FRACTION NOTATION AND TEXTBOOKS IN AUSTRALIA

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Common fractions are frequently introduced to students in Australia through contexts such as sharing food. Shading partitions of common shapes such as circles and squares follows discussions of what constitutes 'half an apple or a quarter of a sandwich'. These partitions of common shapes are described as regional models of fractions. Although textbooks might use regional models to introduce fractions, some students attend to the discrete, countable features of the area models. This can lead to the intended continuous 'parts of a whole' fraction embodiment being interpreted as countable objects. Further, the underpinning idea of area as a quantifiable attribute is frequently not taught before students are expected to make area comparisons through the interpretation of regional models. The standard fraction notation itself encourages a 'count' interpretation of the regional 'parts of a whole' model. When the iterated unit fraction exceeds the whole, the name attributed to the parts can change for some students.

WHAT IS MEANT BY A FRACTION?

Learning the meaning of common fractions and how to operate with them is a traditionally difficult aspect of learning mathematics. The symbol system used to represent fractions, one whole number written above another whole number, $\frac{a}{b}$ where a and b are integers and $b \neq 0$, is not transparent to the meaning of fractions. Moreover, Kieren (1976) argued that from the point of view of curriculum, there was more than a single interpretation of fractions. The seven interpretations proposed by Kieren have been refined by the Rational Number Project (Behr, Harel, Post, & Lesh, 1992; Behr, Post, Silver, & Mierkiewicz, 1980; Behr, Wachsmuth, Post, & Lesh, 1984) to produce five subconstructs of rational number — part-whole, quotient, ratio number, operator, and measure. Olive (1999) cites Nesher's (1985) analysis as the basis of the added fifth subconstruct, part-whole relations. Indeed, Kieren initially used the term whole-part relationships as a description of all rational numbers.

More recently there has been a focus on describing schemes that have proven useful in supporting children's development of fraction based reasoning (Hackenberg & Tillema, 2009; Norton, 2008; Olive, 1999; Steffe, 2002, 2003, 2004; Steffe & Olive, 2010; Tzur, 2004). The various schemes have been proposed as models of students' cognitive structures. Central to these schemes is the way that students operate with units and coordinate units in giving meaning to fractional quantities (Hackenberg, 2007; Norton & Wilkins, 2009; Olive & Vomvoridi, 2006; Watanabe, 1995). The schemes used to characterise students' thinking include the simultaneous partitioning scheme, the part-whole scheme, the equi-partitioning scheme, the partitive fractional scheme, the reversible partitive fractional scheme and the iterative fractional scheme (Norton, 2008).

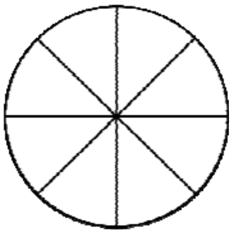
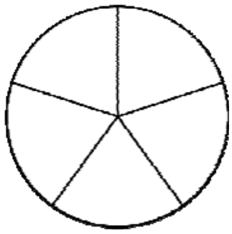
In practice, fractions exist in essentially two forms: embodied representations of comparisons, sometimes called partitioned fractions, and mathematical objects, also known as quantity fractions. A *partitioned fraction* (Isoda, Stephens, Ohara, & Miyakawa, 2007; K. Yoshida, 2004) can be described as the fraction formed when partitioning objects into *b* equal parts and selecting *a* out of *b* parts to arrive at the partitioned fraction *a/b*. A partitioned fraction can be of either discrete or continuous objects but **a partitioned fraction is always a fraction of something**. By comparison, *quantity fractions* express fractional quantities and refer to a universal measurement unit, similar to the way that metres can operate as a standard measurement unit. Asking the question, which is larger, one-half or three-eighths, only makes sense if the question is one of quantity fractions, as you need to know which unit is being referenced. Abstract quantity fractions reference a universal unit, a unique unit-whole, which is independent of any situation. If one-half and three-eighths as mathematical objects do not refer to a universal whole, we cannot compare them.

The transition from partitioned fractions to quantity fractions has not been made explicit for many students learning fractions. It is difficult for students to become aware of a unit-whole when the unit-whole is often implicit in everyday situations involving fractions. To make the transition from partitioned fractions to quantity fractions, students need to develop a sense of the size of fractions. Despite the fundamental value of developing a sense of the size of fractions, it does not appear to be specifically taught or learnt. Studies in several countries (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Hart, 1989; Kerslake, 1986; Ni & Zhou, 2005; H. Yoshida & Kuriyama, 1995) suggest that the underpinning knowledge of 'fractions as mathematical objects' (quantity fractions) is frequently absent from students' concepts.

FRACTIONS IN TEXTBOOKS

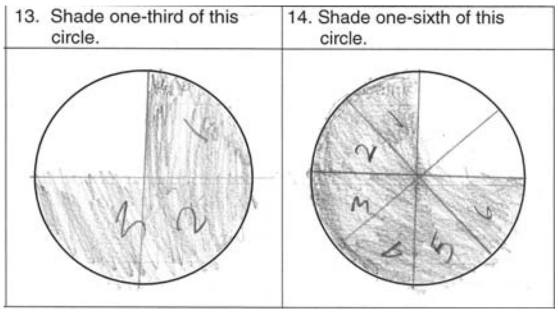
Mathematics textbooks in Australia are developed in an 'open market' and can vary in the level of cognitive challenge provided (Vincent & Stacey, 2008). That is, any person is free to write a textbook and the publishers market the books to schools. These textbooks typically present regional fraction models using pre-partitioned shapes and associate standard fraction notation with these models.

Traditionally, problem analysis of mathematical textbooks looks at three domains: mathematical features, contextual features and performance requirements (Li, 2000; Stigler, Fuson, Ham, & Kim, 1986). The following problems from popular Australian textbooks would be described as requiring a single step and using an illustrative context. The performance requirement could be characterised as a representation. However, rather than having a cognitive requirement of conceptual understanding, I would argue that these tasks require a form of procedural knowledge.

Colour the correct number of equal parts. Colour 	Shade four-fifths ( ) of the following shape. 

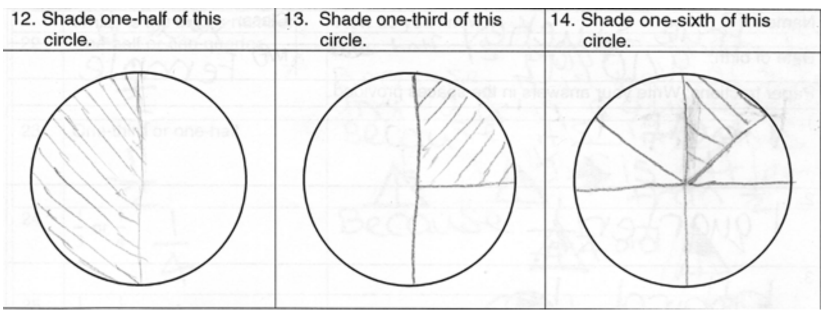
Finding out what students think

When students from Grades 4 to 8 were asked to shade different fractional parts of a circle that had not been partitioned, a wide range of responses and interpretations became evident. For example, the student whose response is shown below has made three equal parts to represent one-third and six equal parts to represent one-sixth. That is, the number of parts corresponding to the denominator appears as the dominant feature of this student's representation of the fraction.



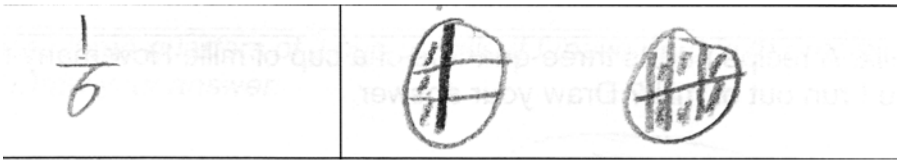
A Year 5 student showing one-third as 3 equal parts and one-sixth as 6 equal parts.

For some students, fractions appear to be defined solely by the number of parts without attention to the equality of all of the parts. In the following example, the student has represented fractional quantities as the number of parts out of the total number of parts. This is not a comparison of areas but rather a comparison of the number of parts.



A Year 6 student's representation of fractions as a number of parts.

Students can also interpret what may appear to be area models as discrete models. In the response below it is clear that the student is attending to the number of parts when drawing subdivided shapes, rather than the relative area of the parts. In determining which is larger, or , the diagram used to represent the reasoning behind the answer of shows circles with three and six parts marked for one-third and one-sixth respectively.

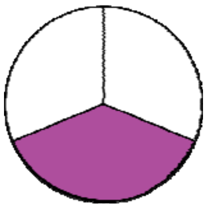


Representing a number of parts rather than the area of the parts.

This representation was clearly about the number of parts rather than the area of the parts. That is, subdivided regional models can be taken as discrete representations of fractions by some students.

Using regional models before students have learnt area

Although textbooks in Australia commonly use regional models to introduce fractions, some students attend to the discrete, countable features of the area models. This can lead to the intended continuous 'parts of a whole' fraction embodiment being interpreted as countable objects. Moreover, the underpinning idea of area as a quantifiable attribute is frequently not taught before students are expected to make area comparisons through the interpretation of regional models. Visually it is very difficult to interpret subdivided circles as indications of 'parts of a whole' fractions based on area.



Interpreting three-eighths of the area of the circle as one-third of the area.

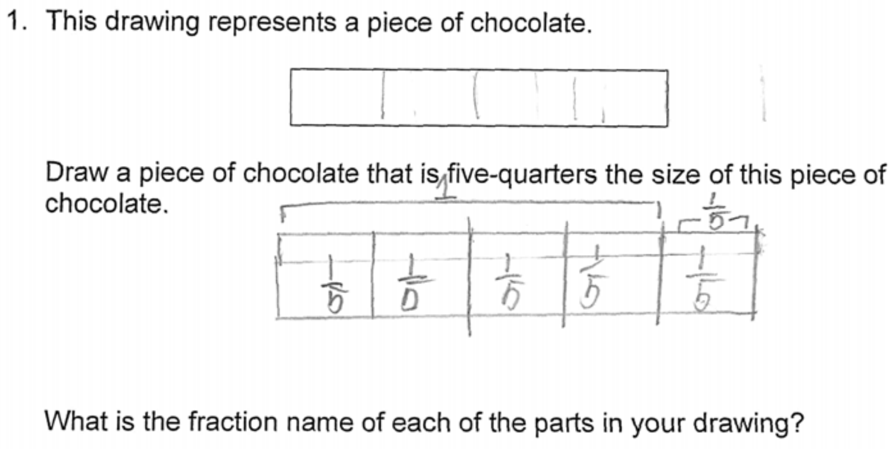
Most adults would interpret the shaded sector of the circle shown above as representing one-third. Visually, using angles to compare data is very difficult and using area is even harder (Cleveland, 1994). In dealing with regional representations of fractions using parts of circles, the parts aren't simply repeated and translated. Rather, an image of a sector needs to be repeated and rotated, and visually rotating an image is very difficult. Consequently, using parts of a circle to represent fractions in textbooks is a futile activity. The part-to-whole comparison of area or angle is not achievable visually.

The standard fraction notation itself encourages a 'count' interpretation of the regional 'parts of a whole' model. Fractions need to be taught in a way that enables students to be aware of the nature of the unit whole and the relationship between sub-units and the whole. As well as using counter-examples to limit the number of unintended features of models students associate with fractions, comparison of length rather than area should be used to introduce fractions. The use of a continuous linear quantity to introduce the fraction concept would emphasise the measurement property as distinct from discrete counts. I agree with Ball (1993) when she asserts, "We need more theoretical and empirical research on representations in teaching particular mathematical content… We need to map out conceptually and study empirically what students might learn from their interactions with [representations]" (p. 190).

Fractions beyond the whole

Students also need opportunities to move beyond the unit-whole to reorganise fractional units in a way that supports working with related units at three levels. Hackenberg (2007) has elaborated on this problem by describing the construction of improper fractions as requiring the interiorisation of three levels of units. For example, conceiving of as an improper fraction means conceiving of it as a unit of 4 units, any of which can be iterated 3 times to produce another unit (the whole), producing a three-levels-of-units structure. Further, Hackenberg states "iteration of a unit or proper fractional amount to produce a fraction greater than one does not necessitate that the child has constructed a structural relationship between the part being iterated, the whole, and the result" (p. 28).

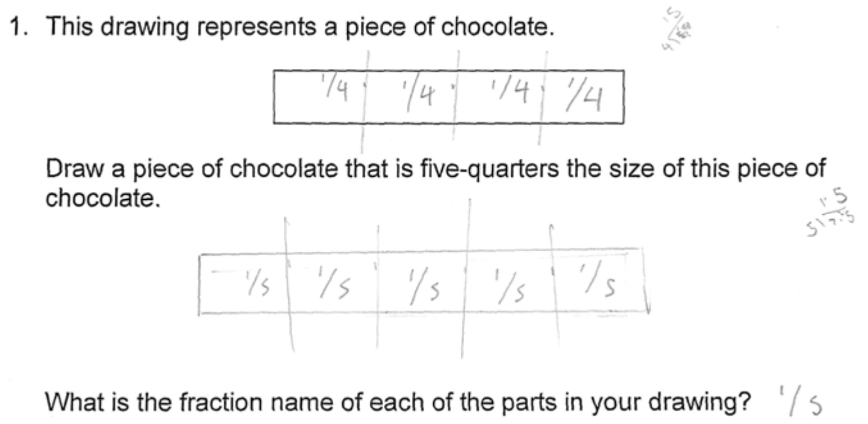
Hackenberg's detailed analysis of four sixth-grade students work with improper fractions (i.e. involving three levels of units) was based on a year-long teaching experiment. One of the tasks analysed involved asking the students to draw seven-fifths of a candy bar if the drawing of the rectangle on their paper represented a candy bar. Although both girls correctly created seven-fifths of the rectangle, when asked about the size of the pieces in the bars they had drawn, the girls maintained that the pieces were sevenths. The same confusion over the names of the fractional parts for improper fractions was noted in a Year 4 class in NSW in 2009 using a paper and pencil task.



Four-fifths representing one whole.

This student's response shows that the initial rectangle has been partitioned into quarters, the size of the quarters effectively replicated in producing the five-quarters requested, with the names of the fractional parts changing to fifths. The size relationships of the fractional parts appear to be maintained but the linguistic tags appears to have changed. What used to be fourths (with four of them still clearly forming the whole) have been given a new name when the quantity exceeds one whole.

The following student's response appears to make use of measurement and division. The initial rectangle was 60 mm long and above the rectangle a division showing 60 divided by 4 can be seen. The resulting length of the rectangle (7.5 cm) is also divided by 5 to verify that the new fractional parts are indeed 1.5 cm long.



Measuring and dividing the unit but renaming the fraction parts

Thinking quantitatively about fractions relies significantly upon equal-partitioning (Lamon, 1996) and the invariance of the whole (H. Yoshida & Sawano, 2002). In representing a number less than one, the whole should be of a fixed size in order to allow comparison of fractions. These ideas are currently missing from Australian mathematics textbooks.

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