more mathematical. We may relate 'more mathematical' in this context to characteristics like general, exact, sure and brief, which suggests mathematical activities such as generalizing, formalizing, proving, and curtailing. Freudenthal (1973) argues, that students can reinvent mathematics by mathematizing, although he also acknowledges that the students cannot simply reinvent the mathematics that took the brightest mathematicians eons to develop. This is why he proposes *guided reinvention*. Teachers and textbooks have to help the students along, while trying to make sure, that the students experience learning mathematics as a process of inventing mathematics for themselves. In order to accomplish this, a reinvention route has to be developed. Therefore, teachers need the help of instructional designers, who in turn may be supported by researchers.

Designing reinvention routes exactly has been the mission of the Freudenthal Institute in the Netherlands, over the past decades. This has resulted in, what we call, a domain-specific instruction theory for realistic mathematics education (RME). RME theory is the result of generalizing over various local instruction theories, which describe how a certain topic can be taught in accordance with Freudenthal's idea of mathematics as a human activity.
In this paper I will focus on the RME design heuristic of emergent modeling. Before doing so, I will present a brief example to illustrate the idea of guided reinvention.

Long division
This idea of guided reinvention has proved itself productive in relation to the written algorithms. I will illustrate this by briefly describing how the long division can be reinvented. I will center this description around a paradigmatic problem about the transfer of supporters of a soccer club (fig. 4).



1128 supporters want to visit the away soccer game of Feijenoord.
The treasurer learns that one bus can carry 38 passengers and that
a reduction will be given for every ten buses.

Figure 4. Feijenoord

Basically the problem can be solved by repeated subtraction, each time a bus is filled with 38 people, you subtract 38 (see fig. 5).

$$\begin{array}{rl} 1296 & \\ \underline{38} & - \quad 1\text{ x} \\ 1258 & \end{array}$$

```
  38  -   1 x
1220
  38  -   1x
1182
  38  -   1 x
1244
  38  -   1x
  ….
```

Figure 5. Repeated subtraction

In addition to this, the information in the task, that a reduction will be given for every ten buses, may work as a suggestion to calculate the number of times you can cash in reductions. Finding out how many times you can fill ten buses, may call the students' attention to the opportunities offered by the decimal system. Even then various solutions are possible (fig. 6).

```
38/ 1296 \   34         38/ 1296 \   34        38/ 1296 \   34
     380 -   10x             380 -   10x           1140 -   30x
     916                     916                    156
     380 -   10x             760 -   20x            152 -   4x
     536                     156                      4
     380 -   10x              76 -   2x
     156                      80
      38 -   1x               76 -   2x
     118                       4
      38 -   1x
      80
      38 -   1x
      42
      38 -   1x
       4
```

Figure 6. Various levels of curtailment

Such leads on the way to the written algorithm are opportunities for students to make discoveries at their own level, to build on their own experiential knowledge and perform short-cuts at their own pace. Working with realistic problems also implies a meaningful approach to the problem of the remainder, i.e., as a real life phenomenon that calls for practical solutions, rather than as a peculiarity of non-terminating divisions. If the context is taken seriously, then '34 remainder 4' is not an acceptable answer. What can we do with these 4 supporters? Well, there are several possibilities, distribute them over the other buses, order an extra bus, or speculate on the withdrawal of at least 4 at the last moment.

Emergent Modeling

Closely connected to the RME principle of guided reinvention, is that of emergent modeling (Gravemeijer, 1999, 2004). With emergent modeling we can, as I said, circumvent the learning paradox. Earlier we discussed the learning paradox as one of the difficulties that arise within an instructional approach that treats the knowledge of experts as an independent body of knowledge, which can be appropriated by students by offering them concrete materials that embody that knowledge.

The emergent modeling approach is in line with Meira's (1995) proposal to circumvent the learning paradox. On the basis of an historical analysis, he suggest a dialectic process of symbolizing and meaning making, in which both the symbolizations and the corresponding meaning develop. Historically, symbols and models did not materialize out of thin air, they are the results of a long processes of inventing, adjusting and refining. So again the conclusion is that instead of trying to help students to make connections with ready-made mathematics, we should try to help students construe mathematics in a more bottom-up manner. This recommendation fits with the idea of emergent modeling. The emergent modeling approach takes its point of departure in the *activity of modeling*. Modeling in this conception is an activity of the students, who are asked to solve a contextual problem. Then the students model the problem, in order to solve it with help of that model. Such a modeling activity might involve making drawings, diagrams, or tables, or it could involve developing informal notations or using conventional mathematical notations. The conjecture is that acting with these models will help the students reinvent the more formal mathematics that is aimed for. So again, the alternative to making connections with a ready-made mathematics, is shaped as an activity of doing mathematics, which is put into service of developing mathematics.

Initially, the models come to the fore as context-specific models. The models refer to concrete or paradigmatic situations, which are experientially real for the students. On this level the model should allow for informal strategies that correspond with situated solution strategies at the level of the situation that is defined in the contextual problem.

From then on, the role of the model begins to change. Then, while the students gather more experience with similar problems, their attention may shift towards the mathematical relations and strategies. As a consequence, the model gets a more object-like character, and becomes more important as a base for mathematical reasoning, than as a way of representing a contextual problem. In this manner, the model starts to become a referential base for the level of formal mathematics. Or in short: a *model of* informal mathematical activity develops into a *model for* more formal mathematical reasoning.

Model-of/model-for

In contrast with the gap metaphor, formal mathematics is not seen as something separate, existing independent of a knowing agent. Instead, formal mathematics is seen as emerging alongside with the model-of /model-for transition. When speaking of formal mathematics, we hasten to say that in RME, formal mathematics is not seen as something 'out there'. Instead, formal mathematics is seen as something that grows out of the students' activity. For us, the notion of 'abstraction' is tied to a progression from informal to more formal mathematical reasoning, which in turn is tied to the creation of new mathematical reality. So instead of 'cutting bonds with (everyday-life) reality', we want to stress 'construction'. Informal, situated knowledge is the basis upon which more formal, abstract mathematical knowledge is build.

Our claim is that the emergent-modeling design heuristic helps instructional designers in developing topic-specific instruction theories and corresponding instructional activities that support learning processes in which students construe new mathematical reality. In order to clarify the emergent modeling heuristic, we will briefly describe two exemplary instructional sequences.

Addition and subtraction as an example

This exemplary sequence, which concerns linear measurement and flexible arithmetic, was developed in connection with a teaching experiment carried out at Vanderbilt University (Stephan, Bowers, Cobb, & Gravemeijer, 2004); Stephan, 1998). The underlying idea is that measuring by iterating measurement units can give rise to the construal of a ruler and that the ruler can subsequently support arithmetical reasoning about problems concerning incrementing, decrementing and comparing measures.

After a series of preparatory activities, the students start measuring with stacks of ten unifix cubes. They first iterate units of ten, then adjust by adding or subtracting ones. In this manner, measuring with tens and ones helps the students in structuring the number sequence up to 100. Next, the students create their own paper strip that is ten unifix cubes long. With that, a basis is being laid for the construction of a measurement strip that comprises ten units of ten; each subdivided into ten units of one cube. The idea is that, thanks to the learning history, measuring with the measurement strip is grounded in the imagery of measuring with units of ten and one. Thus, for the students, measuring with the strip signifies iterating a unit of ten cubes and a unit of one cube. Next, a shift is made from actually measuring items to reasoning about lengths when solving tasks around incrementing, decrementing and comparing lengths of objects that are not physically present (i.e. comparing the measures of the heights of sunflowers in the context of a sunflower contest). These tasks offer opportunities for developing solution methods based on curtailed counting—using the decimal structure as a framework of reference. Numbers close to a decuple, for instance, can be identified by using that decuple as a referent, e.g. 48 = 40 + 8 = 30 + 18 = 50 – 2. These relations can be exploited when analyzing patterns that correspond with jumps of 10.

In this process the ruler is initially used for measuring, and later as a means of support for arithmetical reasoning. For instance, when students are presented with problems such as the following:

We have two planks, one of 48 cm and one of 75 cm.

- How much is the difference?

The students may, of course, use the ruler to count individual units. However, the ruler may also be used as a basis for arithmetical reasoning. A self-evident solution would be to look at the ruler and reason:

48+2=50; 50+10=60; 60+10=70; 70+5=75, so the difference is 2+10+10+5=27.

Another solution might be:

48+20=68; 68+7=75, so the difference is 27.

As a next step, such solutions procedures can be modeled with an empty number line (fig.7).

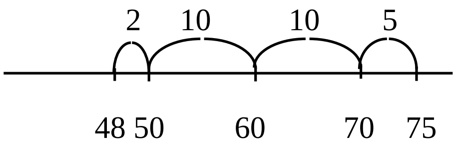


Figure 7. Counting by jumps on the number line

Mark that the use of the empty number line does not only fit these strategies, but also supports them. The number line supports the execution of counting-type methods, by offering a way of scaffolding—to keep track of both partial calculations and partial results. In this way, students adapt the model to their thinking.
Later, the empty number line will be used to depict more sophisticated strategies. Looking at the numbers in the above problem, a student might think of 75-50=25 as a nice familiar number relation. This student might recast the problem in terms of a subtraction task: 75-48=…, which could be solved via 75-50=25; 25+2=27. When justifying his or her strategy, this student might use the number line to show that 'minus 48' equals 'minus 50 plus 2' (see fig. 8).

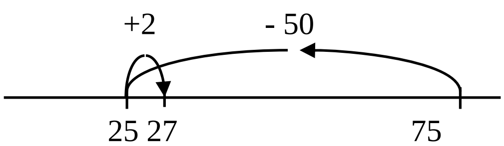


Figure 8. Compensating

We may note that in the latter case, the number line plays a different role than in the earlier cases. Now the number line is used to support the student's reasoning about number relations. Earlier the jumps on the number line were used to model the solution of a contextual problem. This difference is central to the model-of/model-for shift. Let me elaborate this point. Initially, the focus of the students is on the relation between the context problem and the number line. Later the numerical/mathematical relations become more important. In the first stage, the jumps on number line can be explained in terms of the problem situation by (fig. 7). Later on students will start to use the number line to support their reasoning about number relations (fig. 8).

In short, the shift from model/of to model/for concurres with a shift from thinking about the modeled context situation, to thinking about mathematical relations. In the latter phase, number relations give meaning to the use of the number line. In relation to this, we can discern two different types of activity:

(a) *referential activity*, in which acting with the model derives its meaning from activity in the setting described in instructional activities
(b) *general activity*, in which acting with the model derives its meaning from the mathematical relations involved.

These general types of activity can be seen as different levels of activity, which can be completed with a level of activity in the task setting itself at the one hand, and a level of more formal mathematical activity where the student no longer needs a model on the other hand. Together, we can elaborate the model-of/model-for distinction by identifying four general types of activity (Gravemeijer, 1994), as shown in Figure 9.

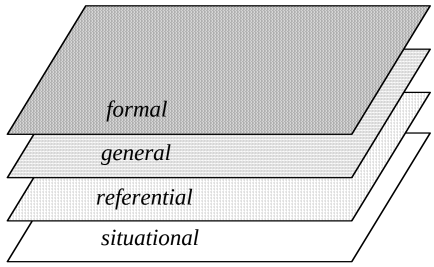


Figure 9. Levels

(1) *activity in the task setting*, in which interpretations and solutions depend on understanding of how to act in the setting
(2) *referential activity*, in which models-of refer to activity in the setting described in instructional activities
(3) *general activity*, in which models-for refer to a framework of mathematical relations
(4) *formal mathematical reasoning* which is no longer dependent on the support of models-for mathematical activity.

At the referential level, models are grounded in students' understandings of experientially-real settings. General activity begins to emerge as the students begin to focus on the mathematical relations involved. Then their reasoning loses its dependency on situation-specific imagery, and the role of models gradually changes as they take on a life of their own.

A crucial aspect of the emergent-models heuristic is that the shift from model-of to model-for is reflexively related with the creation of some mathematical reality. What is expected is, that in the course of the sequence, a shift is taking place in what the numbers signify for the students. Initially, numbers refer to distances, later, numbers start to signify mathematical objects. This shift involves a transition from viewing numbers as tied to identifiable objects or units (i. c. numbers as constituents of magnitudes; '48 centimeter') to viewing numbers as entities on their own ('48'). For the student, a number viewed as a mathematical object still has quantitative meaning, but this meaning is no longer dependent upon its connection with identifiable distances, or with specified countable objects. In the student's experienced world, numbers viewed as mathematical objects derive their meaning from their place in a network of number relations. Such a network may include relations such as 48 = 40 + 8 = 30 + 18 = 50 – 2. The critical aspect of this network is that the students' understanding of these relations transcends individual cases. That is, when students form notions of mathematical objects, they come to view relations like the above as holding for *any* quantity of 48 objects (including a magnitude of 48 units).

This shift from numbers as referents to numbers as mathematical objects is reflexively related to the model *of* to model *for* transition described earlier. On the one hand, the students' actions with 'the model' foster the constitution of a framework of number relations. On the other hand, through the students' development of this framework of number relations, 'the model' *can* take its role as a model for mathematical reasoning. What is aimed for is, that the students come to experience the framework of mathematical relations and the corresponding mathematical objects as some new mathematical reality. This experienced reality corresponds with the perceived body of mathematical knowledge that we identified as the central problem in the 'learning as making connections approach'. This shows the value of the emergent modeling approach: Instead of trying to help students to make connections with a mathematical reality that does not exists for them, the emergent modeling approach helps students in constructing such a mathematical reality by themselves.
In conclusion, we may observe that there are three parallel processes, one involves the model-of/model-for transition, the second concerns the constitution some mathematical objects and a framework of mathematical relations, or some new mathematical reality, the third encompass the series of sub-models or tools that are the concrete correlates of the model. I will discuss how an orientation on those three processes may help the instructional designer by elaborating upon a teaching experiment on data analysis carried out by Cobb, Gravemeijer, McClain and Konold in a 7th-grade classroom in Nashville (USA).

Data analysis as an example
Looking at the three processes mentioned above, we may conclude that the instructional designer will have to consider the choice of 'the model', the new mathematical reality the students are to construe, and the series of symbolizations that will instantiate the model in the concrete instructional activities. We will start the discussion of the instructional sequence that we used in the teaching experiment on data analysis by considering the goals of the instructional sequence. We begin by asking ourselves: *What constitutes the new mathematical reality we want the students to construe, and what are the mathematical relations involved?* Next, we ask ourselves: *What is the overarching model, and what do the underlying sub-models consist of?*

Our answer to the first question is that, what is to be construed as new mathematical reality by the students, may be denoted as 'distribution-as-an-entity'. We want the students to come to view data sets as entities that are distributed within a space of possible values rather than a plurality of values (Hancock, Kaput, & Goldsmith, 1992; Wilensky, 1997). Another argument for choosing distribution as a central statistical idea is that in conventional statistics courses, statistical measures like mean, mode, median, spread, quartiles, (relative) frequency, regression, and correlation are taught as a set of independent definitions. In contrast, these statistical measures come to the fore as characteristics of distributions in the conjectured learning trajectory. Likewise, conventional representations like histogram, and box plot come to the fore as means to characterize distributions. To clarify what we mean by distribution as a learning goal, we may start by observing that for us, distribution is intimately connected with the idea of 'shape'. Like when we colloquially speak of the 'bell shape' of a normal distribution. However, the imagery of a bell-shaped distribution, entails more than a mere figural inscription. In relation to this, we may ask ourselves, what does the curve of figure 10 represent?

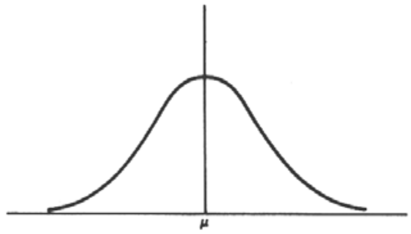


Fig. 10. Bell curve.

At first sight, the height of a point of the curve seems to signify the number of cases that have a measure equal to the corresponding measurement value. However, such a point does not have any width, thus we are working with an endless precision. And there will probably not be a single case with exactly that measure. Actually, the graph has to be viewed as an idealization, or as the limit of a series of (relative frequency) histograms, with their interval widths approaching zero. We believe that approaching distribution from this angle would be far to demanding for 7th-grade students. However, another way to think about such a graph is as a density function. We believe that this offers a way into a qualitative understanding of distribution. In this conceptualization, the height of a point on the graph does not signify a number of cases, but the density of data points around that value. From this perspective, distribution can be thought of in terms of shape and density. Shape and density in turn can be seen as means to organize collections of data points in a space of possible data values. In relation to this, we may mark that Hancock, Kaput, & Goldsmith (1992) found that students tend to see data as attributes of individuals, which implies that students will have to reorganize their thinking to be able to see data as possible values of a variable.

In summary, we may conclude that important mathematical relations concern: shape, density, variable, and data points in a space of possible values. These are the relations that will have to constitute the network of mathematical relations that will be instrumental in the transition from the level of referential activity to the level of general activity. This implies that in order to support this transition, those mathematical relations have to become a topic for discussion in the classroom. To this we may add the issue of multiplicative reasoning, firstly since that is implied by the notions of shape and density, and secondly since it will come to play when comparing data sets of unequal size.

Emergent model and symbolizations

In answer to the second question, we may describe the overarching model as ‘a graphical representation of the shape of a distribution’. Given the tight connection between distribution and shape, it seems self-explanatory that the overarching model is tied to shape. With ‘a graphical representation of the shape of a distribution’ we, of course, do not mean just the figural inscription itself, but also what we hope it will signify for the student. The most common graph of the shape of a distribution is the graph of a density function we discussed earlier. However, we may also think of histograms, box plots, or stem-and-leaf diagrams. To find the graph to start the sequence with, we have looked for a graph that would most closely match an intuitive image of a set of measures. It should be a graph that the students could, in principle, invent themselves. In relation to this, the notion of a scale line came to mind. Especially measures of a linear type, like ‘length’, and ‘time’ are often represented by scale lines. These considerations let to the choice of a graph that consists of value bars, which each value bar signifying a single measure (fig. 11).

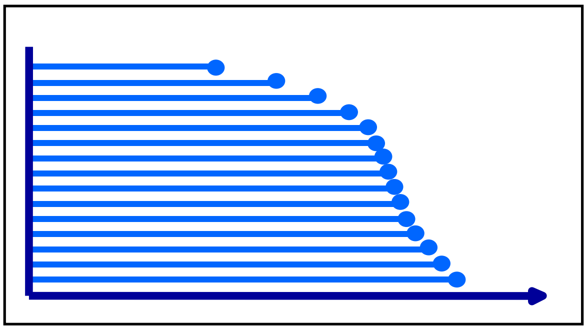


Figure 11. Value-bar graph.

Next we looked for a type of graph that might function as a transition stage between the magnitude-value-bar graph and the graph of a density function. Here we chose a dot plot (fig. 12).

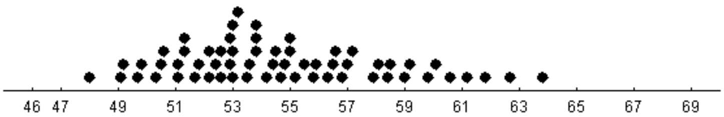


Figure 12. Dot plot.

Within a dot plot, the density of the data points in a given region translates itself in the way the dots are stacked. Consequently, the height of the stacked dots at a given position can be interpreted as a measure for the density at that position. In this sense, the visual shape of the dot plot can be seen as a qualitative precursor to the graph of a density function. On the other hand, the dot plot can be seen as a more condense form of a magnitude-value-bar graph that leaves out the value bars and only keeps the end points.

The aforementioned graphs are embedded in computer (mini)tools that can be used for exploratory data analysis on an elementary level. Point of departure, is a bottom-up approach in which the minitools are perceived by the students as tools that are compatible with their conception of analyzing data, and are experienced as sensible tools in that regard. So for the students the primary function of the minitools is to help them structure and describe data sets in order to make a decision or judgment.
The first minitool displays individual data values as value bars as shown in figure 11. With this minitool, one can display two or three data sets. The minitool has various

options, like sorting and partitioning data, which the students can use to describe and compare data sets.
The second minitool displays individual data values as dots in a dot plot. This minitool can display two data sets at a time, and various tool options are available, to help the student structure the distribution of data points in a dot plot. These options include: making your own groups, making four equal groups, making groups of a certain size, and making equal intervals.

To close, I want to take some instructional activities from the Nashville teaching experiment to explicate what the model-of /model-for transition entail in this sequence.
One of the first tasks of the sequence concerns the comparison the life spans of to brands of batteries, Though Cell and Always Ready. The students do not actually measure life spans, instead the teacher and students talk through the process of data creation. The teacher presents the context of writing about batteries in a consumer report. In relation to this the teacher and students discuss what important characteristics of batteries are. Then, when they have decided on life span as an important characteristic, they discuss how the life span of a battery could be measured. And finally the results of such a measurement for ten Though Cell batteries and ten Always Ready batteries are presented as value bars in the first minitool (fig. 13).

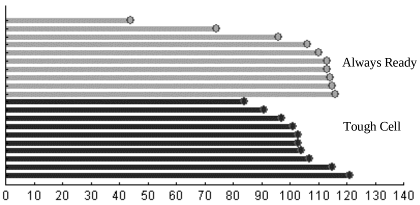


Figure 13. The life span of two brands of batteries.

The students introduced the term ‘consistency’ to argue that they ‘would rather have a consistent battery (...) than one that you just have to try to guess’. We may interpret this argument as referring to the shape of the distribution, which is visible in the way the endpoints of the value bars are distributed in regard to the axis. In relation to this, we can speak of *a graphical representation of the distribution* as a *model of* a set of measures.

Eventually the students used the four-equal-groups display of the second minitool to reason about shape and density (see fig. 14).

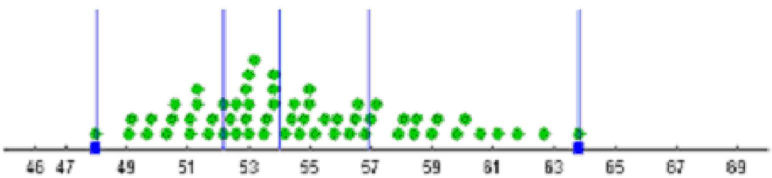


Figure 14. Dot plot structured into four equal groups.

The distance between two vertical bars that mark a quartile can were interpreted as indicating how much the data are ‘bunched up’. Moreover, the median started to function as an indicator of ‘where the hill is’. Finally, the students started to treat distributions as entities. In this regard, we may describe the four-equal groups display as *a graphical representation of the distribution* that started to function as a *model for* a model for reasoning about distributions.

Conclusion

In the above I have tried to show the importance of supporting students in constructing, or reinventing, mathematics on their own accord. I have argued that teachers need the help of local instruction theories to do so. In relation to this I have elaborated one of the key instructional design heuristics that may help designers/researchers to develop such local instruction theories. The emergent modeling heuristic may guide instructional designers/researchers by asking them to

- think through the endpoints of a given instructional sequence in terms of new mathematical reality; to describe what mathematical objects the students are expected to construe, and to explicate the corresponding framework of mathematical relations
- think through the model-of/model-for transition, which for instance means, to consider what informal situated activity is being modeled, and what a potential chain-of-signification might look like.

In connection with the above, the heuristic suggests points of attention for the enactment of the instructional sequence. It highlights that formalizing is not equal to, and cannot be forced by, the use of formal notations. Instead formalizing grows out of a shift of attention towards mathematical relations. The aforementioned considerations will indicate what those relations are, what the mathematical issues are that are to become topics of discussion, and what role the various tools/symbolizations may play. In relation to the latter, I want to point to the central principle that new (sub-)models are to derive their meaning from earlier experiences with earlier (sub-)models.

All in all we may conclude that local instruction theories can only offer a general framework. The teachers will have to respond to the students’ thinking, they have to decide, for instance, which mathematical relations students start to grasp, and which are still to be worked on. Teachers will also have to judge when a new sub-model might be introduced, and check whether that new (sub-)model is experienced as ‘bottom up’, which means that it signifies earlier activities with earlier (sub-)models for the students. In relation to this, I want to close by suggesting that a combination of design research on local instruction theories, and lesson studies that build on those theories might offer a powerful combination for improving mathematics education. In this manner, lesson studies may contribute to the ecological validity of a local instruction theory, as the lesson studies may be considered as follow-up trials with a range of participants in a variety of settings.

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