MATHEMATICAL LITERACY FOR LIVING FROM OECD-PISA PERSPECTIVE

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The OECD/PISA study is spreading all over the globe: 58 countries might participate in 2006. And although PISA has had ample media attention it seems far from clear what PISA actually measures and how. The Literacy aspect is often overlooked, and the discussion is hardly ever about the content, or the instrument. This is an undesirable situation. Given the expansion of PISA it is worth to reflect on its meaning, possibilities and problems, starting with the Mathematical Literacy aspect.

MATHEMATICAL LITERACY

Mathematical Literacy has become a rather common term through the influence from OECD/PISA. But there is quite a history, at least dating back to the seventies, of efforts to cover the idea, often called numeracy or quantitative literacy. As quantitative Literacy has been used rather widely let us first look at a definition of QL. Lynn Arthur Steen (2001) pointed out that there are small but important differences in the several existing definitions and, although he did not suggest the phrase as a definition, referred to QL as the 'capacity to deal effectively with the quantitative aspects of life.' Indeed, most existing definitions Steen mentioned give explicit attention to number, arithmetic, and quantitative situations, either in a rather narrow way as in the National Adult Literacy Survey (NCES, 1993):

> The knowledge and skills required in applying arithmetic operations, either alone or sequentially, using numbers embedded in printed material (e.g., balancing a check book, completing an order form).

or more broadly as in the International Life Skills Survey (ILSS, 2000):

> An aggregate of skills, knowledge, beliefs, dispositions, habits of mind, communication capabilities, and problem solving skills that people need in order to engage effectively in quantitative situations arising in life and work.

The problem we have with these definitions is their apparent emphasis on *quantity*. Mathematical literacy is not restricted to the ability to apply quantitative aspects of mathematics but involves knowledge of mathematics in the broadest sense. As an example, being a foreigner who travels a great deal in the United States, I often ask directions of total strangers. What strikes me in their replies is that people are generally very poor in navigation skills: a realization of where you are, both in a relative and absolute sense. Such skills include map reading and interpretation, spatial awareness, 'grasping space' (Freudenthal, 1973), understanding great circle routes, understanding plans of a new house, and so on. All kinds of visualization belong as well to the literacy aspect of mathematics and constitute an absolutely essential component for literacy, as the three books of Tufte (1983, 1990, 1997) have shown in a very convincing way.

We believe that describing what constitutes *mathematical* literacy necessitates not only this broader definition but also attention to changes within other school disciplines. The Organization for Economic Cooperation and Development (OECD) publication *Measuring Student Knowledge and Skills* (1999) presents as part of *reading* literacy a list of types of texts, the understanding of which in part determines what constitutes literacy. This list comes close, in the narrower sense, to describing many aspects of quantitative literacy. The publication mentions, as examples, texts in various formats:

- Forms: tax forms, immigration forms, visa forms, application forms, questionnaires;
- Information sheets: timetables, price lists, catalogues, programs;
- Vouchers: tickets, invoices;
- Certificates: diplomas, contracts;
- Calls and advertisements;
- Charts and graphs; iconic representations of data;
- Diagrams;
- Tables and matrices;
- Lists;
- Maps.

The definition Steen used in *Mathematics and Democracy: The Case for Quantitative Literacy* (2001) refers to these as ‘document literacy’, following a definition adopted by the National Center for Education Statistics (1993).

Against this background of varying perspectives, we adapt for ‘mathematical literacy’ a definition that is broad but also rather ‘mathematical’:

> Mathematics literacy is an individual’s capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments, and to engage in mathematics in ways that meet the needs of that individual’s current and future life as a constructive, concerned and reflective citizen (OECD, 1999).

This definition was developed by the Expert Group for Mathematics of the Programme for International Student Assessment (PISA), of which the author is chair.

A MATTER OF DEFINITIONS

Having set the context, it seems appropriate now to make clear distinctions among types of literacies so that, at least in this essay, we do not declare things equal that are not equal. For instance, some equate numeracy with quantitative literacy; others equate quantitative and mathematical literacy. To make our definitions functional, we connect them to our phenomenological categories:

Spatial Literacy. We start with the simplest and most neglected: spatial literacy. Spatial literacy supports our understanding of the (three-dimensional) world in which we live and move. To deal with what surrounds us, we must understand properties of objects, the relative positions of objects and the effect thereof on our visual perception, the creation of all kinds of two- and three-dimensional paths and routes, navigational practices, shadows – even the art of Escher.

Numeracy. The next obvious literacy is numeracy, fitting as it does directly into quantity. We can follow, for instance, Treffers' (1991) definition, which stresses the ability to handle numbers and data and to evaluate statements regarding problems and situations that invite mental processing and estimating in real-world contexts.

Quantitative Literacy. When we look at quantitative literacy, we are actually looking at literacy dealing with a cluster of phenomenological categories: quantity, change and relationships, and uncertainty. These categories stress understanding of, and mathematical abilities concerned with, certainties (quantity), uncertainties (quantity as well as uncertainty), and relations (types of, recognition of, changes in, and reasons for those changes).

Mathematical Literacy. We think of mathematical literacy as the overarching literacy comprising all others. Thus we can make a visual representation as follows:

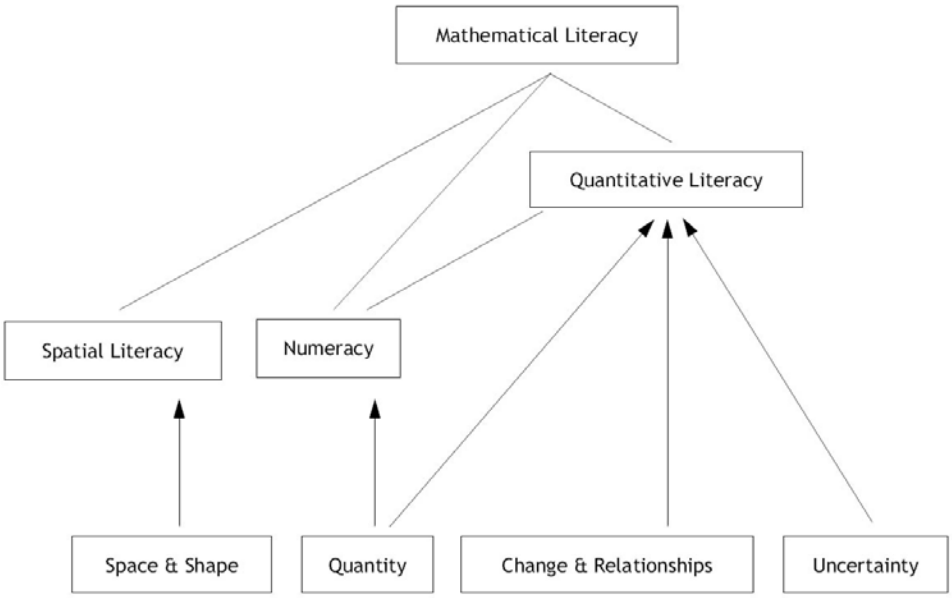

Fig. 1: Tree structure mathematical literacy

PISA LITERACY

The OECD/PISA mathematical literacy domain is concerned with the capacities of students to analyse, reason, and communicate ideas effectively as they pose, formulate, solve and interpret mathematics in a variety of situations. The assessment focuses on real-world problems, moving beyond the kinds of situations and problems typically encountered in school classrooms. In real-world settings, citizens regularly face situations when shopping, travelling, cooking, dealing with personal finances, et cetera, in which mathematical competencies would be of some help in clarifying or solving a problem.

Citizens in every country are increasingly confronted with a myriad of issues involving quantitative, spatial, probabilistic or relational reasoning. The media are full of information that use and misuse tables, charts, graphs and other visual representations to explain or clarify matters regarding the weather, economics, medicine, sports, environment, to name a few. Even closer to the daily life of every citizen are skills involving reading and interpreting bus or train schedules, understanding energy bills, arranging finances at the bank, economising resources, and making good business decisions, whether it is bartering or finding the best buy. Thus, literacy in mathematics is about the functionality of the mathematics you have learned at school. This functionality is important for students to survive in a successful way in the present information and knowledge society.

Such uses of mathematics are based on skills learned and practiced through the kinds of problems that typically appear in textbooks and classrooms. However, they demand the ability to *apply* those skills in a less structured context, where the directions are not so clear. People have to make decisions about what knowledge may be relevant, what process will lead to a possible solution, and how to reflect on the correctness of the answer found.

Some explanatory remarks are in order for the definition of mathematical literacy to become transparent:

1. In using the term 'literacy', we want to emphasize that mathematical knowledge and skills that have been defined and are definable within the context of a mathematics curriculum, do not constitute our primary focus here. Instead, what we have in mind is mathematical knowledge put into functional use in a multitude of contexts in varied, reflective, and insight-based ways.

2. *Mathematical literacy* cannot be reduced to – but certainly presupposes – knowledge of mathematical terminology, facts, and procedures as well as numerous skills in performing certain operations, carrying out certain methods, and so forth. Also, we want to emphasize that the term 'literacy' is not confined to indicating a basic, minimum level of functionality only. On the contrary, we think of literacy as a continuous, multidimensional spectrum ranging from aspects of basic functionality to high-level mastery.

3. A crucial capacity implied by our notion of mathematical literacy is the ability to pose, formulate and solve intra- and extra-mathematical problems within a variety of domains and settings. These range from purely mathematical ones to ones in which no mathematical structure is present from the outset but may be successfully introduced by the problem poser, problem solver, or both.

4. Attitudes and emotions (e.g., self-confidence, curiosity, feelings of interest and relevance, desire to do or understand things) are not components of the definition of mathematical literacy. Nevertheless they are important prerequisites for it. In principle it is possible to possess mathematical literacy without possessing such attitudes and emotions at the same time. In practice,

however, it is not likely that such literacy will be exerted and put into practice by someone who does not have some degree of self-confidence, curiosity, feeling of interest and relevance, and desire to do or understand things that contain mathematical components.

MATHEMATISATION AS KEY TO LITERACY

Mathematical literacy is about dealing with ‘real’ problems. That means that these problems are not ‘purely’ mathematical but are placed in some kind of a ‘situation’. In short, the students have to ‘solve’ a real world problem requiring to use the skills and competencies they have acquired trough schooling and life experiences. A fundamental role in that process is referred to as ‘mathematisation’.

The process of mathematisation starts with a problem situated in reality (1). Next the problem solver tries to identify the relevant mathematics, and reorganizes the problem according to the mathematical concepts identified (2) followed by gradually trimming away the reality (3). These three steps lead us from a real world problem to a mathematical problem. The fourth step may not come as a surprise: solving the mathematical problem (4). Now the question arises: what is the meaning of this strictly mathematical solution in terms of the real world (5)?

The following figure shows the cyclic character of the mathematisation process:

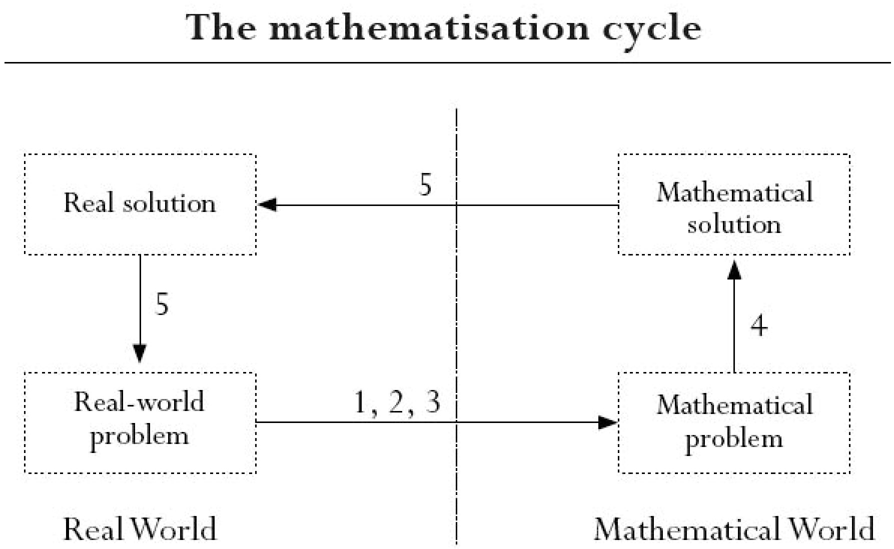

Fig. 2: The mathematisation cycle

In might help the interested reader understanding this cycle better by giving an example. The following item has been field trialled for PISA but was rejected because of the high degree of difficulty. The item has previously been released by OECD. For the purpose of making the mathematisation process clear we have changed the question asked:

HEARTBEAT

For health reasons people should limit their efforts, for instance during sports, in order not to exceed a certain heartbeat frequency.

For years the relationship between a person's recommended maximum heart rate and the person's age was described by the following formula:

Recommended maximum heart rate = 220 – age

Recent research showed that this formula should be modified slightly. The new formula is as follows:

Recommended maximum heart rate = 208 – (0.7 × age)

QUESTION 1: HEARTBEAT

What is the main difference between the two formulas and how do they affect the maximum allowable heart rate?

Fig. 3: Example of item that lends itself to mathematisation

The process of mathematisation starts with a problem situated in reality (1).
As will be clear from the item the realty in this case is physical fitness. An important rule when doing exercises is that one should not go to far in becoming fit, as this may cause health problems. We can jump to this conclusion because of the text: maximal heartbeat.

Next the problem solver *tries to identify the relevant mathematics, and reorganizes the problem according to the mathematical concepts identified* (2).
It seems clear that we have two word formulas that need to be understood, and we are requested to compare these two formulas, and what they really mean in mathematical terms. The formulas give a relation between the advised maximum heart beat rate and the age of a person.

Next we are gradually trimming away the reality (3).
There are different ways to move the problem to a strictly mathematical problem, or trimming away reality. One way to go is to make the word formulas into more formal expressions like:

$y = 220 – x$

$y = 208 – 0.7x$

Of course we have to remind ourselves that y expresses the maximum heart beat in b/m and x the age in years. Another way to go to a 'strictly' mathematical world is drawing the graphs right from the word formulas.

These three steps lead us from a real world problem to a mathematical problem.

The fourth step may not come as a surprise: solving the mathematical problem (4).
The mathematical problem at hand is to compare the two formulas or graphs, and to say something about the differences for people of a certain age. A nice way to start is to find out where the two formulas give equal results or where the two graphs intersect. So we can solve this by solving the equation:

$220 - x = 208 - 0{,}7x$.

This give us $x = 40$ and the corresponding value for y is 180. So the two graphs intersect at the point (40, 180).

And as the slope of the first formula is -1, and of the second one -0,7 we know that the second graph is 'less steep' than the first one. Or the graph of $y - 220 - x$ lies 'above' the graph of $y = 208 - 0{,}7x$ for values of x smaller than 40, and below for values of x larger than 40.

Now let us move to step 5 to see if this solution of the mathematical problem helps us in solving the 'real world' problem.

Now the question arises: what is the meaning of this strictly mathematical solution in terms of the real world (5)?
The meaning is not too difficult if we realize that x is the age of a person and y the maximum heart beat. If one is 40 years old both formulas give the same result: a maximum heart beat of 180. The 'old' rule allows for higher heart rates for younger people: in the extreme, if the age is zero the maximum is 220 in the old formula and only 208 in the new formula.

But for older people, in this case for those over 40, the more recent insights allow for a higher maximum heartbeat. As an example, and again extreme: for an age of 100 years we see that the old formula give us a maximum of 120 and the new one 138.

Of course we have to realize a number of other things: the formulas lack mathematical precision and give a feel of quasi scientific. It is only a rule of thump that should be used with caution. An other point is that for ages at the extreme the outcomes should be taken with even more hesitation.

What this example shows is that even with relatively 'simple' items, in the sense that they can be used within the restrictions of a large international study and can be solved in a short time; we still can identify the full cycle of mathematisation and problem solving.

What the example also shows is that mathematical literacy goes *beyond* most curricular mathematics. We will explore this further next.

MATHEMATICS FOR LITERACY

Mathematics school curricula are organized into strands that classify mathematics as a strictly compartmentalized discipline with an over-emphasis on computation and formulas. This organization makes it almost impossible for students to see mathematics as a continuously growing scientific field that continually spreads into new fields and applications. Students are not positioned to see overarching concepts and relations, so mathematics appears to be a collection of fragmented pieces of factual knowledge.

‘What is mathematics?’ is not a simple question to answer. A person asked at random will most likely answer ‘Mathematics is the study of number.’ Or, if you’re lucky, ‘Mathematics is the science of number.’ And, as Devlin (1994) states in his very successful book *Mathematics: The Science of Patterns*, the former is a huge misconception based on a description of mathematics that ceased to be accurate some 2,500 years ago. Present-day mathematics is a thriving, worldwide activity; it is an essential tool for many other domains like banking, engineering, manufacturing, medicine, social science, and physics. The explosion of mathematical activity that has taken place in the twentieth century has been dramatic. At the turn of the nineteenth century, mathematics could reasonably be regarded as consisting of about 12 distinct subjects: arithmetic, geometry, algebra, calculus, topology and so on. The similarity between this list and the present-day school curricula list is amazing.

A more reasonable figure for today, however, would be between 60 and 70 distinct subjects. Some subjects (e.g., algebra, topology) have split into various sub fields; others (e.g., complexity theory, dynamical systems theory) are completely new areas of study.

Mathematics could be seen as the language that describes patterns – both patterns in nature and patterns invented by the human mind. Those patterns can either be real or imagined, visual or mental, static or dynamic, qualitative or quantitative, purely utilitarian or of little more than recreational interest. They can arise from the world around us, from depth of space and time, or from the inner workings of the human mind.

Since the goal of OECD/PISA is to assess student’s capacity to solve real problems, the strategy has been to define the range of content that will be assessed using a phenomenological approach to describing the mathematical concepts, structures or ideas. This means describing content in relation to the phenomena and the kinds of problems for which it was created. This approach ensures a focus in the assessment that is consistent with domain definition, but covers a range of content that includes what is typically found in other mathematics assessments and in national mathematics curricula.

The concept of a phenomenological organization is not new. In 1990, the Mathematical Sciences Education Board published *On the Shoulders of Giants: New Approaches to Numeracy*, a book that made a strong plea for educators to help students delve deeper to find the concepts that underlie all mathematics and thereby better understand the significance of these concepts in the world. The labelling of phenomenological categories has many varieties. For the PISA study the term ‘overarching idea’ has been chosen.

Many phenomenological categories can be identified and described. In fact the domain of mathematics is so rich and varied that it would not be possible to identify an exhaustive list of phenomenological categories. It is important for purposes of assessment, however, for any selection of *big ideas* that is offered to represent a sufficient variety and depth to reveal the essentials of mathematics and their relations to the traditional strands.

The following list of mathematical phenomenological categories meets this requirement:

Quantity
Space and shape
Change and relationships
Uncertainty

Using these four categories, mathematics content can be organized into a sufficient number of areas to help ensure a spread of items across the curriculum, but also a small enough number to avoid an excessively fine division – which would work against a focus on problems based in real-life situations. Each phenomenological category is an encompassing set of phenomena and concepts that make sense together and may be encountered within and across a multitude of quite different situations. By their very nature, each idea can be perceived as a general notion dealing with a generalized content dimension. This implies that the categories or ideas cannot be sharply delineated vis-à-vis one another. Rather, each represents a certain perspective, or point of view, which can be thought of as possessing a core, a centre of gravity, and a somewhat blurred penumbra that allow intersection with other ideas. In principle, any idea can intersect with any other idea. (For a more detailed description of these four categories or ideas, please refer to the PISA framework (OECD, 2002).)

Quantity

This overarching idea focuses on the need for quantification to organize the world. Important aspects include an understanding of relative size, recognition of numerical patterns, and the ability to use numbers to represent quantifiable attributes of real-world objects (measures). Furthermore, quantity deals with the processing and understanding of numbers that are represented to us in various ways. An important aspect of dealing with quantity is quantitative reasoning, whose essential components are developing and using number sense, representing numbers in various ways, understanding the meaning of operations, having a feel for the magnitude of numbers, writing and understanding mathematically elegant computations, doing mental arithmetic, and estimating.

Space and Shape

Patterns are encountered everywhere around us: in spoken words, music, video, traffic, architecture, and art. Shapes can be regarded as patterns: houses, office buildings, bridges, starfish, snowflakes, town plans, cloverleaf, crystals, and shadows. Geometric patterns can serve as relatively simple models of many kinds of phenomena, and their study is desirable at all levels (Grünbaum, 1985). In the study of shapes and constructions, we look for similarities and differences as we analyze the components of form and recognize shapes in different representations and different dimensions. The study of shapes is closely connected to the concept of 'grasping space' (Freudenthal, 1973) – learning to know, explore, and conquer, in order to live, breathe, and move with more understanding in the space in which we live. To achieve this, we must be able to understand the properties of objects and the relative positions of objects; we must be aware of how we see things and why we see them as we do; and we must learn to navigate through space and through constructions and shapes. This requires understanding the relationship between shapes and images (or visual representations) such as that between a real city and photographs and maps of the same city. It also includes understanding how three-dimensional objects can be represented in two dimensions, how shadows are formed and interpreted, and what perspective is and how it functions.

Change and Relationships

Every natural phenomenon is a manifestation of change, and in the world around us a multitude of temporary and permanent relationships among phenomena are observed: organisms changing as they grow, the cycle of seasons, the ebb and flow of tides, cycles of unemployment, weather changes, stock exchange fluctuations. Some of these change processes can be modelled by straightforward mathematical functions: linear, exponential, periodic or logistic, discrete or continuous. But many relationships fall into different categories, and data analysis is often essential to determine the kind of relationship present. Mathematical relationships often take the shape of equations or inequalities, but relations of a more general nature (e.g., equivalence, divisibility) may appear as well. Functional thinking – that is, thinking in terms of and about relationships – is one of the fundamental disciplinary aims of the teaching of mathematics. Relationships can take a variety of different representations, including symbolic, algebraic, graphic, tabular, and geometric. As a result, translation between representations is often of key importance in dealing with mathematical situations.

Uncertainty

Our information-driven society offers an abundance of data, often presented as accurate and scientific and with a degree of certainty. But in daily life we are confronted with uncertain election results, collapsing bridges, stock market crashes, unreliable weather forecasts, poor predictions of population growth, economic models that do not align, and many other demonstrations of the uncertainty of our world. Uncertainty is intended to suggest two related topics: data and chance, phenomena that are the subject of mathematical study in statistics and probability, respectively. Recent recommendations concerning school curricula are unanimous in suggesting that statistics and probability should occupy a much more prominent place than they have in the past (Cockroft, 1982; LOGSE, 1990; MSEB, 1993; NCTM, 1989, 2000). Specific mathematical concepts and activities that are important in this area include collecting data, data analysis, data display and visualization, probability, and inference.

MATHEMATICS VERSUS MATHEMATICAL LITERACY

In an interview in *Mathematics and Democracy,* Peter T. Ewell (2001) was asked: " *The Case for Quantitative Literacy* argues that quantitative literacy is not merely a euphemism for mathematics but is something significantly different – less formal and more intuitive, less abstract and more contextual, less symbolic and more concrete. Is this a legitimate and helpful distinction? " Ewell answered that indeed this distinction is meaningful and powerful.

The answer to this question depends in large part on the interpretation of what constitutes good mathematics. We can guess that in Ewell's perception, mathematics is formal, abstract, and symbolic – a picture of mathematics still widely held. Ewell continued to say that literacy implies an integrated ability to function seamlessly within a given community of practice. Functionality is surely a key point, both in itself and in relation to a community of practice, which includes the community of mathematicians. Focusing on functionality gives us better opportunity to bridge gaps or identify overlaps. In the same volume, Alan H. Schoenfeld (2001) observed that in the past, literacy and what is learned in mathematics classes were largely disjointed. Now, however, they should be thought of as largely overlapping and taught as largely overlapping. In this approach, which takes into consideration the changing perception of what constitutes mathematics, mathematics and mathematical literacy are positively not disjointed.

For Schoenfeld, the distinction most likely lies in the fact that as a student he never encountered problem-solving situations, that he studied only 'pure' mathematics and, finally, that he never saw or worked with real data. Each of these is absolutely essential for literate citizenship, but none even hints at defining what mathematics is needed for ML, at least not in the traditional school mathematics curricula descriptions of arithmetic, algebra, geometry, and so on.

Again, in *Mathematics and Democracy,* Wade Ellis, Jr. (2001) observes that many algebra teachers provide instruction that constricts rather than expands student thinking. He discovered that students leaving an elementary algebra course could solve fewer real-world problems after the course than before it: after completing the course, they thought that they had to use symbols to solve problems they had previously solved using only simple reasoning and arithmetic. It may come as no surprise that Ellis promotes a new kind of common sense – a quantitative common sense based on mathematical concepts, skills, and know-how. Despite their differences, however, Schoenfeld and Ellis seem to share Treffers' (1991) observation that innumeracy might be caused by a flaw in the structural design of instruction.

These several observers seem to agree that in comparison with traditional school mathematics, ML is less formal and more intuitive, less abstract and more contextual, less symbolic and more concrete. ML also focuses more attention and emphasis on reasoning, thinking, and interpreting as well as on other very mathematical competencies. To get a better picture of what is involved in this distinction, we first need to describe what Steen (2001) called the 'elements' needed for ML. With a working definition of ML and an understanding of the elements (or 'competencies' as they are described in the PISA framework) needed for ML, we might come closer to answering our original question – what mathematics is important? – or formulating a better one.

COMPETENCIES NEEDED FOR MATHEMATICAL LITERACY

The competencies that form the heart of the ML description in OECD/PISA seem, for most parts, in line with the so called 'elements' in Steen (2001). The content part of the competency description relies on the work of Niss (1999) and his Danish colleagues, but similar formulations can be found in the work of many others representing many countries (as indicated by Neubrand et al., 2001). The same holds for the description of the competency clusters. The present description is a further development of work carried out by the author (Boertien and De Lange, 1994; De Lange, 1995, 1999) and his colleagues from the Netherlands and the United States.

1. *Mathematical thinking and reasoning.* Posing questions characteristic of mathematics; knowing the kind of answers that mathematics offers, distinguishing among different kinds of statements; understanding and handling the extent and limits of mathematical concepts.

2. *Mathematical argumentation.* Knowing what proofs are; knowing how proofs differ from other forms of mathematical reasoning; following and assessing chains of arguments; having a feel for heuristics; creating and expressing mathematical arguments.

3. *Mathematical communication.* Expressing oneself in a variety of ways in oral, written, and other visual form; understanding someone else's work.

4. *Modelling.* Structuring the field to be modelled; translating reality into mathematical structures; interpreting mathematical models in terms of context or reality; working with models; validating models; reflecting, analyzing, and offering critiques of models or solutions; reflecting on the modelling process.

5. *Problem posing and solving.* Posing, formulating, defining, and solving problems in a variety of ways.

6. *Representation.* Decoding, encoding, translating, distinguishing between, and interpreting different forms of representations of mathematical objects and situations as well as understanding the relationship among different representations.

7. *Symbols.* Using symbolic, formal, and technical language and operations.

8. *Tools and technology*. Using aids and tools, including technology when appropriate.

To be mathematically literate, individuals need all these competencies to varying degrees, but they also need confidence in their own ability to use mathematics and comfort with quantitative ideas. An appreciation of mathematics from historical, philosophical, and societal points of view is also desirable.

It should be clear from this description why we have included functionality within the mathematician's practice. We also note that to function well as a mathematician, a person needs to be literate. It is not uncommon that someone familiar with a mathematical tool fails to recognize its usefulness in a real-life situation (Steen, 2001). Neither is it uncommon for a mathematician to be unable to use common-sense reasoning (as distinct from the reasoning involved in a mathematical proof).

As Deborah Hughes-Hallett (2001) made clear in her contribution to *Mathematics and Democracy*, one of the reasons that ML is hard to acquire and hard to teach is that it involves insight as well as algorithms. Some algorithms are of course necessary: it is difficult to do much analysis without knowing arithmetic, for example. But learning (or memorizing) algorithms is not enough: insight is an essential component of mathematical understanding. Such insight, Hughes-Hallett noted, connotes an understanding of quantitative relationships and the ability to identify those relationships in an unfamiliar context; its acquisition involves reflection, judgment, and above all, experience. Yet current school curricula seldom emphasize insight and do little to actively support its development at any level. This is very unfortunate. The development of insight into mathematics should be actively supported, starting before children enter school.

Many countries have begun to take quite seriously the problems associated with overemphasizing algorithms and neglecting insight. For example, the Netherlands has had some limited success in trying to reform how mathematics is taught. To outsiders, the relatively high scores on the Third International Mathematics and Science Study (TIMSS) and TIMSS-R by students in the Netherlands appear to prove this, but the results of the Netherlands in the PISA study should provide even more proof.

COMPETENCY CLUSTERS

We do not propose development of test items that assess the above skills individually. When doing real mathematics, it is necessary to draw simultaneously upon many of those skills. In order to operationalise these mathematical competencies, it is helpful to organize the skills into three clusters:

Reproduction (definitions, computations).
Connections (and integration for problem solving).
Reflection, (mathematical thinking, generalization, and insight).

We will elaborate on these clusters next.

Cluster 1. Reproduction

At this first cluster, we deal with the matter dealt with in many standardized tests, as well in comparative international studies, and operationalised mainly in multiple-choice format. It deals with knowledge of facts, representing, recognizing equivalents, recalling mathematical objects and properties, performing routine procedures, applying standard algorithms, and developing technical skills. Dealing and operating with statements and expressions that contain symbols and formulas in ‘standard’ form also relate to this level.

Items at cluster 1 are often in multiple-choice, fill-in-the-blank, matching, or (restricted) open-ended questions format.

Cluster 2. Connections

At this cluster we start making connections between the different strands and domains in mathematics and integrate information in order to solve simple problems in which students have a choice of strategies and a choice in their use of mathematical tools. Although the problems are supposedly non-routine, they require relatively minor mathematisation. Students at this level are also expected to handle different forms of representation according to situation and purpose. The connections aspect requires students to be able to distinguish and relate different statements such as definitions, claims, examples, conditioned assertions, and proof.

From the point of view of mathematical language, another aspect at this cluster is decoding and interpreting symbolic and formal language and understanding its relations to natural language. Items at this cluster are often placed within a context and engage students in mathematical decision making.

Cluster 3. Reflection

At cluster 3, students are asked to mathematise situations (recognize and extract the mathematics embedded in the situation and use mathematics to solve the problem). They must analyse, interpret, develop their own models and strategies, and make mathematical arguments including proofs and generalizations. These competencies include a critical component and analysis of the model and reflection on the process. Students should not only be able to solve problems but also to pose problems.

These competencies function well only if the students are able to communicate properly in different ways (e.g., orally, in written form, using visualizations). Communication is meant to be a two-way process: students should also be able to understand communication with a mathematical component by others. Finally we would like to stress that students also need insight competencies – insight into the nature of mathematics as a science (including the cultural and historical aspect) and understanding of the use of mathematics in other subjects as brought about through mathematical modelling.

As is evident, the competencies at cluster 3 quite often incorporate skills and competencies usually associated with the other two clusters. We note that the whole exercise of defining the three clusters is a somewhat arbitrary activity: There is no clear distinction between different clusters, and both higher- and lower-level skills and competencies often play out at different clusters.

Cluster 3, which goes to the heart of mathematics and mathematical literacy, is difficult to test. Multiple-choice is definitely not the format of choice at cluster 3. Extended response questions with multiple answers (with or without increasing level of complexity) are more likely to be promising formats. But both the design and the judgment of student answers are very, if not extremely, difficult.

It goes without saying that we can describe the eight competencies fitting all three competency clusters. The actual PISA framework does this in detail (meaning eight competencies as they play out at the reproduction cluster, at the connections cluster and at the reflection cluster.

Finally, we want to make the observation that the competencies needed for ML are actually the competencies needed for mathematics *as it should be taught.* Were that the case (with curricula following the suggestions made by Schoenfeld and Hughes-Hallett and extrapolating from experiences in the Netherlands and other countries), the gap between mathematics and mathematical literacy would be much smaller than some people suggest it is at present (Steen, 2001). It must be noted, however, that in most countries this gap is quite large and the need to start thinking and working toward an understanding of what makes up ML is barely recognized. As Neubrand et al. (2001) noted in talking about the situation in Germany: ‘In actual practice of German mathematics education, there is no correspondence between the teaching of mathematics as a discipline and practical applications within a context’ (free translation by author).

AN EXAMPLE: TIDES

Natural phenomena should play a vital role in mathematics for ML. For a country like the Netherlands, with 40 percent of its area below sea level, the tides are very important. The following protocol is taken from a classroom of 16-year-olds (De Lange, 2000):

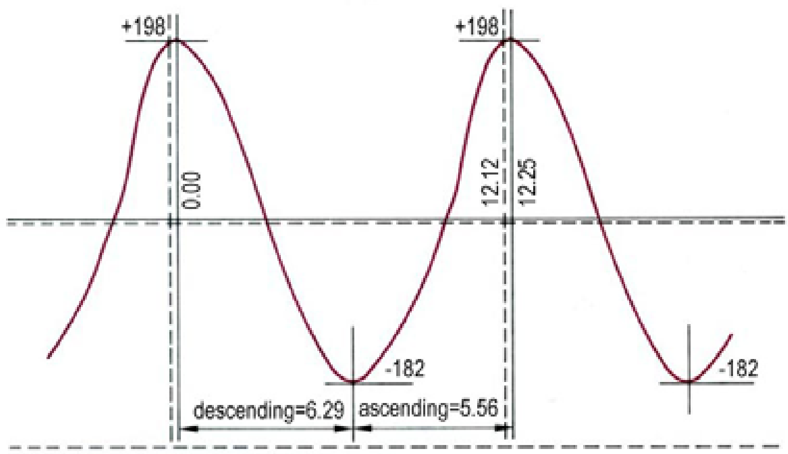

Fig. 4: Tides

Teacher: Let's look at the mean tidal graph of Flushing. What are the essentials?

Student A: High water is 198 cm, and low is -182 cm.

Teacher: And? [pause]

Student A: So it takes longer to go down than up.

Teacher: What do you mean?

Student A: Going down takes 6 hours 29 minutes, up only 5 hours 56 minutes.

Teacher: OK. And how about the period?

Student A: 6 hours 29 and 5 hours 56 makes 12 hours 25 minutes.

Teacher: Now, can we find a simple mathematical model?

Student B: [pause] Maybe 2 sin *x*.

Teacher: What is the period, then?

Student B: 2 pi, that means around 6.28 hours [pause] 6 hours 18 minutes [pause] oh, I see that it must be 2*x*, or, no, *x*/2.

Teacher: So?

Student C: 2 sin (*x*/2) will do.

Teacher: Explain.

Student C: Well, that's simple: the maximum is 2 meters, the low is -2 meters, and the period is around 12 hours 33 minutes or so. That's pretty close, isn't it?

Teacher: [to the class] Isn't it?

Student D: No, I don't agree. I think the model needs to show exactly how high the water can rise. I propose 190 sin (*x*/2) + 8. In that case, the maximum is exactly 198 cm, and the minimum exactly -182 cm. Not bad.

Teacher: Any comments, anyone? [some whispering, some discussion]

Student E: I think it is more important to be precise about the period. 12 hours 33 minutes is too far off to make predictions in the future about when it will be high water. We should be more precise. I think 190 sin [(pi/6.2)*x*] is much better.

Teacher: What's the period in that case?

Student F: 12.4 hours, or 12 hours 24 minutes, only 1 minute off.

Teacher: Perfect. What model do we prefer? [discussion]

Student G: 190 sin [(pi/6.2)*x*] + 8.

The discussion continued with 'What happens if we go to a different city that has smaller amplitudes and where high tides come two hours later?', 'How does this affect the formula?', 'Why is the rate of change so important?'

Why do we consider this a good example of mathematics for ML? Given the *community* in which this problem is part of the curriculum, the relevance for society is immediately clear – and the relevance is rising with global temperatures. The relevance also becomes clear at a different level, however. The mathematical method of trial and error illustrated here not only is interesting by itself, but the combination of the method with the most relevant variables also is interesting: in one problem setting we are interested in the exact time of high water; in another we are interested in the exact height of the water at high tide. Intelligent citizens need insight into the possibilities and limitations of models. This problem worked very well for these students, age sixteen, and the fact that the 'real' model used forty different sine functions did not really make that much difference with respect to students' perceptions.

One important observation has to be made: one should not confuse ML with the study of the discipline of mathematics. ML is about the functionality of the discipline as the students encounter their discipline at school. But the aspects of abstraction, formalization and structure of mathematics are often overlooked at school as well, and this aspect is well worth another study. But the example also shows that some real good traditional mathematics is really needed to become literate in this example, for this certain community of learners.

PROBLEMS AND PROMISES

One of the main aspects of PISA that seems to draw fire a lot is the 'horse-race aspect'.

PISA is seen as a competition with winners and losers and indeed, the report by the OECD is quite supportive of this judgment. On the first pages it is talking about winners and losers. And by doing this it gives authors like Clarke (2003) a good foundation for their critique that with PISA and TIMSS there is a 'current preoccupation with competition'.

Clarke is right of course if we see that the TIMSS-report says: 'Singapore was the top-performing country' (on page 3, in 2003) and the PISA report is even faster: 'Finland is the top performing country', on page 1 (!).

This horserace might be of some interest to the popular press (*Der Spiegel*: 'The Winner is Finland') but it overwhelms the real questions, about the instrument, for instance. Keitel and Kilpatrick (1999) observe, quite correctly, that 'questions of content have usually been seen as secondary'. This is in line with Bonnet's (2002) conclusion that 'one would have wished for more space to be devoted to trying to understand what happens in the classrooms in terms of teaching and learning.' Given my earlier published comments like: 'concerns about content validity resulted in some countries in national options', and: 'the content is a point of concern that gets relatively little attention', I do agree with the earlier quoted authors (De Lange, in press).

But not only the horserace, and the lacking interest in content and instrument aspects is worrying a lot of observers. There are also validity aspects that need further discussion.

A main issue seems to be the choice to measure 'mathematical literacy' instead of 'curricular mathematics' as in TIMSS. One certainly can join OECD in there effort to measure mathematical literacy, as defined in the frameworks of OECD. It seems a worthwhile effort to try not only to define the functionality of mathematics for 15 year old students, but also to design an instrument that tries to operationalize this definition in tasks. But how do we know for sure that these tasks will do the job? Can we really measure mathematical literacy with multiple choice and open questions, with an average answer time of less than 2 minutes? Is it true that we need other instruments as well, like extended tasks (like in PISA Problem Solving). Like group work, like IT use in solving the literacy problems?

And talking about the instrument: how does OECD decide what the cut point will be to be called literate? One can easily find arguments that one test for students of so many countries (in 2006 about 70) cannot suffice the needs of all countries. For instance, in a secondary analysis of the PISA 2003 mathematics result, the researchers suggest that for the Netherlands one can only be called literate if above the level 4 from OECD. OECD has her cut point above level 1.

A point that is often ignored about the horserace that indeed we see Finland reach the finish line first, but do we know the ‘time’ or whether or not it is a world record? In other words, the scores in PISA are *relative* scores: the average over all countries is 500 and the standard deviation is 100. So, although the definition of mathematical literacy is quite elaborated in a whole variety of competencies, there is no criterion referenced testing in PISA. Everything is norm referenced and we have no clue whether or not we do really well even if we are a top-performer as a country. To get a better view on the competencies of students in a country, we need secondary analysis of scores and insight in actual student’s work. OECD encourages the use of their data for this purpose but even if secondary analysis takes place, it never get the media attention it deserves, and does not change in any respect the horserace.

A well acknowledged problem of any international comparative test is the fact that education is culturally embedded. Clarke’s remark: ‘to assess the extend to which Ethiopian students possess the mathematical skills required for effective participation in American society seems both futile and uninformative’ is often embraced but rather futile itself. The mathematical competencies as identified in the Assessment Framework try go beyond the cultural aspects: to think somewhat logically, to be critical, to reason mathematically, to be able to use mathematics are competencies that are useful in any situation or culture. In our opinion the cultural aspect is less about the ‘content’ and much more about the ‘cultural context’ of education in a country.

The following graph, showing socio/economic/geographical clustering of countries participating in TIMSS, tells a story:

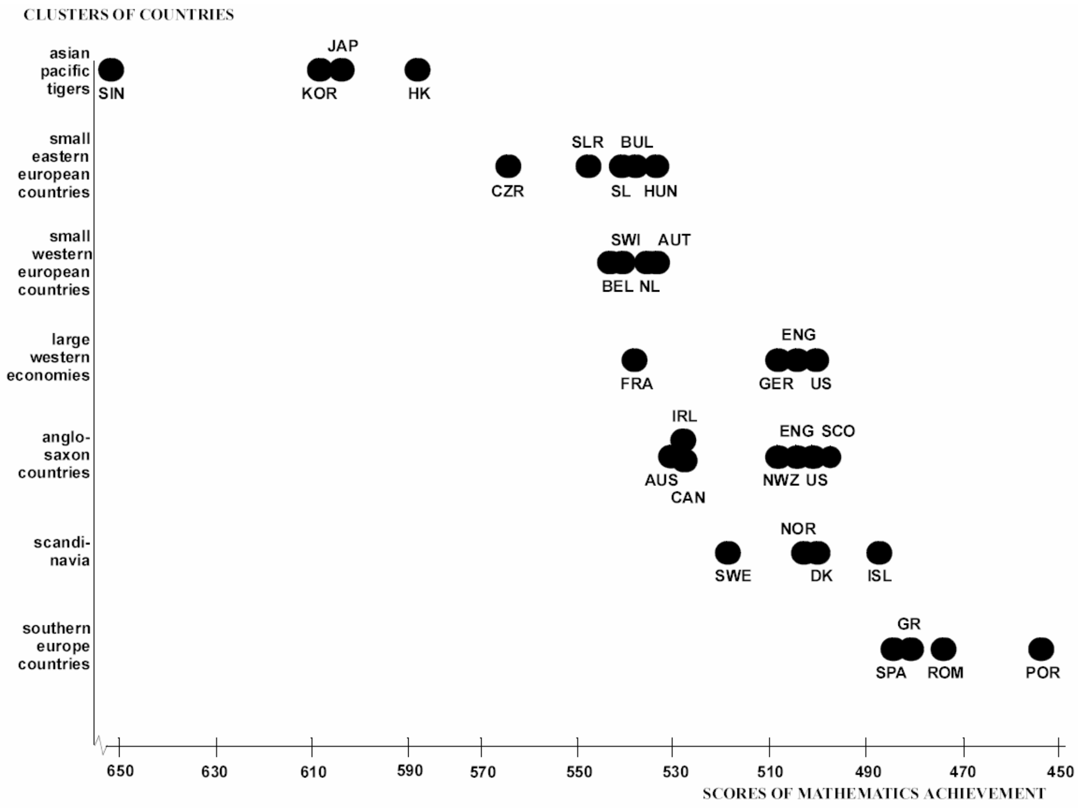

Fig. 5: Mathematical achievement in TIMSS

Authors also get quite excited about the role of contexts in large-scale assessments. There are many good reasons to do so, as we still fail to understand quite often the actual role the context plays in a certain problem. As Feijs and De Lange (2004) concluded: ‘the choices of context in relation to item construction and validation is a very complex one’. And I would like to add: we cannot say anything firm about the relationship ‘context familiarity’ to ‘success rate’.

Officially the PISA study is about measuring or evaluating educational systems. But this is not always simple. Just take the case of Belgium and the Netherlands that do equally well in the horserace but have very different educational systems. The ‘one number tells all principle’ can backfire in bad ways, if we leave it at that one number. This adds to the undesirability of the horserace.

A point of concern is also the choice of what is in the report and what is left out. It is interesting to know that PISA is overseen by a board that consists of officials from the different countries that represent their departments of education. This is understandable as they are paying for the whole PISA enterprise. But at the other side the board can influence the way the results are gathered (multiple-choice is cheap) and which results end up in a prominent place in the reports. As the political scientist Kettle once observed, educational tests are not really about measurement but about political communication.

The Dutch government once asked us to give a reason or two why Singapore outperforms the Netherlands (again: the horserace). One of the reasons might be found if we analyze a graph, that we constructed ourselves, but that was based on available data from TIMSS. Why is this graph not in the report? Who makes the choices?

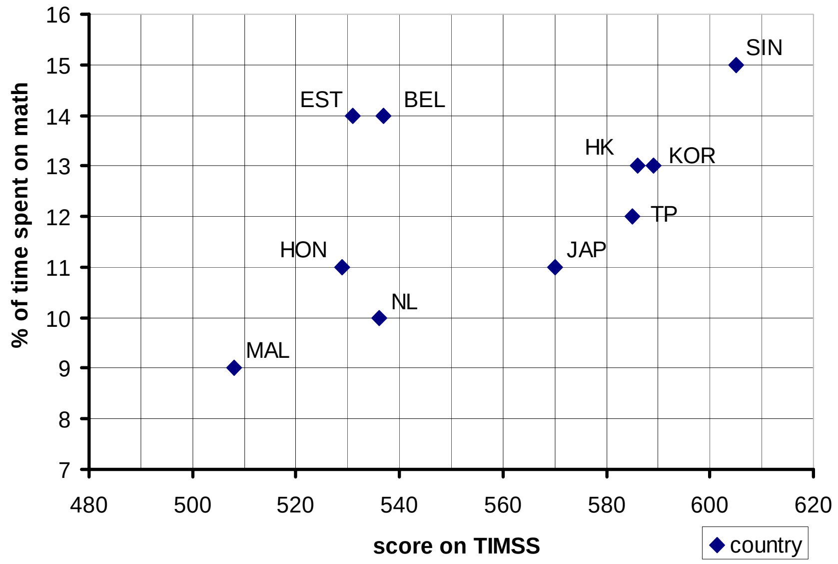

Fig. 6: TIMSS score versus time spent on math

Nevertheless, in our opinion PISA can become an important instrument for measuring mathematical literacy. If it does so at present remains to be proven, but it does some valuable things. One of the most important is putting the volume of ML at the heart of the discussion.

And PISA departs from the usual curricular approach and tries to find out how well our young students are equipped for modern society. It has an innovative and ambitious framework for assessment with the main components: Content, Context and Competencies. It has a nice collection of items that make up the instruments, in comparison with items from previous large scale assessments. It lends itself to extensions that will make the validity better, and make the information more reliable and relevant. It may influence, in a positive way, the thinking about the value of mathematics education in a substantial part of the world, which might in return lead to better communication about the math education problems in the world. It might support the very necessary communication between psychometricians at one side, and math educators at the other. If secondary analyses are carried out the relevance for individual countries will be enhanced considerably.

A FINAL WORD

A final word of warning should be placed here, because heated discussions can arise from being uninformed. OECD/PISA has been clear in pointing out that PISA measures mathematical literacy, which is not the same as curricular mathematics. However, in the popular media this distinction almost never is reported. Therefore, sometimes OECD has been found ‘guilty’ for imposing a new curriculum for mathematics for its member states, and the guests.

PISA reports how students perform on tasks that are intended to measure mathematical literacy, and it is clear therefore that it does not tell how well students ‘master’ their school curricula. TIMSS does that better.

It is also clear that many countries take the outcomes of PISA seriously in the sense that they embrace the idea that the output of an educational process should include a certain amount of ‘functionality’. But it is up to the countries to decide how important this aspect is. And from that policy standpoint will follow how much attention will be given to the functionality in the school curriculum. One country that has taken these steps is Germany: this country made great efforts in introducing more context oriented problems involving good mathematics. The score from 2000 to 2003 has improved, so it will be interesting what Germany will do in 2006 and 2009. Other countries will do similar actions. But we should not forget that the more formal and abstract mathematics, showing the structure of the discipline is *not* what PISA is all about – although some ‘inner-mathematical’ problems are included.

Many countries consider mathematics education having three primary goals:

preparing students for society;
preparing students for future schooling and work;
showing the students the beauty of the discipline.

PISA addresses primarily the first two goals. It would be an additional challenge to see if one can design an international study that addresses the third goal. It seems that the International Mathematics Union could take an initiative in this direction.

References

Boertien, H., & De Lange, J. (1994). *The national option of TIMSS in the Netherlands*. Enschede, the Netherlands: University of Twente.

Bonnet, G. (2002). Reflections in a Critical Eye: On the Pitfalls of International Assessment. *Assessment in Education: Principles, Policy and Practice, 9* (3), 387-399.

Clarke, D. (2003). International Comparative Research in Mathematics Education. In A.J. Bishop, M.A. Clements, C. Keitel, J. Kilpatrick, & F.K.S. Leung (Eds.), *Second International Handbook of Mathematics Education, Part one* (pp. 143-184). Dordrecht/Boston/London: Kluwer Academic Publishers.

Cockroft, W. H. (1982). *Mathematics Counts*. Report of the Committee of Inquiry into the Teaching of Mathematics in Schools. Londen: Her Majesty’s Stationery Office.

De Lange, J. (1992). Higher Order (Un-)Teaching. In *Developments in School Mathematics Education around the World,* edited by I. Wirszup and R. Streit, vol. 3, 49-72. Reston, VA: National Council of Teachers of Mathematics (NCTM).

De Lange, J. (1995). Assessment: No Change Without Problems. In T.A. Romberg (Ed.) *Reform in School Mathematics and Authentic Assessment* (pp. 87-172). New York, NY: State University of New York Press.

De Lange, J. (1999). *Framework for Assessment in Mathematics*. Madison, WI: National Center for Improving Student Learning and Achievement in Mathematics and Science (NCISLA).

De Lange, J. (2000). The Tides They are A-Changing. *UMAP-Journal 21*(1), 15-36.

De Lange, J. (in press). Large-Scale Assessment and Mathematics Education. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*.

Devlin, K. J. (1994). *Mathematics, the science of patterns: The search for order in life, mind, and the universe*. New York: Scientific American Library.

Ellis, Jr., W. (2001). Numerical Common Sense for All. In L.A. Steen (Ed.), *Mathematics and Democracy: The Case for Quantitative Literacy* (pp. 61-65). Princeton, NJ: National Council on Education and the Disciplines.

Ewell, P.T. (2001). Numeracy, Mathematics, and General Education. In L.A. Steen (Ed.), *Mathematics and Democracy: The Case for Quantitative Literacy* (pp. 37-48). Princeton, NJ: National Council on Education and the Disciplines.

Feijs, E. & De Lange, J. (2004). The Design of Open-Open Assessment tasks. In T. Romberg (Ed.), *Standards-based mathematics assessment in middle school: rethinking classroom practice* (pp. 122-136). New York, NY: Teachers College Press.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: Reidel.

Grünbaum, B. (1985). Geometry Strikes Again. *Mathematics Magazine 58*(1), 12-18.

Hughes-Hallett, D. (2001). Achieving Numeracy: The Challenge of Implementation. In L. A. Steen (Ed.), *Mathematics and Democracy: The Case for Quantitative Literacy* (pp. 93-98). Princeton, NJ: National Council on Education and the Disciplines.

International Life Skills Survey (ILSS). (2000). *International Life Skills Survey*. Ottawa, Canada: Statistics Canada, Policy Research Initiative.

Keitel, C., & Kilpatrick, J. (1999). The Rationality and Irrationality of International Comparative Studies. Chapter 16. In G. Kaiser, E. Luna & I. Huntley (Eds.), *International Comparisons in Mathematics Education* (pp. 241-256). London: Falmer Press.

Kennedy, D. (2001). The Emperor's Vanishing Clothes. In L. A. Steen (Ed.), *Mathematics and Democracy: The Case for Quantitative Literacy* (pp. 55-60). Princeton, NJ: National Council on Education and the Disciplines.

Ministerio de Educación y Ciencia. (1990). *Ley de Ordenación General del Sistema Educativo (LOGSE)*. Madrid, Spain: Ministerio de Educación y Ciencia.

Mathematical Sciences Education Board (MSEB). (1993). *Measuring What Counts: A Conceptual Guide for Mathematical Assessment*. Washington, DC: National Academy Press.

National Center for Education Statistics. (1993). *National Adult Literacy Survey*. Washington D.C.: National Center for Education Statistics (NCES).

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics (NCTM).

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics (NCTM).

Neubrand, N. et al. (2001). Grundlager der Ergänzung des Internationalen PISA-Mathematik-Tests in der Deutschen Zusatzerhebung. *ZDM 33*(2), 45-59.

Niss, M. (1999). Kompetencer og Uddannelsesbeskrivelse [Competencies and Subject-Description]. *Uddanneise 9*, 21-29.

Organization for Economic Cooperation and Development (OECD). (1999). *Measuring Student Knowledge and Skills. A New Framework for Assessment*. Paris: OECD.

Organization for Economic Cooperation and Development (OECD). (2002). *Framework for Mathematics Assessment*. Paris: OECD.

Schneider, C.G. (2001). Setting Greater Expectations for Quantitative Learning. In L.A. Steen (Ed.), *Mathematics and Democracy: The Case for Quantitative Literacy* (pp. 99-106). Princeton, NJ: National Council on Education and the Disciplines.

Schoenfeld, A.H. (2001). Reflections on an Impoverished Education. In L.A. Steen (Ed.), *Mathematics and Democracy: The Case for Quantitative Literacy* (pp. 49-54). Princeton, NJ: National Council on Education and the Disciplines.

Steen, L.A. (1990). *On the Shoulders of Giants: New Approaches to Numeracy*. Washington, DC: National Academy Press.

Steen, L.A. (2001). *Mathematics and Democracy: The Case for Quantitative Literacy*. Princeton, NJ: National Council on Education and the Disciplines.

Treffers, A. (1991). Meeting Innumeracy at Primary School. *Educational Studies in Mathematics 22*, 333-352.

Tufte, E.R. (1983). *The Visual Display of Quantitative Information*. Cheshire, CT: Graphic Press.

Tufte, E.R. (1990). *Envisioning Information*. Cheshire, CT: Graphic Press.

Tufte, E.R. (1997). *Visual Explanations – Images and Quantities, Evidence and Narrative*. Cheshire, CT: Graphic Press.