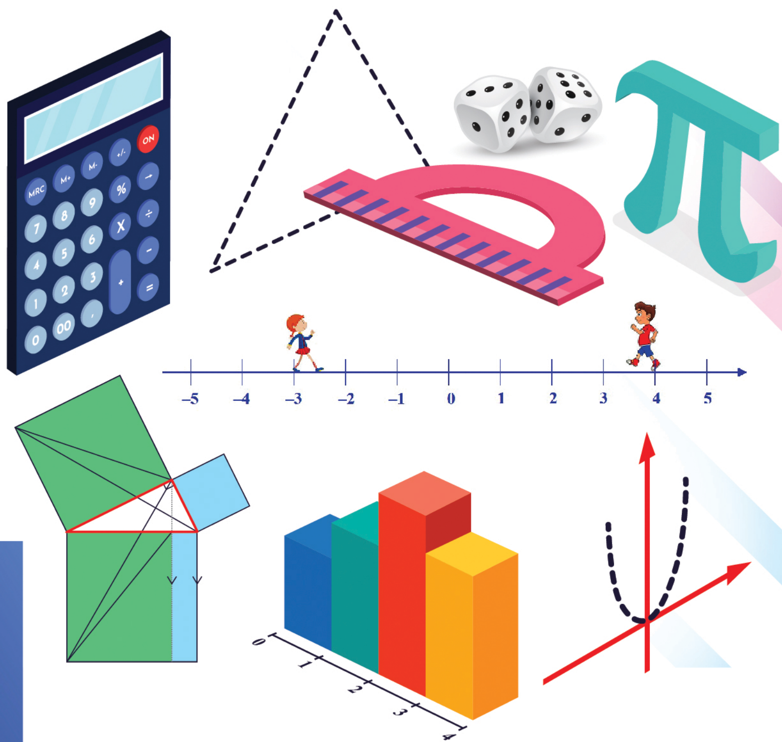
Mathematics Challenges for Classroom Practices at the LOWER SECONDARY LEVEL

Based on SEAMEO Basic Education Standards: Common Core Regional Learning Standards in Mathematics

Second Edition



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GAN Teck Hock
ISODA Masami
TEH Kim Hong

With Support of:

Maitree Inprasitha (KKU)
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Lower Secondary Level

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FOREWORD

Congratulations to SEAMEO RECSAM and CRICED, University of Tsukuba, for another commendable collaboration to publish the second edition of the three guidebooks titled Mathematics Challenges for Classroom Practices at the Lower Primary Level, the Upper Primary Level, and the Lower Secondary Level. I hope the release of these new editions will be shared and benefit teachers/educators in adopting ideas and suggestions to enhance their classroom practices and teacher training.

The first editions of the three guidebooks were written based on the SEAMEO Basic Education Standards: Common Core Regional Learning Standards (SEA-BES CCRLS) in Mathematics, published in 2017. These standards were translated into mathematical tasks to help readers understand their meaning. It was a lengthy effort that began in 2018 for the contributors, including SEAMEO RECSAM, along with academic members from SEAMEO QITEP in Mathematics, the Institute for the Promotion of Teaching Science and Technology (IPST), and Khon Kaen University. Two books were published in 2021 and the third book in 2023.

As the second edition of the SEA-BES CCRLS in Mathematics has also been published, the three guidebooks are also making changes to ensure alignment.

I understand that the three guidebooks have been frequently referred to in the online course conducted under the collaboration of University of Tsukuba and SEAMEO Schools' Network titled "Teaching Mathematics to Develop Mathematical Thinking as Higher order thinking: How do you teach? Why?" With such online sharing, this effort will definitely enrich the learning of many teachers/educators.

In this second edition, the books, which cover Grades 1 to 9, are updated to the current needs and emphasis. They are accessible through the website of the University of Tsukuba. Teachers and educators from SEAMEO Members should treasure the opportunity to make full use of the books for teaching and learning improvements.

I also hope that the Ministries of Education of SEAMEO Member Countries would support and promote this guidebook series among educators in their respective countries. This effort and spirit of cooperation among SEAMEO Member Countries and Associate Member Countries can be realised to bring benefits to classroom practices and mathematics development.

My sincere appreciation and congratulations to CRICED, the University of Tsukuba, for providing professional expertise and funding, SEAMEO RECSAM as the main collaborator, as well as other collaborating partner institutions and individual educators and specialists for their expertise, commitment, and contributions in this continual endeavour.

DATUK DR HABIBAH ABDUL RAHIM
Director, SEAMEO Secretariat, Bangkok, Thailand

FOREWORD

I am very pleased to express my profound appreciation to the Centre for Research on International Cooperation in Educational Development (CRICED), University of Tsukuba for continuously supporting SEAMEO RECSAM in the contribution of the publication, the second edition guidebook series, titled “Mathematics Challenges for Classroom Practices” for the i) Lower Primary Level, ii) Upper Primary Level and iii) Lower Secondary Level. I also express my profound appreciation to SEAMEO Qitep in Mathematics (SEAQIM) for collaboration in this project.

First edition of the three guidebooks were prepared based on the learning standards of SEA-BES CCRLS in Mathematics published in 2017 by SEAMEO RECSAM. Based on records, the guidebook series project started in 2018. Academic educators and specialists of other collaborating partners such as Khon Kaen University, Thailand; the Institute for the Promotion of Teaching Science and Technology (IPST), Thailand, SEAMEO QITEP in Mathematics, Indonesia; mathematics specialists in APEC economies, contributed ideas and preliminary draft of their writings in this guidebook series.

Two of the guidebooks were published in 2021 and another one was published in 2023. I appreciate the project team members who persevere to complete the publication even though it took several years to complete the project.

In the recent development, specialists of SEAMEO RECSAM and Qitep mathematics were invited to review the SEA-BES CCRLS to prepare for the second edition publication. The specialists also reviewed the upper primary level guidebook and some suggestions were proposed. With that alignment to the second edition SEA-BES CCRLS in Mathematics, I am sure new features are included into the guidebook series.

The second edition of the guidebook series has been added with improvement. There includes an extensive list of terminologies on mathematical thinking and processes as well as mathematical values and attitudes are explained to help users understand the mathematics framework used in the books. With the new version, SEAMEO RECSAM will introduce and use the guidebooks more extensively than before to teachers/educators in our training courses and programme. With this effort, I hope the Centre can assist in disseminating the use of this guidebook series.

On behalf of SEAMEO RECSAM, I would like to acknowledge CRICED, University of Tsukuba for supporting this project in funding and professional guidance. The collaboration of Professor Dr Masami Isoda is highly appreciated. He initiated the idea and the guidebook project since 2018. Since many years has passed, but he still keeps the passion to review the three guidebooks to produce the second edition to keep abreast with the latest development and alignment with mathematics curriculum development. I am grateful to the commitment of the writing team members of RECSAM, Ms Teh Kim Hong also coordinated the project, and Mr Gan Teck Hock, who still keep the passion of supporting the project to the current time even after their retirements. I also like to thank the panel reviewers of RECSAM and SEAQIM who provided their constructive suggestions to improve the content of this guidebook series.

There is a lot of hard work and time been invested to produce the three guidebooks. Other than promoting the use of the guidebooks in RECSAM, I hope that this mathematics guidebook series will be promoted by SEAMEO Secretariat and provide their support by promoting this guidebook series in the classrooms of educators and teachers in all SEAMEO member countries and beyond.

Dr. AZMAN BIN JUSOH
Centre Director, SEAMEO RECSAM, Penang, Malaysia

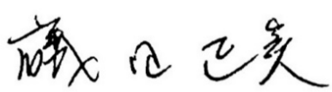
FOREWORD

Since 2009, the University of Tsukuba has been engaged in several projects with SEAMEO Secretariat and Centers as an affiliate member of SEAMEO. As the Director of the Centre for Research on International Cooperation in Educational Development (CRICED), I would like to express my deepest gratitude to SEAMEO RECSAM & SEAQiM, and SEAMEO member countries as well as APEC Lesson Study Network specialists for the cooperations and collaborations granted.

Since 2014, SEAMEO RECSAM had engaged in the SEA-BES projecet. 'SEAMEO Basic Education Standards (SEA-BES): Common Core Regional Learning Standards (CCRLS) in Mathematics and Science (first edition)' was published by RECSAM in 2017. With the support of the Ministry of Education, Culture, Sports, Science and Technology (MEXT), Japan, I had collaborated with Pedro Montecillo Jr., Teh Kim Hong, Mohd Sazali bin Hj. Khalid and other specialists of RECSAM. Comparative analysis of the curriculum documents of SEAMEO countries and other leading countries oriented to the higher-order thinking skills (HOTS) were done. While doing that, international leading scholars provided inputs on the latest trends, and curriculium specialists of member countries produced the framewok which extracted HOTS as mathematical ideas, the ways of thinking, Values, Attitudes and Habits of mind. The mathematical process and humanity standards were embeded into the content of four strands as part of the teaching content. After the publication, we recognised the necessity of concrete examples for general readers to understand the intended meaning of every statement of the standards in the content strands. Many do not visualise concrete images of HOTS in their classroom practices even though SEA-BES CCRLS in Mathematics extracted it as mathematical thinking skills, values, attitudes and habits of mind. Turning constraints into an opportunity, the First Edition of the guidebook series 'Mathematics Challenges for Classroom Practices' was prepared, particularly for teacher educators, textbook authors, and curriculum developers; with the suppport of Ms. Teh Kim Hong, Mr. Gan Tech Hock, RECSAM and collaboration of CRME-IRDTP at KKU (Thailand), IPST (Thailand) and SEAMEO QITEP in Mathematics (Indonesia) and also with the support of the coordinators. In the elaboration of the guidebook series, HOTS were extracted as mathematical thinking for the content of teaching.

The Second Editions are prepared to align with the Era of the Forth Industrial Revolution, especially the Era of Artificial Intelligence (AI) and Digital Transformation. In editing the first edition, we did minor revision to the SEA-BES CCRLS in Mathematics. In recent years, SEA-BES CCRLS in Mathematics has gained traction. SEAQiM conducted the MaRWA project, and CRICED, SEAMEO RECSAM and SEAQiM had collaborated the Unpluged Computational Thinking Projects under the leadership of SEAMEO STEM-Ed and published the Guidebook for Unpluged Computational Thinking (2024) based on the CCRLS in mathematics framework. Similarly, the APEC Lesson Study project was extended to Project InMside (2021) with Roberto Araya. The re-edited first edition of the guidebooks have been used for the online courses to develop mathematical thinking by CRICED with SEAMEO School Network. In the online courses, constructive feedbacks from APEC Lesson Study Network specialists and participants were received. Based on these initiatives and efforts, we publish the second edition of the Guidebook series 'Mathematics Challenges for Classroom Practices'.

Last but not least, I would like to convey my sincere appreciation to all contributors in the contributors' list, Gakko Tosho (the Japanese Textbook Publisher: 'Study with your friends Mathematics' Series in English Editions) who provided us with innovative ideas for Teaching, and Japan Society of Promotion of Science (JSPS) for providing us the grant for Research and Development.



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PREFACE

Realising reform in school curriculum beyond the 21st century and revitalising teacher education have been set as prioritised agendas in SEAMEO countries. On this demand, SEAMEO Basic Education Standards: Common Core Regional Learning Standards (SEA-BES:CCRLS) in Mathematic was published in 2017 for the main purpose of strengthening collaboration on curriculum standards and learning assessment across different educational systems in SEAMEO countries. In order for this document to be understood beyond the curriculum developers, supporting materials need to be developed for helping other users such as classroom teachers and teacher educators to acquire a deeper understanding of the standards. This book is an initiative to provide such support. With this support, it is anticipated that teachers and teacher educators will be able to innovate their classroom practices for developing competency and professional development aligning with the trends of the 4th Industrial Revolution.

Mathematics Challenges for Classroom Practices at the Lower Secondary Level consists of mathematical tasks for the following strands:

- Numbers and Algebra
- Relations and Functions
- Space and Geometry
- Statistics and Probability

These tasks are prepared to be used for pre-service and in-service mathematics teacher education. Its main purpose is to help readers develop mathematical knowledge for teaching (MKT) which consists of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) (see, Ball, Thames, and Phelps, 2008)[1]. In developing the tasks, the English edtion of Japanese mathematics textbooks publised by Gakko Tosho[2] had been used as the main reference. These textbooks provided major guides for learning mathematics through problem solving approach to develope mathematical thinking. As such, the tasks in this book are focused on mathematical ideas and ways of thinking. Basically, the tasks are developed in three ways: (a) analysing misconception of ideas, (b) developing ideas from previously learnt knowledge, and (c) using inquiry-based investigation to learn new ideas. In designing the tasks, the importance of local contexts of the SEAMEO community had been considered. However, some essential elements of Japanese school mathematics were also incorporated into the tasks. It is hoped that these elements will set off a new breath of mathematics learning in the SEAMEO community, shaping our students to be critical and creative thinkers in the era of artificial intelligence and data science.

Each task is written based on a standard in SEA-BES: CCRLS in Mathematics and it serves to clarify (a) the curriculum knowledge of teaching in PCK, and (b) the mathematical ideas and ways of thinking on SMK related to the standard. Apart from that, SEA-BES CCRLS in Mathematics is used as the basic source for MKT. Since SEA-BES CCRLS in Mathematics was developed with curriculum specialists of SEAMEO countries, solving the tasks will also provide a bird's eye view of their national curriculum to the readers. Furthermore, it will broaden their perspective of mathematical ideas, ways of thinking and curriculum sequence with respect to their use of local textbooks.

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1. Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What makes it special? Journal of Teacher Education, 49(5), 389-407.
2. Hitotsumatsu, S. et al (2005). Study with your friends: mathematics for elementary school (G. 1-6). Tokyo: Gakko Tosho./ Isoda, M., Murata, A., Yap, A. (2011, 2015, 2020) Study with your friends: mathematics for elementary school (G. 1-6). Tokyo: Gakko Tosho/ Isoda, M., Tall, D. (2019). Junior High School Mathematics (G. 1-3). Tokyo: Gakko Tosho. For developing tasks in this book, authors are inspired by the tasks and task sequence of these mathematics textbooks.

Although the targeted readers of this book are teachers and teacher educators, most of the tasks can also be solved by students in classrooms as they are also aligned with school mathematics curriculum. In addition, teachers and teacher educators are expected to solve them without much difficulties. However, studying the tasks carefully will raise the awareness of the depth of SMK and PCK required to complete the tasks. This will triggered off a need to upskill their understanding of mathematical ideas and function effectively to develop students mathematical competency. Thus, solving tasks in this book will provide readers the opportunities to relearn the mathematics content for teaching. Furthermore, it will also help them to identify (a) the objectives of teaching the content, (b) the gap between students' prior knowledge and what is to be learnt, (c) what and how students reorganise the content knowledge of their learning, (d) students' difficulties in learning the content, and (e) what ideas will be developed through their new learning.

Readers may also choose to work with any task according to their interests. It is not necessary to work out all the tasks according to the sequence in the book. However, for a deeper understanding of the mathematical ideas embeded in a task, it is recommended that readers should solve the task in the following manner: (a) solve the task by themselves and read the related standards, (b) communicate their solutions with others to identify what is really new content for them, and (c) paraphrase and summarise the communication with others based on the perpective of mathematical ideas and ways of thinking that align with the framework of SEA-BES CCRLS as described in Chapter One.

This book is recommended for use in many ways and various contexts. Firstly, as all the tasks were designed based on school mathematics curriculum, so they can be used directly by students as learning tasks. In addition, teachers can also use the book as a quick guide to create similar mathematical tasks that incorporate mathematical thinking. Secondly, when the book is used in the context of in-service teacher education such as in lesson study, teachers can solve the tasks in this book as a step to gain a deep understanding of the mathematical ideas in order to prepare a unit of lesson plan based on the standard chosen. This may help to improve the effectiveness of lesson planning and anticipate responses of students to the tasks. Thirdly, in the context of pre-service teacher education, the tasks in this book can serve as a mean to acquire MKT which may be required for any teacher employment examination or entrace examination for an education graduate programme. Fourthly, in the context of mathematics education research, this book can be used as a reference for MKT. Last but not least, when the book is used in the context of curriculum reform and textbook revision, it could serve as a guide to formulate new objectives and tasks which are not existing in their current curriculum and textbooks.

Mathematics Challenges for Classroom Practices at the Lower Secondary Level is the outcome of many contributions of educators and academia from different leading institutions. In order to ensure good quality of content and streamline the presentation of the writings, many rounds of editing and rewriting were unavoidable. It is our hope that the mathematical ideas and ways of thinking promoted through this book will enhance the teachers' capacities to develop their students' potentials in facing the challenging and demanding era ahead.

Editors
GAN Teck Hock
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The authors/editors would like to thank and express our gratitude to the following contributors, institutions and organisations in making this project and publication possible:

Dr Ethel Agnes Pascua-Valenzuela, Director of SEAMEO Secretariat, Bangkok, Thailand for providing CRICED the opportunities to collaborate with RECSAM and other SEAMEO centres;

Dr Shah Jahan Bin Assanarkutty, the Centre Director of SEAMEO RECSAM for his encouragement and support in realising the publication of the books;

Dr Johan Bin Zakaria, the ex-Centre Director of SEAMEO RECSAM for his confidence and generosity of granting permission to use the SEABES: CCRLS in mathematics to develop this guidebook series;

The University of Tsukuba generously supported the funding of this guidebook series project. The books will be disseminated to SEAMEO member countries and recommended for use by teachers, teacher educators for the benefit of the SEAMEO community;

Assoc. Professor Dr Maitree Inprasitha, the Vice President of Khon Kaen University(KKU) and Director of the Institute for Research and Development in Teaching Profession (IRDTP) for ASEAN, Thailand, for providing the support in organising a workshop to lead the KKU and other affiliated Universities lecturers in contributing writings to this guidebook series;

Dr Suppattra Pativisa, Senior Professional Expert of The Institute for the Promotion of Teaching Science and Technology (IPST), Thailand, in lending the support and led the lecturers in contributing writings to the guidebook series;

Dr Wahyudi, the former director of SEAMEO QITEP in Mathematics (SEAQIM) and currently the Deputy Director of SEAMEO Secretariat, together with Dr Sumardyono, the current Director of SEAQIM, provided the support to lead their mathematics lecturers in contributing the writings;

The specialists of APEC economies of the year September 2017 during the 13th APEC-Khon Kaen University Conference, and the year February 2018, in providing suggestions and contributions of resources in helping to conceptualise the content and outcome of the books;

All curriculum specialists and educators who attended the SEA-BES CCRLS in Mathematics workshop in March 2017, held in SEAMEO RECSAM, contributed ideas and suggestions on shaping the outcome of the guidebooks, as well as those who followed the sessions on how to write the tasks and submitted the writings for considerations;

Pedro Lucis Montecillo Jr, mathematics specialist of RECSAM (until May 2018) for assisting the coordination at the early stage of the project;

All staff of SEAMEO RECSAM who supported the production of these books by providing ideas, reviewing the contents, coordination and manage the distribution of books to other institutions.

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CHAPTER 1

Guide to the SEA-BES CCRLS Framework in Mathematics

The SEAMEO Basic Education: Common Core Regional Learning Standards (SEA-BES CCRLS) was developed and directed to create a harmonious SEAMEO Member community in the era of artificial intelligence and data science through mutual understanding. In this respect, the CCRLS framework in Mathematics (2017) outlines three basic components towards developing creative, competent and productive global citizens essential for achieving this aim. The comprehensive illustration of the framework is attached in Appendix A. The revised framework in Figure 1 shows the interconnection of the three components.

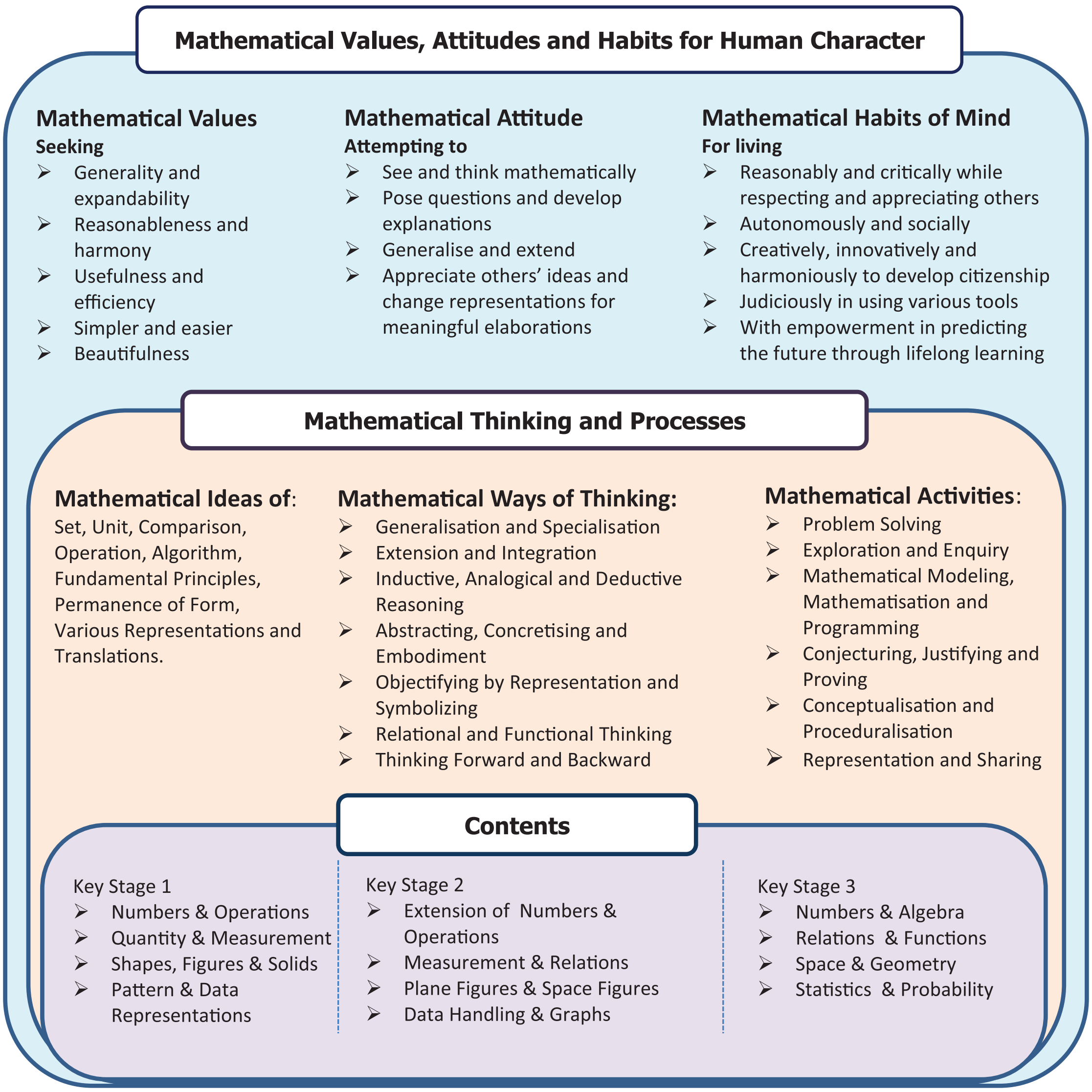


Figure 1. Revised CCRLS Framework in Mathematics

This book is written to guide readers acquire a better understanding of this framework particularly on mathematical thinking and processes which are embedded in all the tasks. The detailed explanation of mathematical ideas, mathematical ways of thinking and mathematical activities can be found in Appendix B. The standards for the strand on Mathematical Processes-Humanity is attached in Appendix C to provide readers with challenging activities to promote metacognitive thinking at different level of arguments to make sense of mathematics. In order to understand the development and progression of learning from the primary level to the secondary level (Key Stage 3), the learning standards of Key Stage 1 and Key Stage 2 can be referred in Appendix D and Appendix E, respectively.

The interconnection of the three components is shown in Figure 2. The ultimate aim of the CCRLS framework is to develop mathematical values, attitudes and human characters which are the essence of a harmonious society. This component is closely related to the affective domain of human character traits which correspond to soft skills that can be developed through appreciation. In relation to this, acquisition of mathematics contents as hard skills and reflection on the thinking processes are needed to inculcate the capability of appreciation. The reflection is necessary for learners to recognise their cognitive skills derived through the contents. Even though contents appeared to be learned independently through acquisition, the mathematical thinking and process, and the appreciation of mathematical values, attitudes and habits for human character is possible to be developed through reflective experiences.

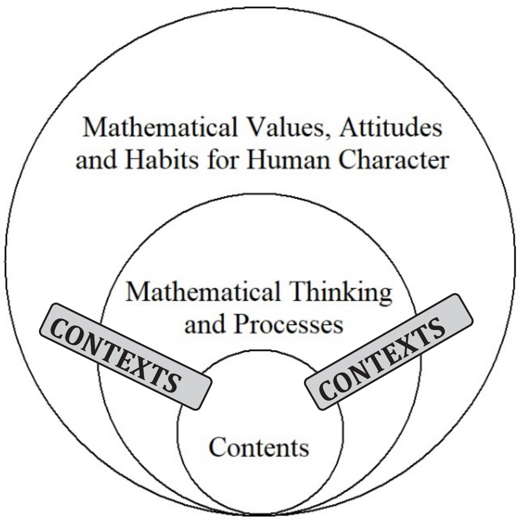


Figure 2. Interconnection of Components in CCRLS Framework in Mathematics

The three components will not be ideally operationalised without appropriate contexts. The tasks in this book provide the contexts for developing the mathematical thinking and processes, which are the key learning objectives. Completing the tasks correspond to gauging the readers' acquired mathematical knowledge for teaching. Thus, it is recommended that readers should constantly reflect on the appropriateness of their solutions to the tasks. Other than this, comparing solutions and discussion with others should always be done habitually in order to gain a deeper understanding of the mathematical processes. This may enable the readers to discover any hidden mathematical ideas and structures in the tasks with appreciation. The tasks are specifically designed to cater for this purpose.

In a nutshell, an important target of solving the tasks in this book is to enable readers to acquire a better insight of the learning standards. This insight will in turn help them to understand and appreciate their national mathematics curriculum from the perspective of SEABES CCRLS. Furthermore, since the learning standards are developed based on the framework which emphasised on the components of contents, thinking and processes, as well as values of mathematics, ultimately, readers will be able to acquire mathematical teaching knowledge with appreciation.

CHAPTER 2

Numbers and Algebra [K3NA]

Topic 1: Extending Numbers to Positive and Negative Numbers [K3NA1]

Standard 1.1: [K3NA1-1]

Extending numbers to positive and negative numbers and integrate four operations into addition and multiplication

i. Understand the necessity and significance of extending numbers to positive and negative numbers in relation to directed numbers with quantity
ii. Compare numbers which is greater or less than on the extended number line and use absolute value for distance from zero
iii. Extend operations to positive and negative numbers and explain the reason
iv. Get efficiency on calculation in relation to algebraic sum

Sample Tasks for Understanding the Standards

Task 1: Positive and Negative Numbers in the Real World

Altitude or Depth of a Location

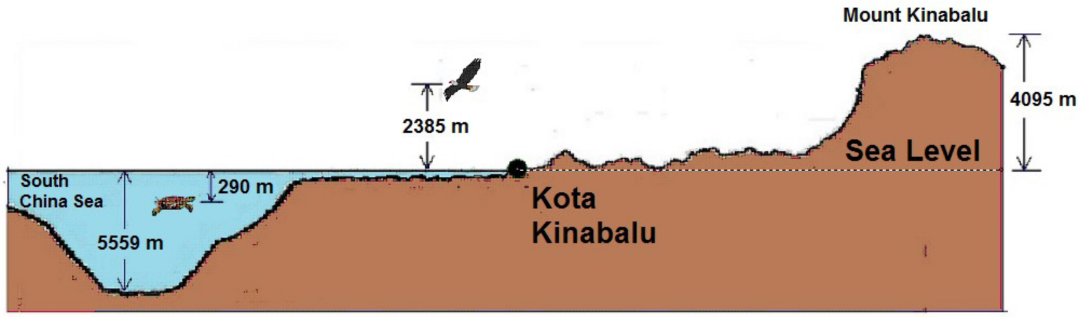


Diagram 1

Altitude is the vertical distance above sea level and depth is the vertical distance below sea level.

Diagram 1 shows the cross sectional view of Mount Kinabalu to its nearest city, Kota Kinabalu which is next to the South China Sea.

i. Use numbers to indicate the altitude or depth of the following locations:

- The summit of Mount Kinabalu
- The eagle
- Kota Kinabalu City
- The sea turtle
- The seabed of South China Sea

ii. How can numbers be used to differentiate locations above and below the sea level?

Elevator with Positive and Negative Floors

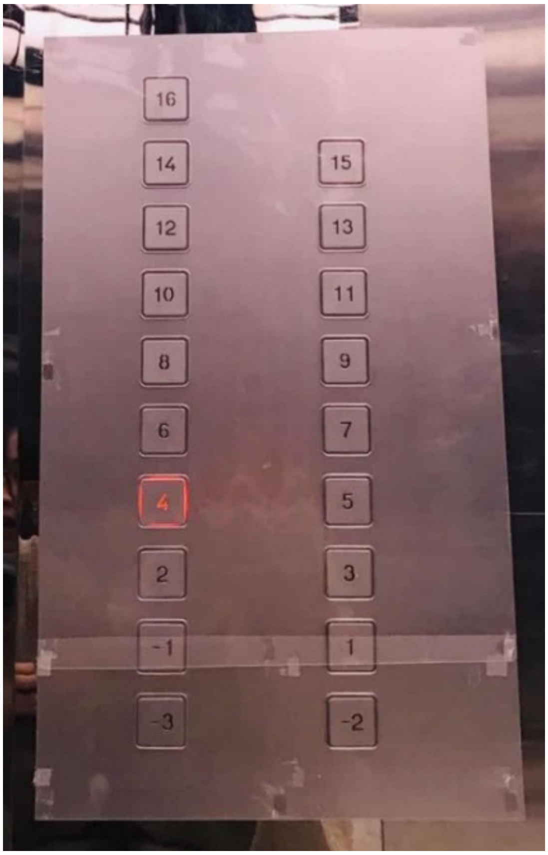
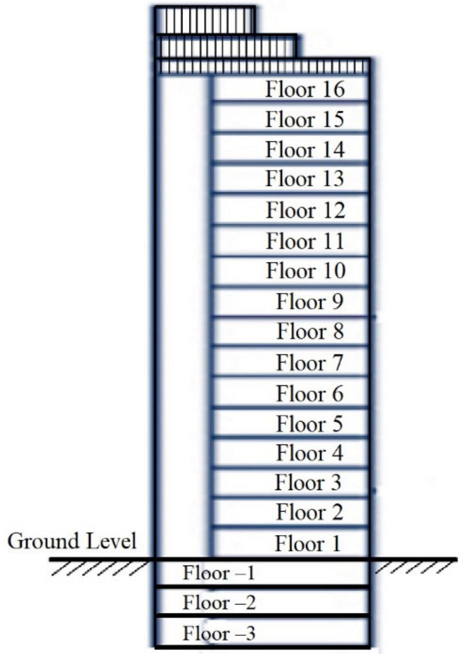


Diagram 2

Diagram 2 shows a button panel of an elevator. In this elevator, positive numbers are used to label floors on and above the ground, whereas negative numbers are used to label basement floors as shown below.



i. Are the positive and negative numbers used correctly in labelling the floors? Why or why not?

ii. What modifications to the button panel would you suggest?

iii. Explain the reasons for your suggestions.

Task 2: Comparing Positive and Negative Numbers

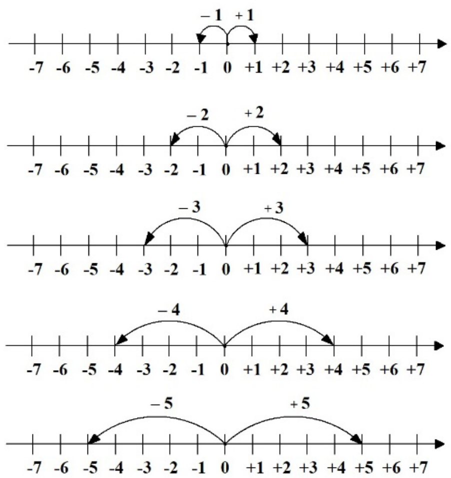


Diagram 3

i. Diagram 3 shows five number lines used to represent the numbers +1, +2, +3, +4, +5, –1, –2, –3, –4 and –5.

- Describe how positive and negative numbers are represented in a number line.
- With the use of real-world examples, explain why +5 > +3, but –5 < –3.

ii. Diagram 4 shows a number line for zero and positive numbers whereas Diagram 5 shows an extended number line to include the negative numbers.

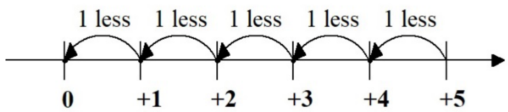


Diagram 4

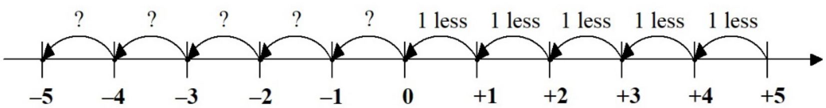


Diagram 5

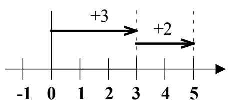
As shown in Diagram 4, any number on the number line is always 1 less than the number to its right.

- Does the same rule still apply for the extended number line?
- Based on this rule, what can you conclude on using a number line to compare positive and negative numbers?
- Delete the wrong word in the (   ).
    - ❖ All positive numbers are (less/greater) than zero.
    - ❖ All negative numbers are (less/greater) than zero.
    - ❖ All positive numbers are (less/greater) than negative numbers.
    - ❖ For any two positive numbers, the one with larger absolute value is (smaller/ larger).
    - ❖ For any two negative numbers, the one with larger absolute value is (smaller/ larger).

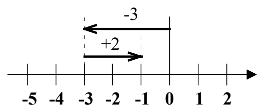
iii. Why is number line a good tool to represent positive and negative numbers?

Task 3: Addition and Subtraction of Positive and Negative Numbers Using Number Line

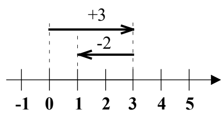
Addition Using Number Line



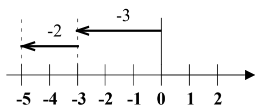
(a) $(+3) + (+2) = \square$



(b) $(-3) + (+2) = \square$



(c) $(+3) + (-2) = \square$



(d) $(-3) + (-2) = \square$

Diagram 6

On a number line, adding a positive or negative number can be performed by moving the number of steps toward the positive (right) or the negative (left) direction.

i. Diagram 6 shows four additions involving positive and negative numbers performed using number lines.
   - Draw an arrow on each of the number lines to show the answer □ for each of the additions.

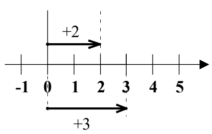
ii. A student claims that "adding a negative number is the same as subtracting the positive part of the number."
   - Justify that this claim is always true.

Subtraction Using Number Line

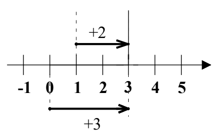
(a) $(+3) - (+2) = \square$

Think as

$(+2) + \square = (+3)$



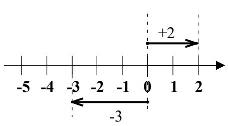
$\square + (+2) = (+3)$



(b) $(-3) - (+2) = \square$

Think as

$(+2) + \square = (-3)$



$\square + (+2) = (-3)$

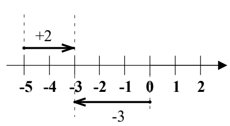


Diagram 7

Subtraction is the inverse operation of addition. Thus, a subtraction can be thought of as an addition with a missing addend. As an example, (+3) – (+2) = □ can either be thought of as

(+2) + □ = (+3) or □ + (+2) = (+3).

i. Diagram 7 shows two subtractions involving positive and negative numbers performed using number lines.
   - Draw an arrow on each of the number lines to show the answer □ for the two ways of thinking for each of the subtractions.

ii. Perform the following subtractions using number lines:
   - (+3) – (–2)
   - (–3) – (–2)

iii. Which of the two ways of thinking about subtraction do you find it easier to perform subtraction involving positive and negative numbers using number line? Why?

Task 4: Addition and Subtraction of Positive and Negative Numbers Using Playing Cards

Representing Numbers with Playing Cards

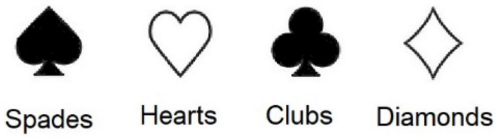


Diagram 8

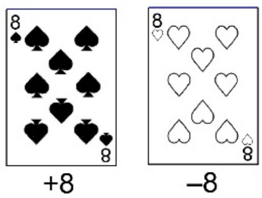


Diagram 9

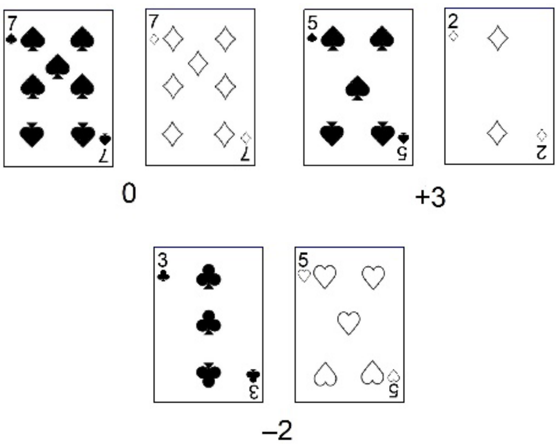


Diagram 10

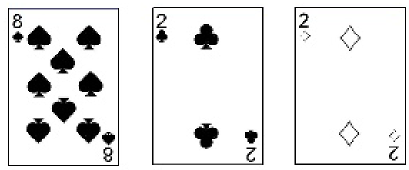
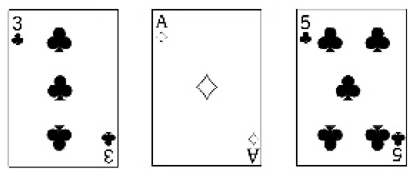
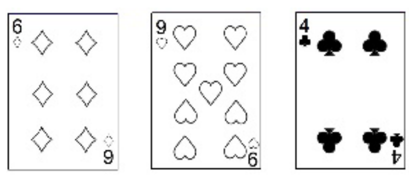
Diagram 8 shows the shapes of the four suits in a set of playing cards.

Black shapes (Spades & Clubs) are used to represent positive numbers, and red shapes (Hearts & Diamonds) are used to represent negative numbers.

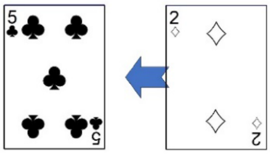
Diagram 9 shows +8 and –8 represented by 8 of Spades and 8 of Hearts respectively.

When two or more cards are used to represent a number, equal numbers of black and red shapes will cancel off and become zero. Diagram 10 shows three examples of numbers represented by two cards.

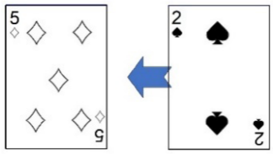
i. What numbers are represented by the following sets of cards?

- 
- 
- 

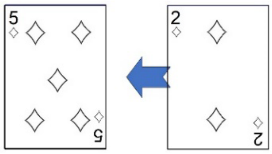
Addition Using Playing Cards



(+5) + (–2) = +3



(–5) + (+2) = –3



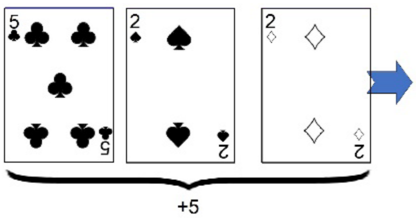
(–5) + (-2) = –7

Diagram 11

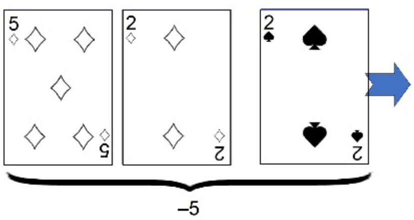
When using playing cards, addition can be performed by “putting in” cards. Diagram 11 shows three examples of addition performed using playing cards.

i. Use playing card to perform the following calculations:
   - (+8) + (–5)
   - (–4) + (–2)
   - (–1) + (+4)

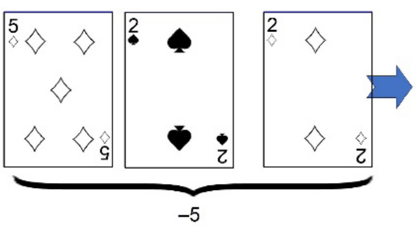
Subtraction Using Playing Cards



(+5) – (–2) = +7



(–5) – (+2) = –7



(–5) – (–2) = –3

Diagram 12

Subtraction is the inverse operation of addition. So, it can be performed by “taking away” cards. Diagram 12 shows three examples of subtraction performed using playing cards.

i. Use playing card to perform the following calculations:
   - (+7) – (+2)
   - (+3) – (–6)
   - (–3) – (–8)
   - (–5) – (+7)

Task 5: Comparing Number-Line Method and Playing-Card Method

Number lines and playing cards are two methods of performing addition and subtraction. These two methods have some distinct similarities and differences in terms of the mathematical ideas involved.

i. Compare and contrast these two methods.

ii. Which of these two methods do you think will be easier for your students? Why?

iii. Which of these two methods do you think can help your students learn other mathematical ideas such as vector in their future grades?

Task 6: Addition and Subtraction Rules Involving Negative Numbers

$$
\begin{array}{cclcl}
(+5) & + & (+2) & = & (+7) \\
 & & \downarrow \text{1 less} & & \downarrow \text{1 less} \\
(+5) & + & (+1) & = & (+6) \\
 & & \downarrow \text{1 less} & & \downarrow \text{1 less} \\
(+5) & + & 0 & = & (+5) \\
 & & \downarrow \text{1 less} & & \downarrow \text{1 less} \\
(+5) & + & (-1) & = & (+4)
\end{array}
$$

Diagram 13

A student was asked to find the sum (+5) + (-1) and he gave the answer +4. When asked to explain his method to get the answer, the student's reasoning is shown in Diagram 13. He further explained that "adding a negative number is the same as subtracting the corresponding positive number."

i. Is the student's reasoning valid? Why or why not?

ii. Using the example (+5) – (–1) = (+6), how would you use the same reasoning to convince other students that "subtracting a negative number is the same as adding the corresponding positive number"?

Task 7: Application of Negative Numbers in Real World

In an international athletics competition, depending on its direction of blow, a tailwind is considered either assisting or opposing the effort of the runners. According to the International Association of Athletics Federation (IAAF) Competition Rules, a tailwind of more than +2 m/s will make the results not considered as a record although the results are considered valid for that competition.

Table 1
Current World Records for Man's 100 m Sprint

Record Holder & Country	Usain Bolt, Jamaica
Time	9.58 s
Tailwind	+0.9 m/s
Venue	World Athletics Championship, Berlin, Germany
Date	16 Aug 2009

i. Table 1 shows the current world records for 100 m sprint for the man category.

- Why was the tailwind of +0.9 m/s considered as an advantage to Usain Bolt?
- Usain Bolt's run was the result of his own running speed assisted by the tailwind. Calculate the resultant speed of Usain Bolt's run.
- Assuming the tailwind was pushing Usain Bolt's whole body forward at the speed of +0.9 m/s. Calculate Usain Bold's own running speed without the assistance of the tailwind.

ii. The current woman record holder for South America Continent is Rosângela Santos of Brazil. Her record is 10.91 s with a tailwind of – 0.2 m/s.

- Why was this tailwind of – 0.2 m/s considered as a disadvantage to Rosângela Santo?
- Calculate the resultant speed of Rosângela Santo with the influence of the tailwind.
- Calculate the actual speed of Rosângela Santo without the influence of the tailwind.

Task 8: Real-World Multiplication and Division Involving Positive and Negative Numbers

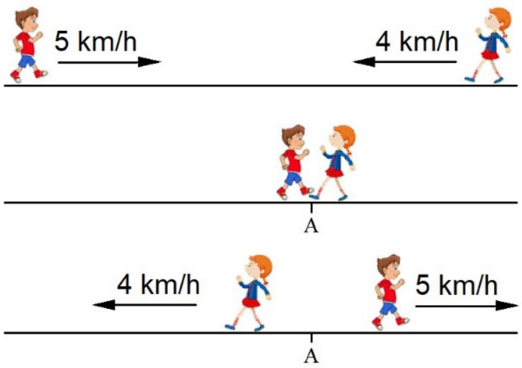


Diagram 14

Diagram 14 shows a boy and a girl walking along a straight track. The boy walks to the right from one end of the track with a speed of 5 km/h, and the girl walks in the opposite direction from another end with a speed of 4 km/h.

Both the boy and the girl cross at location A and continue their walking.

i. Use multiplication sentences to represent the respective distance of the boy and the girl from location A at the following times:

- 2 hours after they crossed.
- 2 hours before they crossed
  [Hint: Consider the direction of movement (right or left) and time (after or before).]

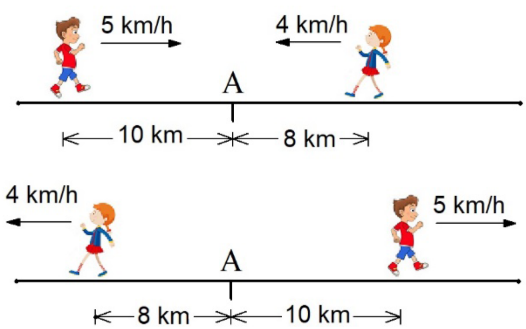


Diagram 15

ii. Diagram 15 shows two instances of the boy and girl during the walk. Use division sentences to represent the respective time for the two instances.

Task 9: Multiplication and Division Rules Involving Positive and Negative Numbers

Justify by the Distributive Rule of Multiplication

(-3) x 0 = 0 … *any number multiplied by* 0 *is* 0

(-3) x [(+2) + (-2)] = 0 ... *since* $(+2) + (-2) = 0$

Expand by Distributive Rule,

[(-3) x (+2)] + [(-3) x (-2)] = 0

(-6) + [(-3) x (-2)] = 0 ... *since* $(-3) \times (+2) = (-6)$

∴ [(-3) x (-2)] = (+6) ... *since* $(-6) + (+6) = 0$

Justify by Permanence of Form

(-3) x (+2) = (-6)

↓ 1 less ↓ +3

(-3) x (+1) = (-3)

↓ 1 less ↓ +3

(-3) x 0 = 0

↓ 1 less ↓ +3

(-3) x (-1) = (+3)

↓ 1 less ↓ +3

(-3) x (-2) = (+6)

Diagram 16

$24 \div 3 \times (-2)$
$= 24 \div (-6)$
$= -4$

$24 \div 3 \times (-2)$
$= 24 \times \frac{1}{3} \times (-2)$
$= -(24 \times \frac{1}{3} \times 2)$
$= -16$

Diagram 17

i. Diagram 16 shows two different approaches used by two students to justify that $(-3) \times (-2) = (+6)$.

- Which of these approaches do you think is easier to convince your students? Why?
- Justify that $(+3) \times (-2) = (-6)$.
- Justify that $(-8) \div (-2) = (+4)$.

ii. Two students calculated $24 \div 3 \times (-2)$ in two different ways and got different answers as shown in Diagram 17.

- Which student is correct? Explain your reasons.
- What was the mistake made by the student who got the wrong answer?
- How would you help your students to avoid such mistake?

iii. Calculate

- $(-12) \div (-2) \times 5$
- $4 \times (-5) \div 12$
- $(-36) \div (-8) \div (-3)$
- $\frac{2}{3} \div (-\frac{9}{5}) \div 5$

Task 10: Calculation Involving Algebraic Sum

$(+2) - (+5) = (+2) + (-5)$

... because subtracting a positive number is the same as adding the corresponding negative number.

$(+2) + (-5)$ is the algebraic sum of two terms, the positive term is $(+2)$ and the negative term is (-5).

So, the sum of $(+2)$ and (-5) is (-3).

$(+2) - (-5) = (+2) + (+5)$

... because subtracting a negative number is the same as adding the corresponding positive number.

So the sum of $(+2)$ and $(+5)$ is $(+7)$.

Diagram 18

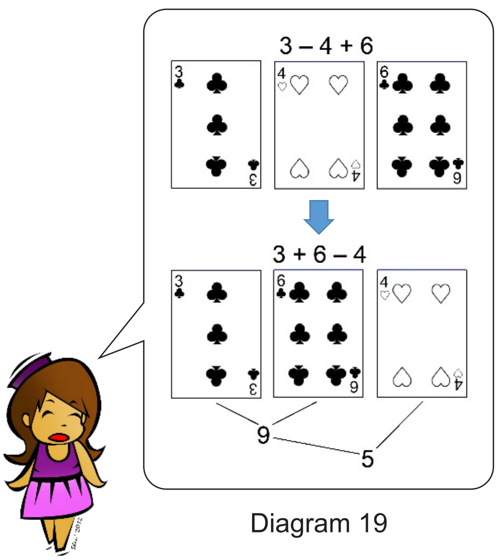


Diagram 19

i. A student claims that subtraction involving positive and negative numbers is easier to calculate if the subtraction is change to addition. Diagram 18 shows the explanation given by the student based on two examples.
- Is the student's claim valid? Why or why not?

ii. Change the following into addition-only mathematics expressions and then find the sum for each expression.
- $(+8) - (+2)$
- $(+6) - (-4)$
- $(-9) + (-3) - (-5)$
- $(-5) - (-2) - (-8)$
- $(+2) + (-0.6) - (+1.8)$

iii. Find the value of $(a + b + c)$ for each of the following sets of a, b and c.
- $a = 5, b = -8, c = -3$
- $a = -6.5, b = 2, c = 4$
- $a = -3\frac{1}{2}, b = -4, c = -\frac{1}{2}$

iv. Another student claims that the commutative property, $a + b = b + a$, which only hold true for addition of any real numbers a, b and c can also be used to do calculation involving subtraction. She gives the following example to support her claim.

$$\boxed{3 - 4 + 6 = 3 + 6 - 4}$$

Then, she uses playing card to explain her example as shown in Diagram 19.
- Is the second student's claim valid? Why or why not?

v. Calculate
- $12 - 18 + 14$
- $-12 + 3 - 4 + 6$
- $2 - 5 + 7 - 1$
- $-2.3 - 5.7$
- $1 - 1.8 - 0.4$
- $-\frac{1}{3} + \frac{2}{3} - (-\frac{1}{6})$

Topic 2: Utilising Letters for Algebraic Expressions and Equations [K3NA2]

Standard 2.1: [K3NA2-1]

Extending the utilisation of letters for general representation of situations and find ways to simplify algebraic expressions

i. Appreciate the utilisation of letter for general representation of situations to see the expression as process and value
ii. Find ways to simplify expressions using distributive law and figural explanations, establish the calculation with like and unlike letter
iii. Acquire fluency of simplifying expression and appreciate it for representing the pattern of situation

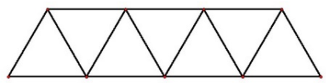
Sample Tasks for Understanding the Standards

Task 1: Representing Number Patterns by Algebraic Expressions

Truss-Bridge Problem

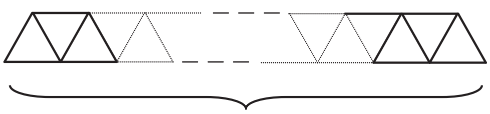
Diagram 1

i. Diagram 1 shows a truss bridge made up of many triangles. 15 steel beams are needed to build the following bridge with 7 triangles.



- How many steel beams are needed to build a truss bridge with 11 triangles?

i. There are n number of triangles in the following truss bridge.



n Δ

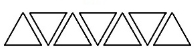
(Note: n is an odd number.)

- Write an algebraic expression in term of n to represent the number of steel beams needed to build the bridge.

ii. Diagrams 2, 3 and 4 show the reasoning of three students in solving the Truss-Bridge Problem.

Student A's Reasoning

For 7 triangles

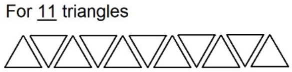


(7 x 3) – 6 = 15 total beams
7 sets of △ beams take away 6 overlaping beams

For 9 triangles

(9 x 3) – 8 = 19 total beams
9 sets of △ beams take away 8 overlaping beams

For 11 triangles



(11 x 3) – 10 = 23 total beams
11 sets of △ beams take away 10 overlaping beams

So, for n triangles ⟶ (n x 3) – (n – 1) total beams
n sets of △ beams and take away n less 1 overlaping beams

Diagram 2

Student B's Reasoning

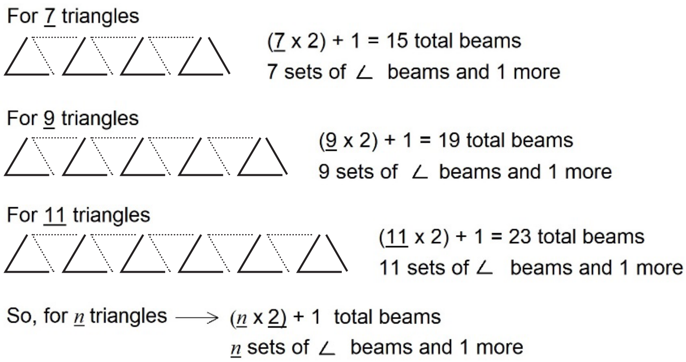


Diagram 3

Student C's Reasoning

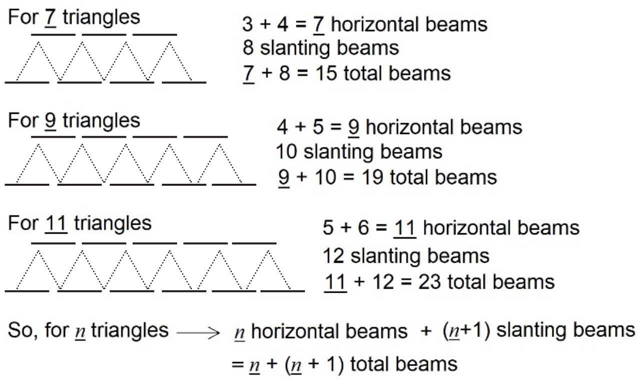


Diagram 4

The three students viewed n to represent different things in their reasoning and ended up in getting three different expressions in term of n.

Student A: $n \text{ x } 3 - (n - 1)$
Student B: $(n \text{ x } 2) + 1$
Student C: $n + (n + 1)$

- What was n representing in Student A's reasoning?
- What was n representing in Student B's reasoning?
- What was n representing in Student C's reasoning?

iii. If the students have learned the simplification of algebraic expressions, they will see that these three different expressions will be simplified to be a same expression, that is $2n + 1$. However, if the students have not learned how to simplify algebraic expressions, it will be difficult for them to view the three expressions as representing a same solution to the Truss-Bridge problem.

- Without simplifying the expressions, how would you help your students see that the three different expressions are representing a same solution?

Task 2: Simplifying Algebraic Expressions

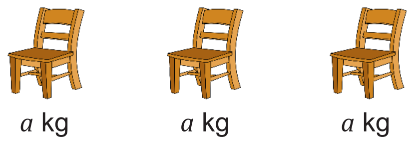


Diagram 5

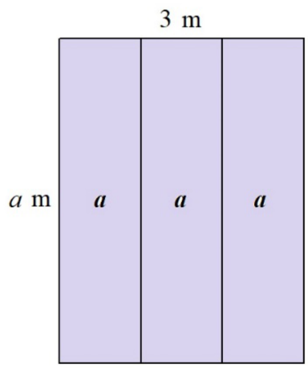


Diagram 6

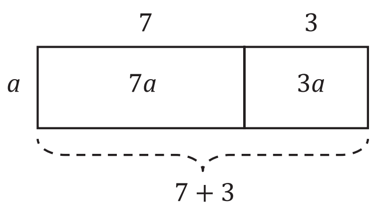


Diagram 7

$$\frac{6x+1}{2} = \frac{\cancel{6}^{3}x+1}{\cancel{2}^{1}} = 3x+1$$

Diagram 8

i. Diagram 5 shows 3 chairs each weighing a kg.

- What does $3a$ mean with respect to the chairs?
- Write an addition sentence involving $3a$.

ii. Diagram 6 shows a floor mat which is 3 m wide and a m long.

- What does $3a$ mean with respect to the floor mat?
- Write a multiplication sentence involving $3a$.
- How does this multiplication sentence relate to the addition sentence in (i)?

iii. A student is having difficulty to understand that $7a + 3a = 10a$. Diagram 7 shows a drawing used by a teacher to help the student overcome the difficulty.

- Show how the distributive property of multiplication can be used to verify that $7a + 3a = 10a$.
- Show how the distributive property of multiplication can be used to verify that $7a - 3a = 4a$.

iv. Diagram 8 shows a student's work when simplifying an algebraic expression.

- What is the student's misconception?
- Simplify $\frac{6x+1}{2}$ by using the following methods.
  - ❖ Draw a diagram.
  - ❖ Use the distributive property of multiplication.

v. A student claims that $3a + 2b = 5ab$.

- What is the student's misconception?
- How would you help the student to correct the misconception?

Standard 2.2: [K3NA2-2]
Thinking about set of numbers in algebraic expression with letters as variables and represent them with equality and inequality

i. Recognise numbers as positive and negative numbers, and explain integers as a part of numbers
ii. Represent a set of numbers using variables with equality and inequality
iii. Translate given sets of numbers on the number lines using interval and inequality notations
iv. Appreciate redefining of even and odd numbers using letters to represent different sets of variables

Sample Tasks for Understanding the Standards

Task 1: Sets of Numbers in Algebraic Expressions

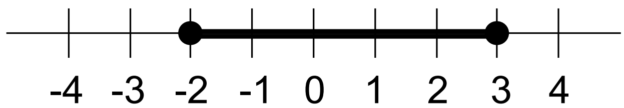


Diagram 1

The dark line segment in Diagram 1 represents the value of x for $-2 \leq x \leq 3$.

Given that x is an integer.

i. Why is the diagram **not** correctly representing the values of x?

ii. Make correction to the diagram.

iii. What if x is a real number and $-2 \leq x < 3$?

Task 2: Redefining Even and Odd Numbers

A and B are the sets of positive odd and even numbers respectively.

- A $= \{1, 3, 5, 7, 9, 11, 13, ...\}$
- B $= \{2, 4, 6, 8, 10, 12, 14, ...\}$

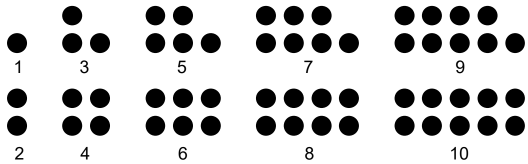


Diagram 2

Diagram 2 shows the first five odd and first five even numbers represented by black dots.

i. Given that n is an integer.
- Write in terms of n, an algebraic expression to represent
  - ❖ Set A
  - ❖ Set B
- State clearly the possible values of n for each algebraic expression in (i).

ii. Prove that:
- The sum of two even numbers is always even.
- The sum of two odd numbers is always even.

If n is an integer, then

- $2n$ is an even number.
- both $2n + 1$ and $2n - 1$ are odd numbers.

Diagram 3

iii. Diagram 3 shows a student's confusion when he read the statement about both $2n + 1$ and $2n - 1$ are odd numbers for any integer n.
- Using specific values of n, construct a table to verify the statement to the student.

iv. In this case, the letter n functions as a variable and the expression $2n$ represents the condition of the set of even numbers, whereas the expression $2n + 1$ and $2n - 1$ represent the condition of the set of odd numbers.
- What is a variable?
- Why is variable important in Algebra?

Standard 2.3 : [K3NA2-3]
Thinking about how to solve simple linear equation

i. Review the answers of equations from the set of numbers and thinking backward
ii. Know the properties of equations which keep the set of answers of equation
iii. Appreciate the efficient use of the properties of equations to solve linear equation
iv. Use equations based on life situations to develop fluency, to solve equation, and interpret the solution

Sample Tasks for Understanding the Standards

Task 1: Equality and Inequality

Diagram 1

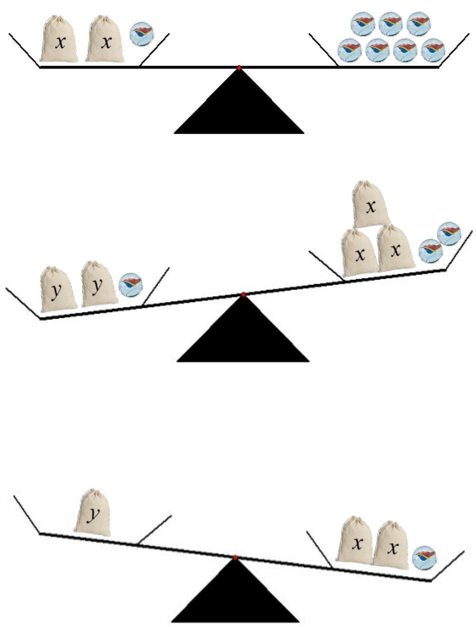


Diagram 2

Diagram 1 shows two bags and a marbles. One bag has x number of marbles in it and the other bag has y number of marbles. All marbles are the same.

Diagram 2 shows three situations comparing the bags and marbles.

i. Find the number of marbles in

- Bag x
- Bag y

ii. Describe your process of finding the solutions for (i).

Task 2: Properties of Equality

i. Solve the equations by inspection:

- $a + 2 = 6$ Equation 1
- $b - 2 = 6$ Equation 2
- $3x = 12$ Equation 3
- $\frac{y}{2} = 3$ Equation 4

ii. Diagram 3 shows the processes of solving the four equations in (i).

- Fill in the equation for each $\boxed{?}$.

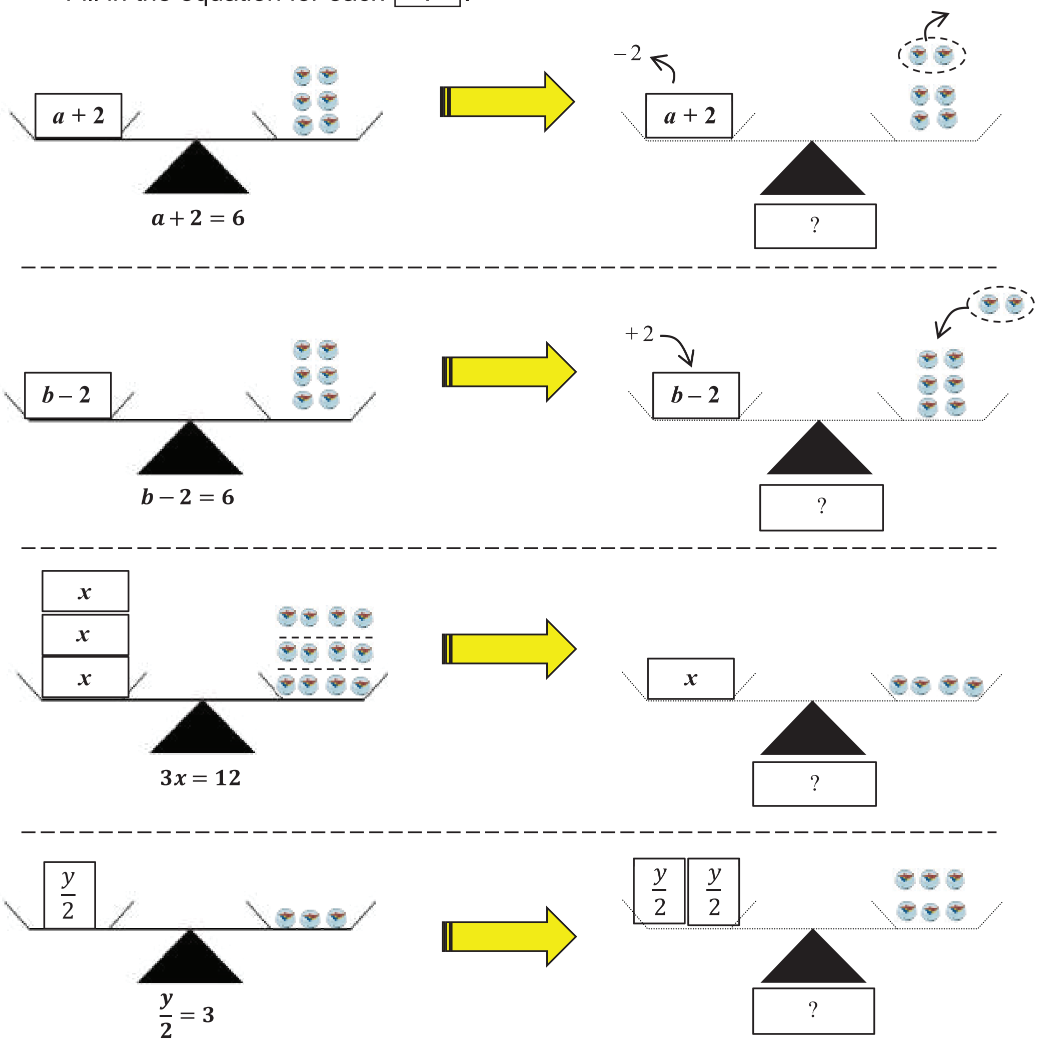


Diagram 3

iii. The process of solving equations 1 to 4 as illustrated in Diagram 3 reflect four properties of equality.

- State the four properties of equality.

Task 3: Applying Properties of Equation in Solving Equations

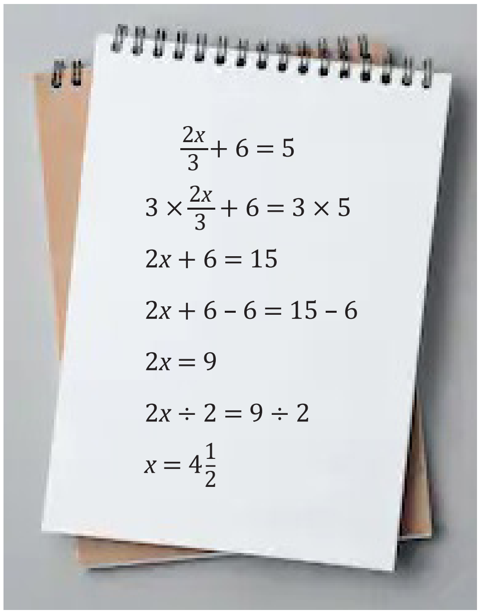


Diagram 4

i. What does it mean by solving an equation?

ii. Diagram 4 shows the work of a student solving the equation $\frac{2x}{3} + 6 = 5$.

- What is the error made by the student?
- How would you guide the student to realise that $x = 4\frac{1}{2}$ is a wrong solution?
- How would you help the student to correct his error?

Topic 3: Algebraic Expressions, Monomials and Polynomials and Simultaneous Equations [K3NA3]

Standard 3.1: [K3NA3-1]

Thinking about the calculations of monomial and polynomials, simple case

i. Introduce terms, monomials and polynomials
ii. Introduce a number raised to the power of two as square, and a number raised to the power third as cube
iii. Get fluency for the calculation of polynomials such as combining like terms and the use of the four operations in simple cases

Sample Tasks for Understanding the Standards

Task 1: Monomials and Polynomial Expressions

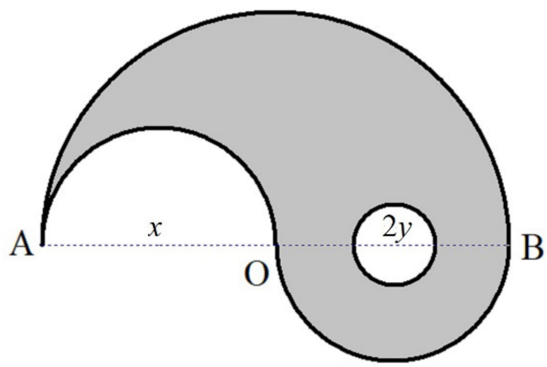


Diagram 1

Diagram 1 shows a figure formed by a big semicircle, two smaller semicircles and a small circle.

Given that AOB is the diameter of the big semicircle and $\mathrm{AO} = \mathrm{OB} = x$ cm.

i. If the diameter of the small circle is $2y$ cm, find its area and circumference as monomial expressions in terms of y.

ii. Show that the outer perimeter of the shaded region is $2\pi x$.

iii. Find the area of the shaded region as a polynomial expression in terms of x and y.

iv. If both x and y are doubled, what will happen to the outer perimeter and area of the shaded region?

Task 2: Square and Cube

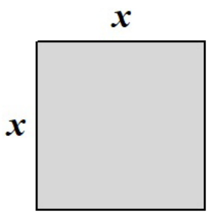


Diagram 2

Diagram 2 shows a square metal sheet with side length x units.

i. Find the area of the metal sheet in term of x.

ii. Calculate the area of the metal sheet for the following values of x.

- x = 10 cm
- x = ½ m
- x = 5.2 m

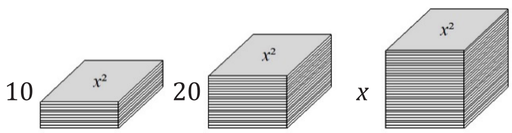


Diagram 3

iii. Identical metal sheets with area x^2 square units are stacked up to 10 cm, 20 cm and x cm high as shown in Diagram 3.

- Find the volume of the three stacks of metal sheets in terms of x, respectively.
- Calculate the volume of each stack of metal sheets for the same values of x in (ii).

Student A's Solution

$$(4x + 1) - (x - 5)$$
$$= 4x + 1 - x - 5$$
$$= 3x - 4$$

Student B's Solution

$$(4x + 1) - (x - 5)$$
$$= 4x + 1 - x + 5$$
$$= 3x + 6$$
$$= 9x$$

Diagram 4

i. Diagram 4 shows the solutions of two students, A and B, when doing calculation involving polynomials.

- For each of the student work, show where the mistake is and explain why.

Task 3: Calculation Involving Polynomials

Student A's Method

$$\frac{x+2y}{2} - \frac{x-y}{3}$$
$$= \frac{3(x+2y)}{6} - \frac{2(x-y)}{6}$$
$$= \frac{3(x+2y)-2(x-y)}{6}$$
$$= \frac{3x+6y-2x-2y}{6}$$
$$= \frac{x+8y}{6}$$

Student B's Method

$$\frac{x+2y}{2} - \frac{x-y}{3}$$
$$= \frac{1}{2}(x+2y) - \frac{1}{3}(x-y)$$
$$= \frac{1}{2}x + y - \frac{1}{3}x + \frac{1}{3}y$$
$$= \frac{3}{6}x - \frac{2}{6}x + \frac{3}{3}y + \frac{1}{3}y$$
$$= \frac{1}{6}x + \frac{4}{3}x$$

Diagram 5

i. Diagram 5 shows two students' methods to do calculation involving polynomials.

- Are the answers obtained using the two methods the same or different? Explain your reasons.
- Which method do you think is easier for your students? Explain your reasons.

ii. Simplify.

- $-2(3x-4)+8x-3$
- $a - \frac{a+5b}{3}$
- $\frac{4x-y}{3} + \frac{x-3y}{2}$

Task 4: A Mathematical Trick

1	2	3	4	5
6	7	8	9	

- Pick any two cards.
  Example: $\boxed{3}$ and $\boxed{5}$.
- Form two 2-digit numbers using the digits.
  Example: 35 and 53.
- Find the difference between the two numbers.
  Example: 53 – 35 = 18

Diagram 6

Diagram 6 shows the rule to play a trick using the number cards 1 to 9.

On his first trial, a student picked 4 and 5 and obtained the following result:

54 – 45 = 9

On his second trial, the students picked 4 and 7 and obtained the following result:

74 – 47 = 27

Based on these results, the student concluded that the difference between the two numbers is always a multiple of 9.

i. Prove that the student's conclusion is always true.
[Hint: Any 2-digit number ab can be expressed as a polynomial $10a + b$, where a and b are any digit from 1 to 9.]

Standard 3.2: [K3NA3-2]
Thinking about how to solve simultaneous equation in the case of linear equations

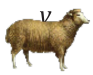
i. Understand the meaning of solution of linear equations and simultaneous linear equations as a pair of numbers
ii. Know the substitution and elimination methods of solving simultaneous linear equations
iii. Get fluency for selecting the methods from the form of the simultaneous linear equations
iv. Appreciate simultaneous linear equation in the situations

Sample Tasks for Understanding the Standards

Task 1: Solving Simultaneous Linear Equations with Two Variables

Chicken-and-Sheep Problem

Table 1
Solving Chicken-and-Sheep Problem

 x	 y	Total Animal Heads	Total Animal Feet

Old MacDonald had a chicken-and-sheep farm. There are x number of chickens and y number of sheep in the farm.

i. One day, he counted all the animals in his farm and announced to his wife that:

   "*There are altogether 18 animal heads in our farm.*"

   - Write an equation in terms of x and y to represent the relationship.
   - Fill in Table 1 to show all the possible values of x, y and total number of animal heads.

ii. Later, his wife counted all the animals and announced to him that:

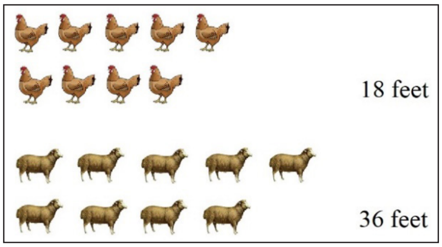
   "*There are altogether 60 animal feet in our farm.*"

   - Write another equation in terms of x and y to represent the second relationship.
   - Complete Table 1 to show all the possible numbers of animal feet.

iii. How many chicken and how many sheep are there in Old McDonald's farm?

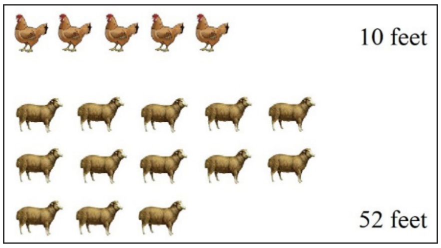
iv. What does it mean by solving two equations simultaneously?

Assume there are 9 chickens.
There will be 9 sheep.



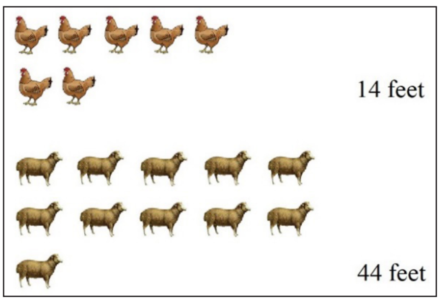
18 heads and 54 feet

Not enough feet. Assume there are 5 chickens. There will be 13 sheep.



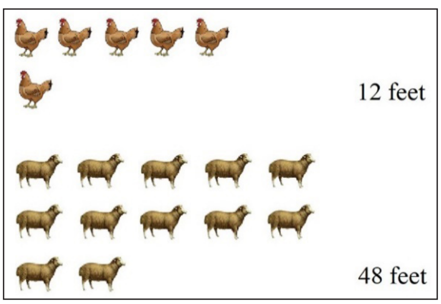
18 heads and 62 feet

Too many feet. Assume there are 7 chickens. There will be 11 sheep.



18 heads and 58 feet

Not enough feet. Assume there are 6 chicken. There will be 12 sheep.



18 heads and 60 heads

Diagram 1

v. A student solved the Chicken-and-Sheep problem by first setting a tentative number of chickens, then proceeded to find the solution by a repeated process as shown in Diagram 1.

- Explain the student's reasoning in solving the problem.
- Using this student's method, solve

  $x + y = 9$

  $2x + y = 16$

Apple-and-Orange Problem

In a supermarket in Bangkok, a packet of 2 apples and 1 orange is sold at 31 Bhat, but a packet of 1 apple and 2 oranges is sold at 26 Baht.
What is the price of an apple and an orange respectively?

vi. Solve the Apple-and-Orange problem by your own reasoning.

vii. Three students solved the Apple-and-Orange problem.

- Diagrams 2 shows the reasoning of the first student in solving the problem.

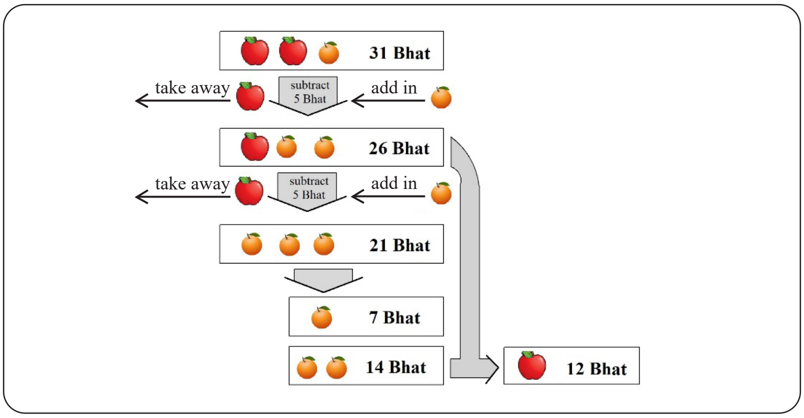


Diagram 2

❖ Explain the first student's reasoning.
❖ Does your own reasoning used in solving the problem match the students' reasoning? If yes, explain how the reasoning matched? If no, represent your reasoning with a diagram.

- The second student represented his reasoning while solving the problem with a diagram as shown in Diagram 3 and explained it using a series of equations.

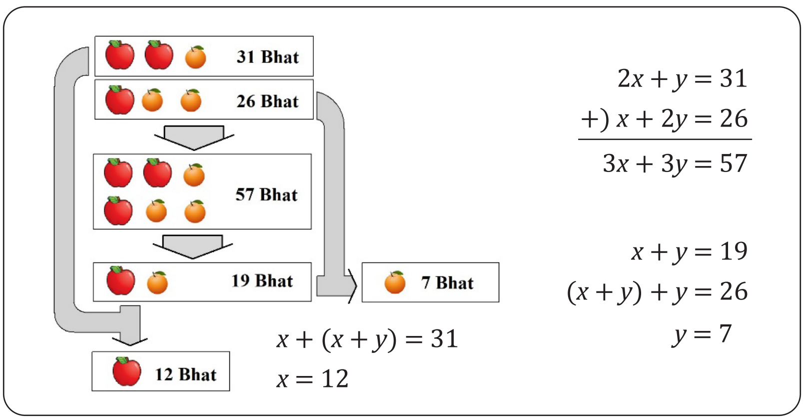


Diagram 3

❖ Explain how the equations are connected to the student's reasoning.

- The third student represented his reasoning while solving the problem with a diagram as shown in Diagram 4.

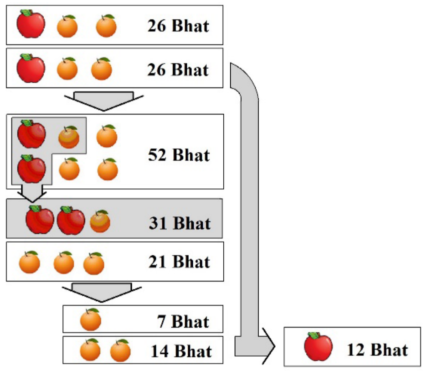
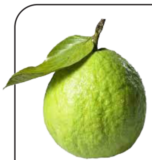


Diagram 4

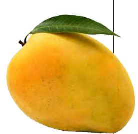
❖ Explain the third student's reasoning using a series of equations.

Guava-and-Mango Problem



Guavas and mangoes are packed into seperate boxes. Each box has a same number of guavas or mangoes.

Somchai bought 3 boxes of guavas and 5 boxes of mangoes. He counted the fruits and found that there are altogether 86 fruits. Then, he gave away 3 boxes of mangoes to his parents and he had 56 fruits left.



How many fruits in each box of guavas and mangoes, respectively?

Diagram 5

i. Diagram 5 shows a problem involving the number of fruits in a box.

- Represent your reasoning in solving the Guava-and-Mango problem with a diagram. Explain your diagram.

Set A	Set B
$x + y = 12$ $2x + 3y = 28$	$a + b + c = 15$ $a + 2b + 3c = 24$

Diagram 6

ii. Diagram 6 shows two sets of equations.

- When these two sets of equations are solved simultaneously, what can you say about the number of solutions for each set of equations?

- What is the relationship between the number of variables and the number of equations when solving a set of equations simultaneously?

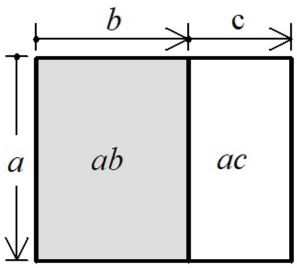
Topic 4: Expansion and Factorisation of Polynomials [K3NA4]

Standard 4.1: [K3NA4-1]
Acquisition to see the polynomials in second degree with expansion and factorisation and use it

i. Use the distributive law to explain the formulae for expansion and explain them on diagrams
ii. Acquire proficiency for selecting and using the appropriate formulae
iii. Use the expansion formulae to factorise the second degree expression and recognize both formulae with inverse operation
iv. Solve simple second degree equation using the factorisation and apply in life situations

Sample Tasks for Understanding the Standards

Task 1: The Distributive Property of Multiplication



$a \times (b + c) = ab + ac$

Diagram 1

i. Diagram 1 shows the distributive property involving the multiplication of any three numbers a, b and c.

- Verify that $a \times (b + c) = ab + ac$ holds true for both positive and negative numbers.
- Draw another diagram to verify that $(b + c) \times a = ab + ac = a \times (b + c)$.

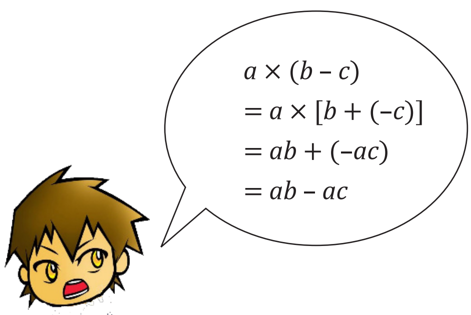


Diagram 2

ii. A student claimed that "*since subtracting a number is the same as adding the corresponding negative number, therefore* $a \times (b - c) = ab - ac$." The student's reasoning is shown in Diagram 2.

- Draw a diagram to support the student's claim.

Task 2: Multiplication and Division of Algebraic Expressions

Multiplication and Division of Polynomials by Monomials

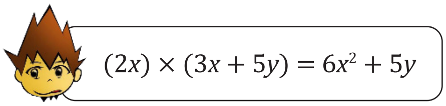


Diagram 3

i. When asked to multiply the monomial $2x$ with the polynomial $3x + 5y$, a student gave a wrong answer as shown in Diagram 3.

- What is the possible cause for the student's mistake?
- Draw a diagram to help the student correct the mistake.

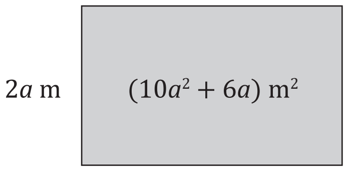


Diagram 4

ii. Diagram 4 shows a plot of rectangular land with length $(2a)$ m and area $(10a^2 + 6a)$ m^2

- Write an algebraic expression relating the width of the land to its length and area.
- Find the width of the land in terms of a.

Multiplication of Polynomials by Polynomials

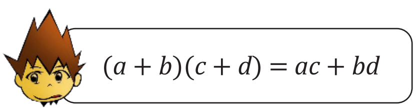


Diagram 5

i. When asked to multiply the polynomials $(a + b)$ with $(c + d)$, a student gave a wrong answer as shown in Diagram 5.

- What is the possible cause for the student's mistake?

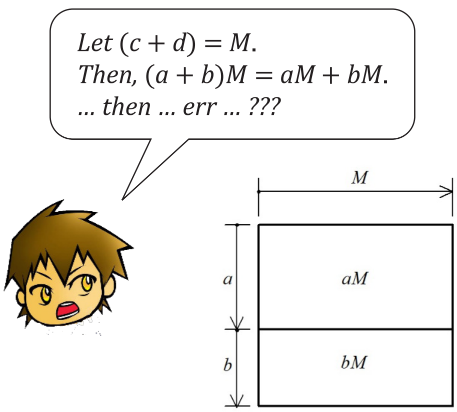


Diagram 6

ii. Another student tried to explain to the first student by drawing a diagram as shown in Diagram 6. However, this student got stuck halfway and could not complete his explanation.

- Complete the explanation using the diagram drawn by the second student.

Task 3: Expansion of Algebraic Expressions

Expansion Formulae

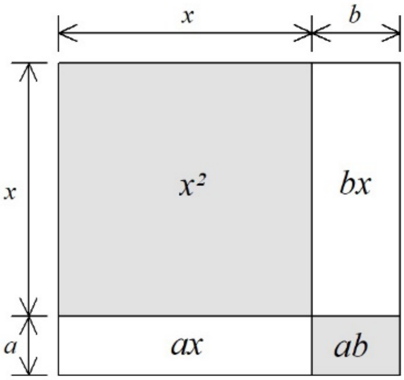


Diagram 7

Diagram 7 shows the expansion of $(x+a)(x+b)=x^2+(a+b)x+ab$ diagrammatically.

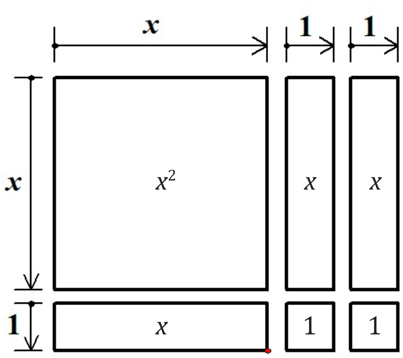
i. Explain the process of expansion using the distributive property of multiplication.

ii. Explain how the following expansion formulae could be derive from the expansion of $(x+a)(x+b)$.
   - $(x+a)^2$
   - $(x-a)^2$
   - $(x+a)(x-a)$

iii. Use the expansion formulae to expand
   - $(2x+3)(2x+5)$
   - $(3b+2)^2$
   - $(2x-3y)^2$
   - $(3c-5)(3c+5)$
   - $(a-3b)(-2a+4b)$

iv. Expand
   - $4(a+2b-3c)$
   - $(a+b)(x-y+3)$

Task 4: Factorisation of Algebraic Expressions



$$x^2+3x+2 \underset{\text{expanding}}{\overset{\text{factorising}}{\rightleftarrows}} (x+1)(x+2)$$

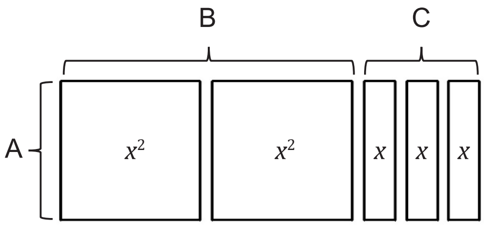
Diagram 8

Diagram 8 shows how the polynomial x^2+3x+2 is factorised to be $(x+1)(x+2)$.

i. What does Diagram 8 tell you about factorisation of polynomial expressions?

ii. Use the cut-out pieces (x^2-piece, x-piece, 1-piece) in Material Sheet 1 at the end of this topic to factorise:
   - x^2+5x+6
   - $2x^2+7x+3$

Task 5: Factorise by Common Factor



Factorising $2x^2 + 3x$

Diagram 9

i. Diagram 9 shows how the polynomial $2x^2 + 3x$ is factorise using the cut-out pieces.
   - What is the value of A, B and C respectively?

ii. The factorisation of $2x^2 + 3x$ involves a common factor.
   - What is the common factor?
   - Use the cut-out pieces to factorise $2x^2 + 6x$.

iii. There are two different ways to arrange the cut-out pieces in the factorisation of $2x^2 + 6x$.
   - What is the same and what is different about the results of these two arrangements?

iv. How would you use the cut-out pieces to factorise $2x^2 - 6x$?

v. Factorise
   - $2ax - 6ay$
   - $3x^2 + 6xy$
   - $x^2y + xy^2$
   - $a^2 - 4ab + 3a$

Task 6: Factorise by Formulae

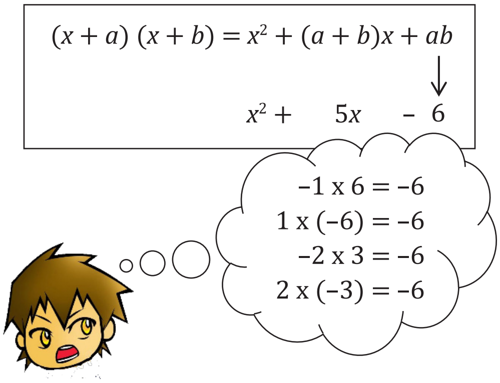


Diagram 10

i. When using the expansion formula for $(x + a)(x + b)$ to factorise the polynomial expression $x^2 + 5x - 6$, a student made a comparison between the formula and the expression as shown in Diagram 10.

The student found four possible combinations of numbers a and b to get the product -6. He then decided that (-1) and 6 was the correct combination and complete the factorisation as $x^2 + 5x - 6 = (x - 1)(x + 6)$.

- Did the student make the correct decision? Explain your reasons.

ii. Expand

- $(x + a)(x + b)$
- $(x + a)^2$
- $(x - a)^2$
- $(x + a)(x - a)$
- Use any of these expansion formulae to factorise the following.
  - ❖ $4x^2 - 9$
  - ❖ $x^2 - 6x + 9$
  - ❖ $a^2 + 10a + 25$
  - ❖ $y^2 - y - 12$

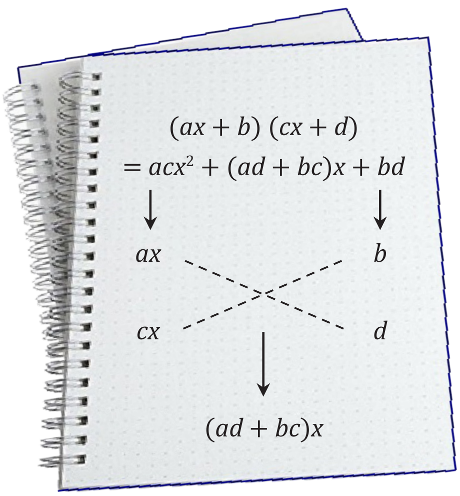


Diagram 11

iii. From the expansion of $(ax + b)(cx + d)$, a student made a note on how to factorise some second degree polynomials as shown in Diagram 11.

- Explain the student's method.
- Factorise
  - ❖ $2x^2 + 5x + 3$
  - ❖ $9x^2 - 12x + 4$

Material Sheet 1: Cut-Out Pieces (x^2-piece, x-piece, 1-piece)

<table>
<tr><td colspan="3">x^2</td><td colspan="3">x^2</td></tr>
<tr><td colspan="3">x</td><td colspan="3">x</td></tr>
<tr><td colspan="3">x</td><td colspan="3">x</td></tr>
<tr><td colspan="3">x</td><td colspan="3">x</td></tr>
<tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
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</table>

Topic 5: Extending Numbers with Square Roots [K3NA5]

Standard 5.1: [K3NA5-1]

Extending numbers with square roots and calculate the square roots algebraically

i. Define square root and discuss ways to estimate the nearest value of a square root by Sandwich Theorem
ii. Understand that some square roots cannot be represented as fractions
iii. Compare square roots using number line and understand that the order does not change but the differences between two consecutive square roots varied
iv. Think about multiplication and addition of square root and understand the algebraic way of calculation which is similar to polynomial
v. Appreciate square roots in applying to situations in life

Sample Tasks for Understanding the Standards

Task 1: Square Root

Meaning of Square

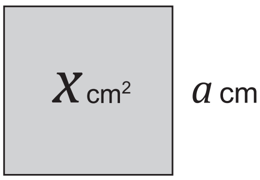


Diagram 1

Diagram 1 shows a square with side length a cm and area x cm^2.

i. How would you explain that $\sqrt{x} = a$?

ii. We know that $\sqrt{4} = 2$. Why is –2 also a square root of 4?

Plotting Square Roots on a Double Number Line

Table 1
Estimated Values of $\sqrt{x}$

x	$\sqrt{x}$
0	0.0
1	1.0
2	1.4
3	
4	2.0
5	
6	
7	
8	2.8
9	3.0
10	3.2

i. Table 1 shows some positive estimated values of $\sqrt{x}$ to one decimal place for $0 \leq x \leq 10$ where x is an integer.

- Complete the table.
  (Use an electronic calculator to help you.)

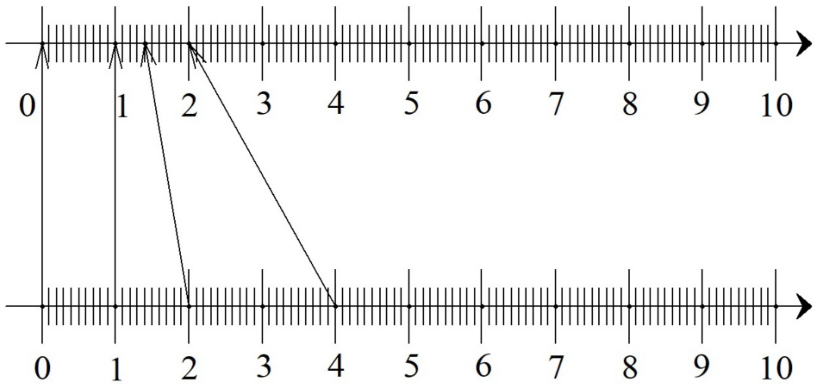


Diagram 2

ii. Diagram 2 show a double number line for $0 \leq x \leq 10$. The arrows show the mapping of x onto $\sqrt{x}$ for $\sqrt{0}$, $\sqrt{1}$, $\sqrt{2}$ and $\sqrt{4}$.
- Complete the mapping of x onto $\sqrt{x}$ on the double number line for the other values of x in Table 1.
- What do you observe about the lengths between two consecutive dots on
  - ❖ the x number line?
  - ❖ the $\sqrt{x}$ number line?
  - ❖ What could you conclude about the order and values of $\sqrt{x}$?

iii. Between 8, and 9, which number is closer to $\sqrt{72.3}$? Explain how do you make the decision.

Estimate Square Roots Using Drawing

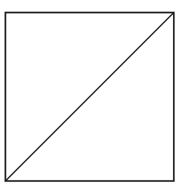


Diagram 3

Diagram 3 shows a square with one of its diagonals. The side length of the square is 1 m.

i. Show that the length of the diagonal is $\sqrt{2}$ m**.**

ii. Prove that $\sqrt{2}$ **cannot** be express in the form $\frac{a}{b}$, for a, b are integers and $b \neq 0$.

iii. Using a scale of 10 cm to represent 1 m, draw the square on a graph paper.
- How would you use your drawing to estimate the value of $\sqrt{2}$?
- Show that $1.4 < \sqrt{2} < 1.5$ on your drawing.

iv. Draw another diagram to estimate the value of $\sqrt{5}$ to one decimal place. Check your estimate with a calculator.

Estimate Square Roots using Electronic Spread Sheet

Table 2
Estimated Value of $\sqrt{2}$

b^2	$\sqrt{b^2} = b$
1.00000	**1.00000**
2.25000	1.50000
1.96000	1.40000
2.10250	1.45000
4.00000	**2.00000**

If a, b and c are positive numbers and $a < b < c$, then $\sqrt{a} < \sqrt{b} < \sqrt{c}$. This relationship can be used to estimate the values of square roots. For example, since we know that $\sqrt{1} = 1$ and $\sqrt{4} = 2$, then $1 < \sqrt{2} < 2$.

i. Table 2 shows some values of b and b^2 for $1 < b < 2$ and $1 < b^2 < 4$. As shown in the table, $1.40 < \sqrt{2} < 1.45$ So, the estimated value of $\sqrt{2}$ to one decimal place has to be 1.4.

- Use an electronic spread sheet to construct Table 2. Continue to input the values of b to estimate the value of $\sqrt{2}$ to four decimal places. Check your estimate with an electronic calculator.

ii. Construct another table to estimate the value of $\sqrt{3}$ to three decimal places. Check your estimate with an electronic calculator.

Task 2: Calculation Involving Square Roots

i. A square root, $\sqrt{x}$, is simplified if x cannot be divided by a perfect square other than 1. For examples, $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ are simplified, but $\sqrt{8}$ and $\sqrt{18}$ are not simplified.
- Simplify each expression.
  - ❖ $\sqrt{8}$
  - ❖ $\sqrt{18}$
  - ❖ $\sqrt{75} - 4\sqrt{3}$
- What property of square roots is used to simplify the expressions?

ii. Rationalise the denominator of each expression.
- $\dfrac{10}{\sqrt{5}}$
- $\dfrac{3}{\sqrt{5}+1}$

iii. What properties of square roots are used to rationalise the denominator of each expression in (ii)?

Task 3: Applying Square Roots to Situations in Life

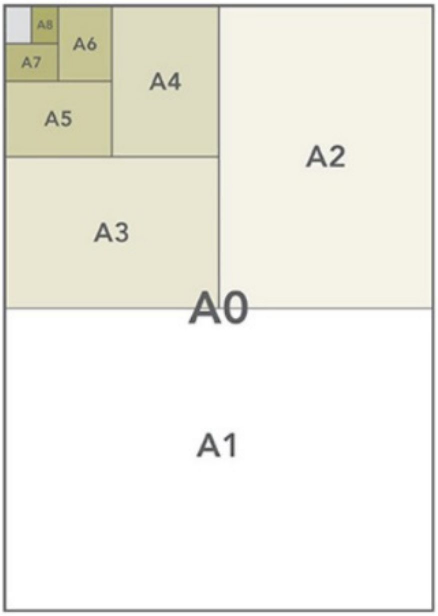


Diagram 4

Diagram 4 shows how the A series of international paper sizes A1 to A8 are derived from paper size A0. Each size is half the paper size before it. For examples, A1 is half the size of A0, A2 is half the size of A1, and so on.

Table 3 shows the measurements of the A0 to A5 sizes given as (width x length) in millimitres.

i. Calculate the ratio of $length : width$ for each size.

ii. Show that the ratios for the various paper size are all approximately equal to $\sqrt{2}$.

- Why is it so?

Table 3
International A Series Paper Size

Size	$width$ (mm) x $length$ (mm)
A0	841 x 1188
A1	594 x 841
A2	420 x 594
A3	297 x 420
A4	210 x 297
A5	148 x 210

Topic 6: Solving Quadratic Equations [K3NA6]

Standard 6.1: [K3NA6-1]
Solving simple second degree equation using the factorisation and apply on the situation

i. Find the answers of simple second degree quadratic equations by substitution and explore by completing the square, quadratic formula and factorisation
ii. Get fluency to select the appropriate ways for solving quadratic equations
iii. Apply quadratic equations in life situations

Sample Tasks for Understanding the Standards

Task 1: Forming Quadratic Equations

Diagram 1

Uncle Ben has a rectangular plot of land as shown in Diagram 1. Given that the length of the land is x m and its perimeter is 48 m.

i. What is the width of the plot of land in terms of x?

ii. If the area of the plot of land is 140 m^2, write an equation in the form of $ax^2 + bx + c = 0$.

iii. What do we need to do with the equation if we want to find the length and width of Uncle Ben's land?

Task 2: Solving Quadratic Equations of Various Forms

Zero-Factor Property

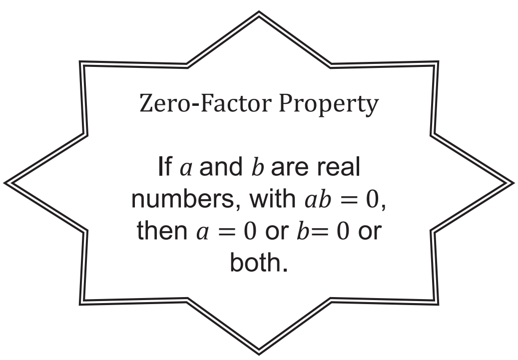
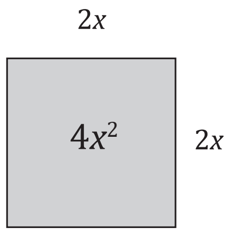


Diagram 2

i. Generally, a quadratic equation is in the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.
   - Why is it that a cannot be zero?
   - If b = 0, then c cannot be more than zero. Explain with examples.
   - If c = 0, then the quadratic equation can be solved by factorising its quadratic expression. Explain with examples.

ii. The zero-factor property, as stated in Diagram 2, is an important idea in solving quadratic equation by factorising its quadratic expression.
   - Solve the quadratic equations
     - ❖ $5x^2 - 10 = 0$
     - ❖ $3x^2 + 15x = 0$
     - ❖ $2m^2 - 3m - 5= 0$

Solving Quadratic Equations of the form $(x+q)^2 = r$



Area = $(2x) \times (2x) = (2x)^2 = 4x^2$

Diagram 3

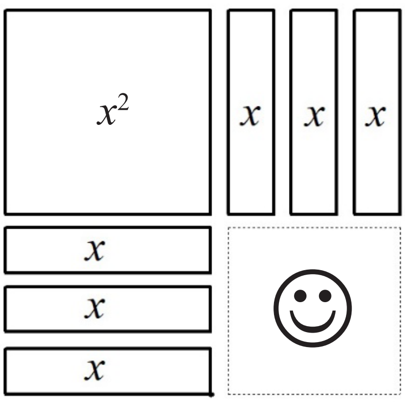
i. Diagram 3 shows a square with length $2x$ units and area $4x^2$ square units.

- What is the value of x, if the area of the square is 20 square unit?

ii. Any quadratic equation in the form $(x+q)^2 = r$ may be solved using the idea of square roots.

- Solve the quadratic equations
  - ❖ $(a - 3)^2 = 25$
  - ❖ $(2x + 3)^2 = 49$
  - ❖ $x^2 + 6x + 9 = 36$

$x^2 + 6x - 3 = 0 \rightarrow x^2 + 6x = 3$



$x^2 + 6x + \boxed{☺} = (x+3)^2$

Diagram 4

iii. Any quadratic equation in the general form $x^2 + bx + c = 0$ may be solved if it could be transformed into the form $(x+q)^2 = r$. Diagram 4 shows part of the process to transform $x^2 + 6x - 3 = 0$ into the form $(x+q)^2 = r$.

- What is the number in ☺ ?
- Explain how the quadratic equation $x^2 + 6x - 3 = 0$ is transformed into $(x+3)^2 = 12$.
- In general, what number needs to be added to the expression $(x^2 + bx)$ in order to change it to be $(x+q)^2$?
- Transform each quadratic equation into the form $(x+q)^2 = r$
  - ❖ $x^2 + 8x - 7 = 0$
  - ❖ $x^2 - 5x + 2 = 0$
  - ❖ $3x^2 + 6x - 9 = 0$

Quadratic Formula

$$x^2 + bx + c = 0$$
$$x^2 + bx = \boxed{}$$
$$x^2 + bx + (\frac{b}{2})^2 = \boxed{} - c$$
$$(x + \frac{b}{2})^2 = \boxed{}$$
$$\boxed{} = \pm\sqrt{\frac{b^2 - 4c}{4}}$$
$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Diagram 5

i. Diagram 5 shows how a quadratic formula can be derived to solve any quadratic equation in the general form $x^2 + bx + c = 0$.

- Fill in the missing parts.
- What can we say about the solutions of a quadratic equation $x^2 + bx + c = 0$ if
  - ❖ $b^2 - 4c > 0$
  - ❖ $b^2 - 4c = 0$
  - ❖ $b^2 - 4c < 0$

ii. Derive a general formula for solving $ax^2 + bx + c = 0$

Task 3: Real-World Problem Solving with Quadratic Equations

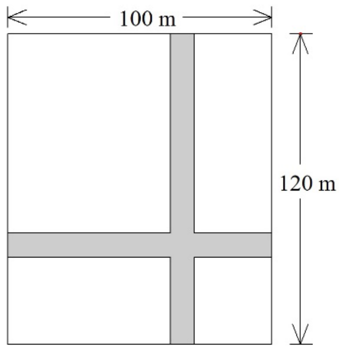


Diagram 6

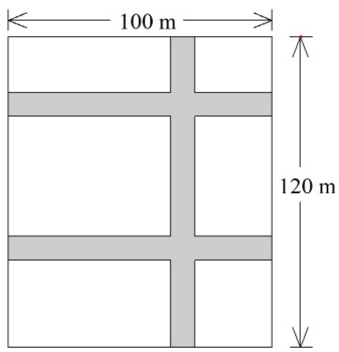


Diagram 7

Uncle Manat has a piece of rectangular land. He wants to create paths with equal width as shown in Diagram 6. The rest of the land will be made into gardens.

Let x be the width of the paths.

The total area of the gardens is to be 11 781 m^2.

i. Write a quadratic equation in terms of x to show the relationship.

ii. Solve the equation to find the width of the paths.

iii. There are two solutions to the quadratic equation in (ii), but only one is the width of the path. Why is the other solution not the width of the path?

iv. What if Uncle Manat wants to add another path of equal width as shown in Diagram 7, but still keeps the same total area for the gardens?

CHAPTER 3

Relations and Functions

Topic 1: Extending Proportion and Inverse Proportion to Functions with Variables [K3RF1]

Standard 1.1: [K3RF1-1]

Extending proportion and inverse proportion to functions with variables on positive and negative numbers

i. Extend proportions to positive and negative numbers, using tables and equations on situations
ii. Plot set of points as graph for proportions defined in ordered pairs (x, y) in the coordinate plane using appropriate scales precisely
iii. Introduce inverse proportion using tables, equations and graphs
iv. Introduce function as correspondences of two variables in situations
v. Explore the property of proportional function with comparison of inverse proportional function
vi. Appreciate proportion and inverse proportion functions in life

Sample Tasks for Understanding the Standards

Task 1: Function

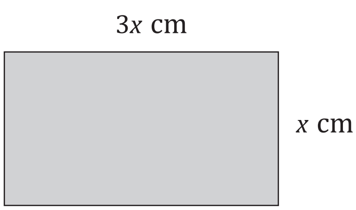


Diagram 1

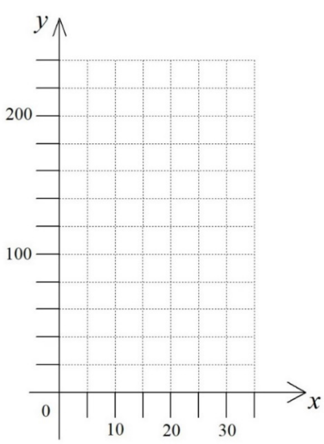


Diagram 2

Diagram 1 shows a rectangle with a width which is thrice its length, x.

i. Given that y is the perimeter of the rectangle.

- Express y in terms of x using an equation.
  ❖ What is the relationship between x and y?

- Complete the following table for the values of x and y.

x (cm)	5	10	15	20	25	30
y (cm)	40					

- Plot the graph of y against x on Diagram 2.

- Explain why y is a function of x.

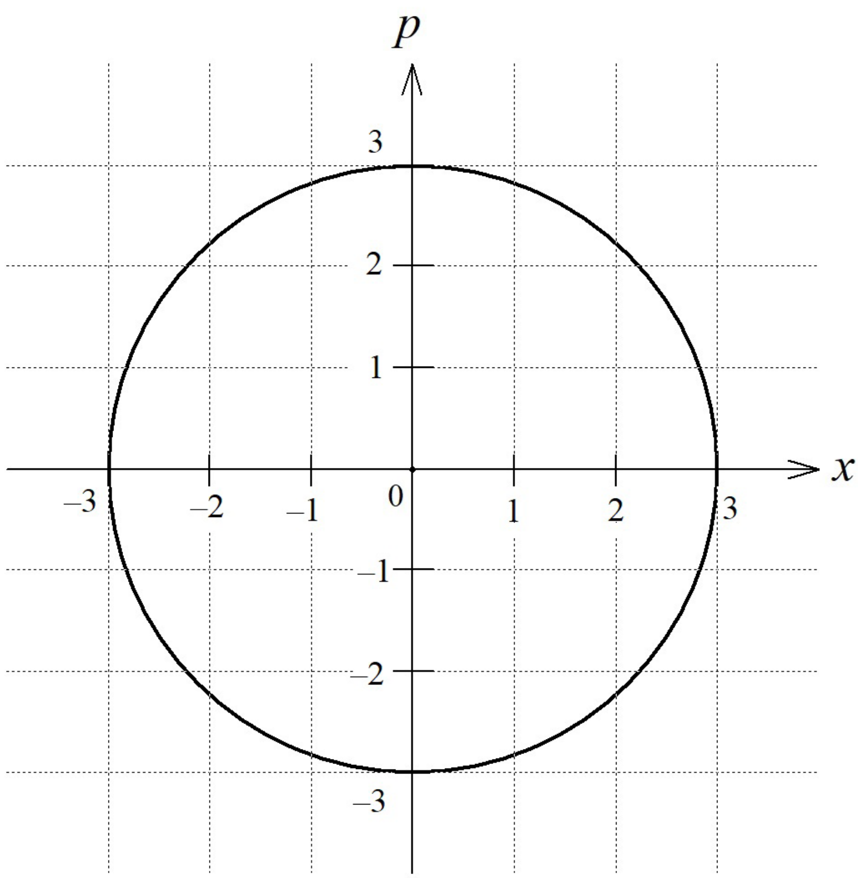


Diagram 3

ii. Diagram 3 shows the graph of $p^2 + x^2 = 9$.

- Is p a function of x? Explain your reasons.

Task 2: Direct Proportions

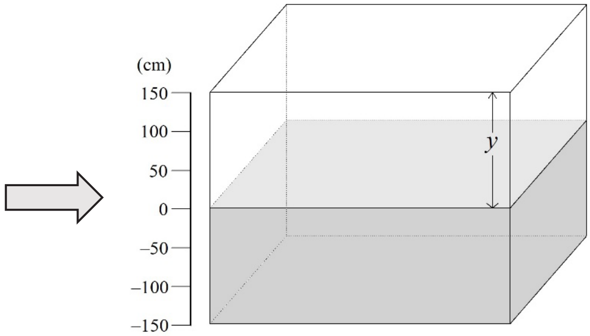


Diagram 4

Diagram 4 shows a water tank with a height of 300 cm and its orthogonal-projection drawing. The tank is being filled with water and the water level increases 2 cm per minute.

i. Let the half-tank water level be the reference point and y cm as the water level after x minutes.
- What is the water level 15 minutes later?
- What is the water level 10 minutes before?
- Complete the following table.

x (min)	-15	-10	-5	0	5	10	15	20
y (cm)			-10	0	10			

- When $x \neq 0$, find the value of $\frac{y}{x}$ for each order pairs (x, y).
- Express y in terms of x as an equation.
- What does it mean by y is directly proportional to x?
- What is the constant of proportion for this case?
- Plot the graph of y against x.

ii. After the tank is fully filled, water is being removed where the water level decreases 3 cm per minute. Again, let the half-tank level be the reference point.
- Complete the following table for the removal of water.

x (min)	-30	-20	-10	0	10	20	30	40
y (cm)			30	0		-60		

- Explain the meaning of
  - ❖ $y = 30$ cm when $x = -10$ min
  - ❖ $y = -60$ cm when $x = 20$ min
- Plot the graph of y against x for the removal of water.
- In this case, is y still directly proportional to x? Explain your reasons.

Task 3: Inverse Proportion

x cm

y cm

Diagram 5

Diagram 5 shows a rectangle with width x cm and length y cm.

i. Given that the area of the reactangle is 18 cm^2.

- Express y in terms of x using an equation.
- Explain why $x \neq 0$ and $y \neq 0$.
- Find the values of y for the values of x as follows:
  - ❖ $x = \frac{1}{4}$
  - ❖ $x = \frac{1}{2}$
  - ❖ $x = \frac{3}{4}$
  - ❖ $x = 5$

ii. Complete the following table for the values of x and y.

x (cm)	1	2	3	4	5½	6
y (cm)	18	9	6			

iii. Study the order pairs (1, 18) and (2, 9). As the value of x increases 2 times, what can you say about the value of y ?

- What about (1, 18) and (3, 6)?
- Can you say the same for all other order pairs?

iv. What does it mean by y is inversely proportional to x?

- What is the constant of proportion for this case?

v. Explain what will happen to the rectangle and the values of y when $x < 0$.

- Extend the table to include the values of $-6 \leq x < 0$.
- Plot the graph of y against x for the domain $-6 \leq x \leq 6$.
- Why is the point when $x = 0$ cannot be plotted on the graph?

Task 4: Application of Proportion

Proportional Relationship of a Circle

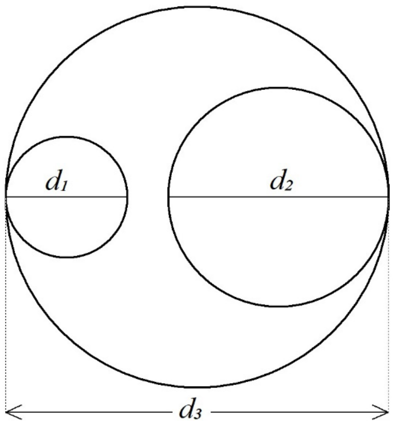


Diagram 6

Diagram 6 shows three circles with diameters d_1, d_2 and d_3.

i. From practical measurements, we know that $\frac{C_1}{d_1} = \frac{C_2}{d_2} = \frac{C_3}{d_3} = k$, where C_1, C_2 and C_3 are circumferences of the three corresponding circles, and k is a constant.

- What is the proportional relationship between C and d ?

ii. We also know that the formula to calculate the area of a circle is $A = \pi r^2$, where A is the area and r is the radius of the circle.

- What is the proportional relationship between A and r?

Problem Solving with Weighing

Diagram 7

Pak Agus wanted to use 300 pieces of iron nails but he found it very troublesome to count the nails. So, he decided to weigh the nails instead. He counted 10 identical nails of the same mass and weighed the iron nails as shown in Diagram 7.

He then added more iron nails until he got a lump of the iron nails.

i. The lump of iron nails was 800 g. Did Pak Agus get enough pieces of iron nails? Justify your answer.

Burning Insence Stick

Diagram 8

Diagram 8 shows an insence stick burning at a rate of -1.2 cm/min. After burning for few minutes, the length of the insence stick was found to be 24 cm.

i. What was the length of the insence stick 2 minutes before this?

ii. What will the length of the insence stick be 5 minutes after this?

iii. How many more minutes will it take the insence stick to burn completely?

Interlocking Gears

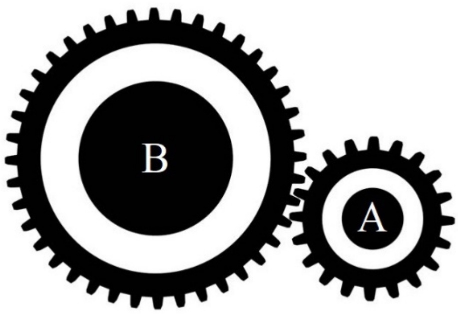


Diagram 9

Diagram 9 shows two gears, A and B that are rotating while interlocking with each other. Gear A has 20 teeth and rotates 8 times per second. Gear A is fixed, but Gear B can be fitted with different gears.

i. If Gear B has 40 teeth, how many times will it rotate in one second?

ii. If we want Gear B to rotate 1 time in one second, how many teeth must it have?

Walking Speed

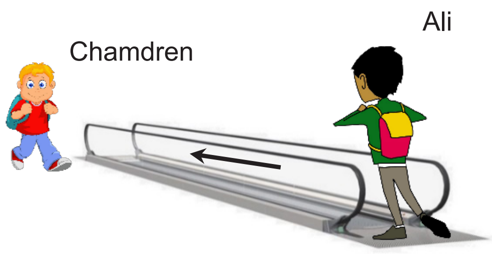


Diagram 10

Diagram 10 shows a moving walkway which is 60 m long and moves at a speed of 1 m per second.

As Ali was getting onto the moving walkway, he saw his classmate, Chamdren, started walking next to the moving walkway from the opposite direction at a speed of 1.2 m per second.

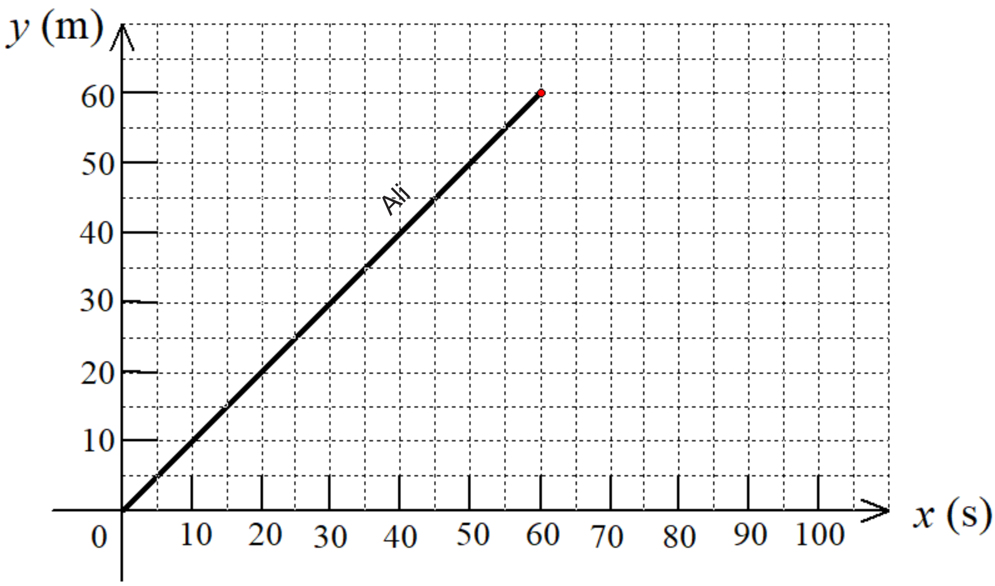


Diagram 11

Diagram 11 shows the graph that represent Ali's movement.

i. Plot another graph on Diagram 11 to represent Chamdren's movement.

Use the graphs in Diagram 11 to answer questions (ii) to (iv).

ii. 20 s after they both started walking, how far apart were Ali and Chamdren?
iii. After how many seconds of walking would Chamdren walk pass Ali?
iv. When Chamdren arrived at the end of the moving walkway, how many m was Ali from the other end of the walkway?

Topic 2: Exploring Linear Function in Relation to Proportions [K3RF2]

Standard 2.1: [K3RF2-1]
Exploring linear function in relation to proportion and inverse proportions

i. Identify linear functions based on situations represented by tables and compare it with proportional functions
ii. Explore properties of linear function represented by tables, equations and graphs and compare it with direct and inverse proportional functions
iii. Acquire fluency to translate the rate of change of a linear function represented in table, as coefficient in an expression and gradient in a graph
iv. Acquire fluency to translate y values of $x = 0$ in a table, constant in an expression, and y intercept in graph
v. Apply the graphs of linear functions to solve simultaneous equations
vi. Apply the linear function for data representation on situations to determine best fit line

Sample Tasks for Understanding the Standards

Task 1: Linear Function

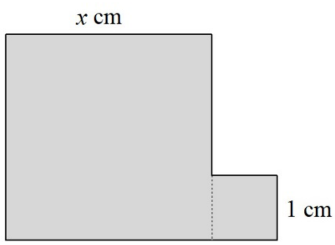


Diagram 1

i. A square with side x cm is joined to another square with side 1 cm to form a new shape as shown in Diagram 1.
- What will happen to the shape if $x = 1$?
- What if $0 < x < 1$?
- What if $x = 0$?

ii. Given that y is the perimeter of the shape and z is its area.
- Complete the following table for the domain $0 \leq x \leq 10$.

x (cm)	0	¾	1	2	6	7½	9	10
y (cm)								
z (cm^2)								

iii. Given that y is a function of x.
- Express y as a function of x using an equation for the domains (a) $1 \leq x \leq 10$ and (b) $0 \leq x \leq 1$.
- What do you notice about function y for the two domains?

iv. Given that z is also a function of x.
- Express z as a function of x using an equation for the domains (a) $1 \leq x \leq 10$ and (b) $0 \leq x \leq 1$.
- What do you notice about function z for the two domains?

v. Sketch the graphs of y against x and z against x respectively.
- What is the same and what is different about the two graphs?

vi. What will happen to functions y and z if the values of x are extended to the domain $x < 0$? Illustrate your answer with a sketch of the new shape.

vii. Given that y is a linear function but z is not.
- What is a linear function?

Task 2: Linear Functions and Proportional Functions

Water Temperature

Room temperature
= 22 °C
Rate of heating
= 12 °C per minute

Diagram 2

Diagram 2 shows a kettle of water being heated up at room temperature.

i. Explain what does it mean by the increase in water temperature is proportional to the time of heating.

ii. Assuming y is the increment of water temperature in °C and x is the time of heating in minute.

- Express y as a function of x.
- What is the constant of proportion?
- What is the increment of water temperature after 5 minutes of heating?

iii. Given that the water temperature is p° C after x minutes of heating.

- Express p as a function of x.
- Why is function y different from function p?
- Is p a linear function? Explain your reasons.

Running Speed

Diagram 3

Diagram 3 shows a runner reaching the finishing line in a marathon run.

i. A runner takes 6 hours to finish the run, what is his average speed of running in km per hour?

ii. A runner is running at an average speed of 8 km per hour. How many hours will he take to finish the run?

iii. Explain what does it mean by the average speed is inversely proportional to the time of running.

iv. Assuming x is the time in hours and s is the average speed in km per hour for a runner to run the 42.2 km marathon race.

- Express s as a function of x.
- What is the constant of proportion?
- Is s a linear function? Explain your reasons.

Task 3: Rate of Change of a Function

Linear Function

Table 1
Values of x and y of a Linear Function

x-increment

$+1 \quad +1 \quad +½ \quad +1 \quad +1½$

x	-2	-1	0	$½$	$1½$	3
y	-4	-1	2	$3½$	$6½$	11

? ? ? ? ?

y-increment

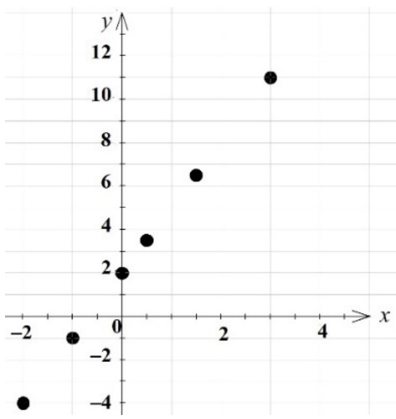


Diagram 4

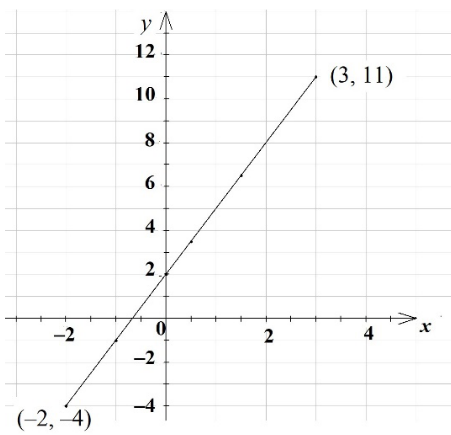


Diagram 5

Table 1 shows the values of x, y and the respective x-increment of a linear function y for the domain $-2 \leq x \leq 3$.

i. When x increases from -2 to -1, the x-increment is $+1$. What is the corresponding y-increment?

ii. Let the ratio $\frac{y\text{-increment}}{x\text{-increment}} = k$.

- Find the value of k for $x = -2$ to $x = -1$.
- Find all other corresponding values of y-increment and k for the x-increment shown in Table 1.
- Verify that the value of k is a constant.
- The constant k is known as the rate of change of y with respect to x. Explain the meaning of rate of change.

iii. The 6 ordered pairs of (x, y) from Table 1 are plotted on a graph as shown in Diagram 4. These 6 points are then joined to form a straight line as shown in Diagram 5.

- The 6 points plotted are finite, but a straight line is formed by infinite points. Although the 6 points look like forming a straight line, can we say that the graph of linear function y plotted against x is a straight line? Explain your reasons.
- Justify that the graph is a straight line by using similarity of triangles.
  [Hint: Take a common point such as A on the line. Then, consider corresponding angles of various similar triangles going through point A.]
- Find the gradient of the graph of linear function y.

iv. Express y in terms of x using an equation.

Nonlinear Function

Table 2
Inverse Proportion $y = \frac{12}{x}$

x	...	-3	-2	-1	0	1	2	3	...
y	...				–				...

This task is to investigate the rate of change of an inverse proportion function.

i. Given an inverse proportion $y = \frac{12}{x}$.

- What is the constant of proportion?
- Complete Table 2.
- Find the rate of change of y with respect to x for the following intervals of x:
  - ❖ $x = -3$ to $x = -2$
  - ❖ $x = -2$ to $x = -1$
  - ❖ $x = 1$ to $x = 2$
  - ❖ $x = 2$ to $x = 3$
- Verify that the rate of change of $y = \frac{12}{x}$ is not a constant.
- Why is $y = \frac{12}{x}$ considered a nonlinear function? Explain your reasons.

ii. Is each of the following statements true or false? Explain your reasons.

- **All** linear functions are direct proportions.
- **All** nonlinear functions are **not** direct proportions.
- A linear functions is **not** an inverse proportion.

Task 4: Graphs of Linear Functions

Table 3
Values of x and y for Three Linear Functions

x	...	-4	-3	-2	-1	0	1	2	3	4	5	...
$y_1 = 2x$	...				-2	0	2					...
$y_2 = 2x + 6$	...				4	6	8					...
$y_3 = 2x - 4$	...				-6	-4	-2					...

Any linear function can be represented in the form $y = ax + c$. Table 3 shows some values of x and y for $y_1 = 2x$, $y_2 = 2x + 6$ and $y_3 = 2x - 4$.

i. Complete the table.

ii. Plot the graphs of y_1, y_2 and y_3 on Diagram 5.
   - What does the value of a and c tell us respectively?
   - What will happen to the graph when $a = 0$?

iii. A student claims that "when a increases positively, the graph $y = ax + 4$ will rotate counter-clockwise." What does the student's claim mean?

iv. Plot the graphs of $y_4 = -x + 4$, $y_5 = -2x + 4$, and $y_6 = -3x + 4$ on Diagram 6.
   - What will happen to the graph when a decreases negatively?

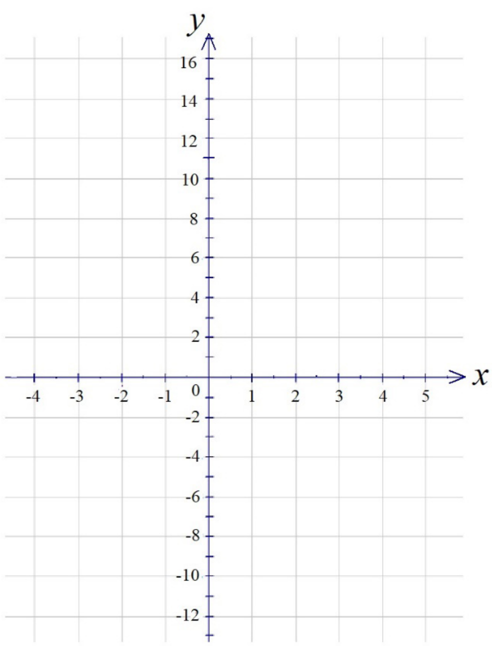


Diagram 5

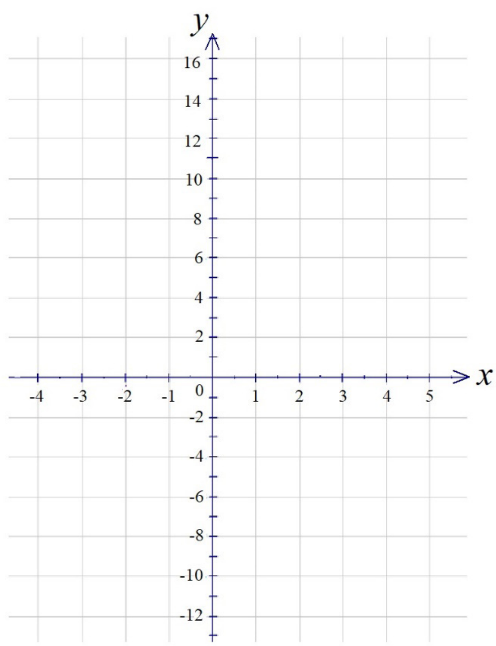


Diagram 6

Task 5: Using Graphs of Linear Functions to Solve Simultaneous Equations

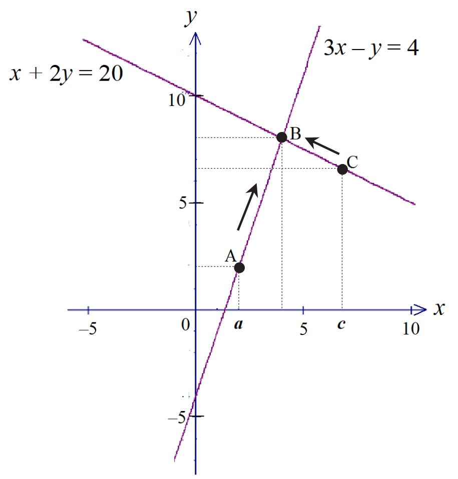


Diagram 7

i. Given a system of linear equations

$x + 2y = 20$

$3x - y = 4$

- Solve the simultaneous equations.
- Diagram 7 shows the graphs of the two equations. A and C are two points on the lines $3x - y = 4$ and $x + 2y = 20$ respectively. If A and C move in the directions as shown by the arrows, the two points will meet at B, which is the intersection point of the two lines. The x-coordinates of A and C are a and c, respectively.
  - ❖ What is the y-coordinate of A in terms of a?
  - ❖ Find the y-coordinate of A as the value of a changes as follows:

    $a = 2\frac{1}{3} \rightarrow a = 3\frac{1}{3} \rightarrow a = 4\frac{1}{3}$
  - ❖ What is the y-coordinate of C in terms of c?
  - ❖ Find the y-coordinate of C as the value of c changes as follow:

    $c = 6 \rightarrow c = 5 \rightarrow c = 3$
  - ❖ What will happen to the values of a and c when A and C meet at B? What about the y-coordinates of A and C?
- Find the coordinates of B.
- Compare the coordinates of B with the solution of simultaneous equation in (i). What could you conclude?

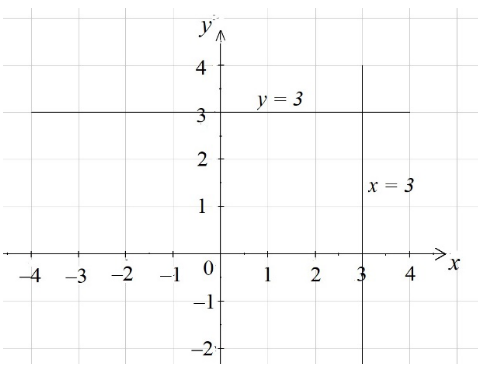


Diagram 8

ii. Diagram 8 shows the graphs of $x = 3$ and $y = 3$. Both $x = 3$ and $y = 3$ are straight lines on the graph but not both are linear functions.

- Why is $y = 3$ a linear function but $x = 3$ is not?

Task 6: Application of Linear Functions

Scientific Investigation

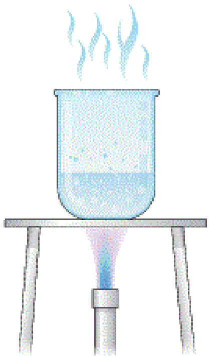


Diagram 9

Table 4
Water Temperature After x Minutes of Heating

x (min)	0	1	2	3	4
y (ºC)	25.2	35.3	45.0	57.1	68.2

Diagram 9 shows a beaker of water at room temperature is being heated. Table 4 shows the water temperature, y ºC, after it was heated for x minutes.

i. Based on Table 4, plot all the points (x, y) on a piece of graph paper. Use a scale of 2 cm = 10 °C for y-axis and 2 cm = 1 min for x-axis.

ii. Assuming y is a linear function of x, draw the straight line of best fit for the points. Use the graph to answer the following questions.
   - How many minutes after heating will the water temperature reach 80 ºC?.
   - Find the rate of increase in water temperature with respect to time.
   - Find the equation of the line of best fit.

Study of Movement

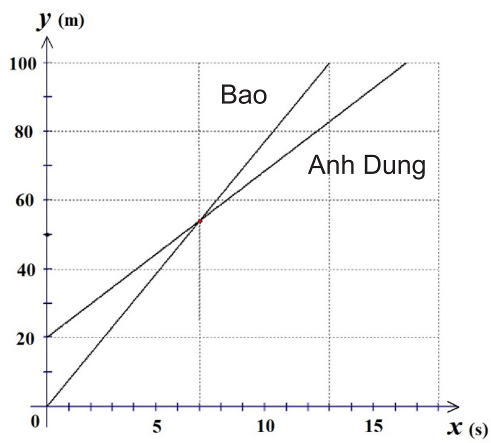


Diagram 10

i. Diagram 10 shows the relationship of distance (y metres) and time (x seconds) for a 100-metre race between two friends, Anh Dung and Bao.
   - Explain what happen at the start?
   - What is the running speed (in m/s) of Anh Dung and Bao respectively?
   - At what time and distance from the start did Bao overtake Anh Dung?
   - At what time did Bao complete the race?
   - How many metres away from the ending point was Anh Dung when Bao completed the race?

ii. 5 seconds after the race started, Chinh ride a bicycle to chase after the two friends along the same track at a speed of 10 m/s. Draw the graph that represent Chinh's movement on Diagram 10.
   - Did Chinh manage to overtake (a) Anh Dung, and (b) Bao respectively? If yes, at what distance from the ending point?

Determining the Best Deal

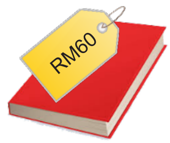
Printing Company	Printing Cost
A	 RM100 for printing each book.
B	 +  There is an initial payment of RM100 and then RM60 for printing each book.
C	 If the number of books is not more than 60, the cost is RM400 no matter how many books are printed.

Diagram 11

RECSAM wants to print a guide book for Mathematics teachers. Diagram 11 shows the quotations from three printing companies.

Let the cost of printing x copies of books be y ringgit (RM).

i. Express y in terms of x for each of the quotations.

Plot the graphs of y against x for the three quotations, then use the graphs to answer the following questions.

ii. If RECSAM wants to print 25 copies of the book, at what cost will each company offer?

iii. Which company offers the cheapest cost for printing
- 15 copies of the books?
- 35 copies of the books?
- 55 copies of the books?

Topic 3: Exploring Simple Quadratic Function [K3RF3]

Standard 3.1: [K3RF3-1]
Exploring quadratic function $y = ax^2$ in relation to linear function

i. Identify the quadratic function on situations using tables and compare it with linear function
ii. Explore properties of quadratic function using tables, equations as well as graphs and comparing it with linear function
iii. Apply the quadratic function on situations in daily life and appreciate it

Sample Tasks for Understanding the Standards

Task 1: Simple Quadratic Function, $y = ax^2$

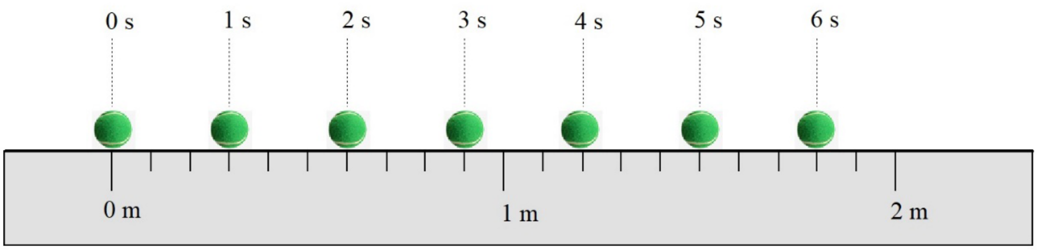


Diagram 1

i. A tennis ball was rolled on a flat runway. The stroboscopic photograph in Diagram 1 shows the position of the tennis ball every second after it was rolled. Let y be the distance in metre travelled by the tennis ball after x seconds.

Table 1
Distance Travelled for Flat Runway

x (s)	0	1	2	3	4	5	6
y (m)	0	0.3					

- Complete Table 1 for the flat runway.
- Plot the graph of y against x.
  - ❖ Is y proportional to x? Explain your reasons.
- Write an equation to represent the relationship of y as a function of x for the flat runway.
- Where will the position of the tennis ball be after it rolled for 7 s?

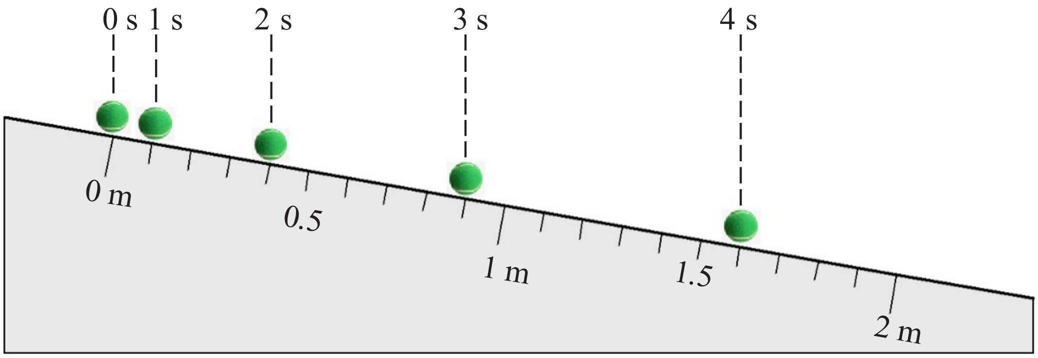


Diagram 2

ii. Later, the runway was tilted and the tennis ball was let go to roll down the slope. The stroboscopic photograph in Diagram 2 shows the position of the tennis ball every second after it was let go.

Table 2
Distance Travelled for Tilted Runway

x (s)	0	1	2	3	4
y (m)	0	0.1			

- Complete Table 2 for the tilted runway.
- Plot the graph of y against x.
  - ❖ Is y proportional to x? Explain your reason.
- Plot the graph of y against x^2.
  - ❖ Is y proportional to x^2? Explain your reason.
- Write an equation to represent the relationship of y as a function of x for the tilted runway.
- Where would the position of the tennis ball be after it rolled for 5 s?

Task 2: Properties of Quadratic Function

Table 3
Quadratic Function $y = x^2$

x	0	1	2	3	4	5	6	7	8
x^2	0	1	4	9	16	25	36	49	64
A	–	1	3	5	7				
B	–	–	2						

Table 3 shows the values of x^2, A and B for the interval $0 \leq x \leq 8$.

i. The numbers in row A are derived from the numbers in row x^2. Study the numbers 1, 3, 5, 7 in row A carefully.
   - How are the numbers 1, 3, 5, 7 derived from the numbers in row x^2?
   - Complete row A.

ii. The numbers in row B are derived from the numbers in row A by a similar way.
   - How is the number 2 in row B derived from the numbers in row A?
   - Complete row B.
   - What do you notice about the numbers in row B?

Table 4
Quadratic Function

x	0	1	2	3	4	5	6	7	8
$3x^2$	0	3	12	27	48	75			
A	–	3	9	15	21				
B	–	–	6						

Table 4 shows the values of $3x^2$, A and B for the interval $0 \leq x \leq 8$.

iii. The numbers in rows A and B are derived the same way as in Table 3.
   - Complete Table 4
   - What do you notice about the numbers in row B?

iv. Construct a similar table for the quadratic function $y = 5x^2$.
   - What will the number in row B be?
   - Explain your reasons.

v. What will the number in row B be for the following quadratic functions?
   - $y = \frac{1}{2}x^2$?
   - $y = -2x^2$
   - $y = 2x^2 + 3$

vi. Why does the number in row B becomes a constant for a particular quadratic function?

Task 3: The Graph of Function

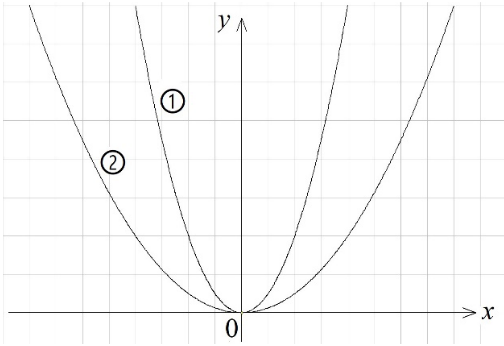


Diagram 3

The graph of quadratic function $y = ax^2$ is always in the shape of a parabola irrespective of the values of a.

i. Diagram 3 shows two graphs of $y = ax^2$ for $a > 0$.
   - Match each of the graphs with the corresponding function. Justify your decision.
     - ❖ $y = \frac{1}{2}x^2$
     - ❖ $y = 2x^2$
   - What will happen to the graph when $a < 0$?
   - What will happen to the function and its graph when $a = 0$?

ii. The parabolas shown in Diagram 3 may look different. However, the two parabolas are basically a same shape because all parabolas are similar shapes.
   - Prove that the two parabolas are similar.
     [Hint: Consider enlargement of the parabolas with origin as the centre of enlargement.]

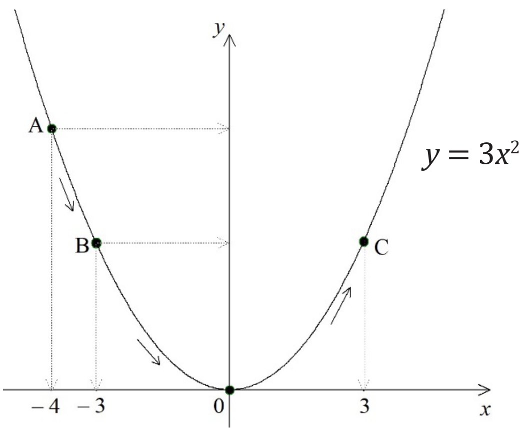


Diagram 4

iii. Diagram 4 shows the graphs of $y = 3x^2$ passing through the origin. A, B and C are three points on the graph.
   - What will happen to the values of x and y as point A moves in the direction indicated by the arrows toward point C passing through point B and the origin?
   - At what value of x is y minimum?
   - What is the minimum value of y?
   - What is the rate of change of y with respect to x between points A and B?
   - What will happen to this rate of change as point A moves towards C?

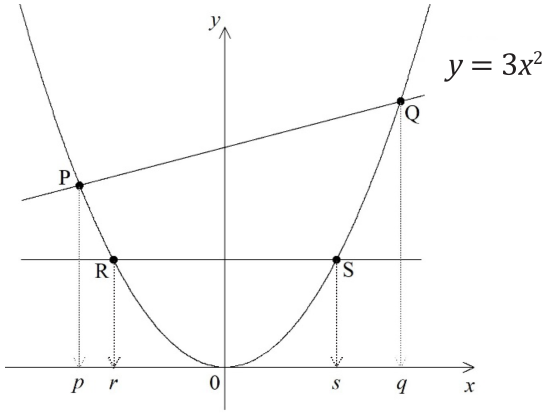


Diagram 5

iv. Diagram 5 shows the graph of $y = 3x^2$ again but this time a slanting line is drawn to intersect with the graph at points P and Q. Another line parallel to the x-axis is drawn to intersect with the graph at points R and S. The x-coordinates of point P, Q, R, S are p, q, r, s respectively.

- What is the numerical relationships between the absolute values $|p|$ and $|q|$?
- What is the numerical relationship between the absolute value $|r|$ and $|s|$?
- Explain your reasons.

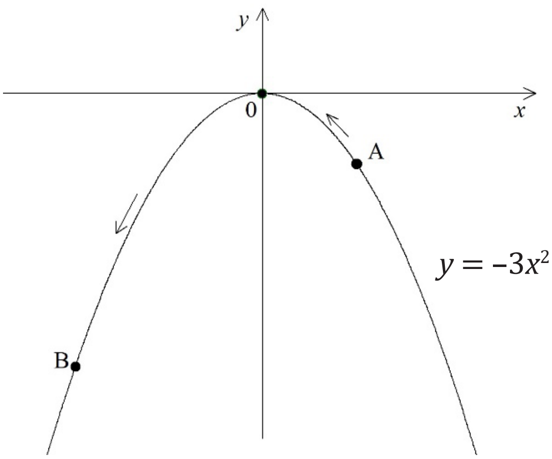


Diagram 6

v. Diagram 6 shows the graph of $y = -3x^2$ passing through the origin. A and B are two points on the graph.

- What will happen to the values of x and y as point A moves in the direction indicated by the arrows toward point B passing through the origin?
- At what value of x is y maximum?
- What is the maximum value of y?
- What will happen to the rate of change of y with respect to x as point A moves toward point B?

v. Compare the graphs of quadratic function $y = ax^2$ with linear function $y = ax$. Describe the differences between the two graphs from the following aspects:

- Shape of the graph, when (a) $a > 0$, (b) $a < 0$.
- Change in the value of y as the value of x changes, when (a) $a > 0$, (b) $a < 0$.
- Rate of change of y with respect to x when (a) $a > 0$, (b) $a < 0$.

Task 4: Applications of $y = ax^2$ in Real World

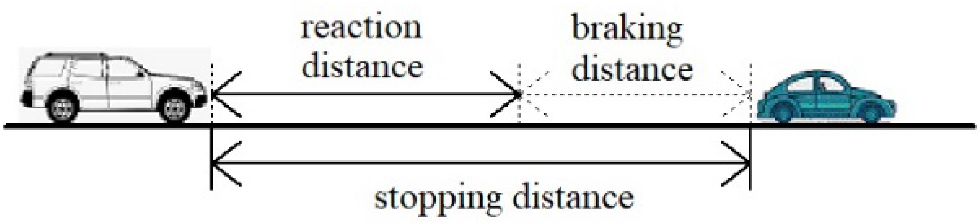


Diagram 7

When you follow behind a car during a driving journey, it is important to keep a safe stopping distance. Stoping distance is the distance it takes you to bring your car to a complete stop in an emergency. As show in Diagram 7,

Stopping distance	=	Reaction distance	+	Braking distance

Reaction distance is the distance your car travels between something happening on the road up ahead and you reacting to it. This distance is directly proportional to the speed of your car, x.

$$\text{Reaction distance} = ax$$

where a is your reaction time.

Braking distance is the distance your car travels after you hit the brakes before it comes to a complete stop. This distance is directly proportional to the square of the speed of your car, x^2.

$$\text{Braking distance} = kx^2,$$

where k is a constant of proportion depending on the breaking system of your car.

i. Given that your reaction time is 0.6 s. When your car is running at a speed of 40 km per hour, its' breaking distance is 10 m.

- If your car is travelling at 90 km per hour, calculate the safe stopping distance.
- What if your car is travelling at 110 km per hour?

Topic 4: Generalising Functions [K3RF4]

Standard 4.1: [K3RF4-1]
Generalising functions with various representations on situations

i. Distinguish domain, range and intervals and is appropriately use for explaining function
ii. Use various situation for generalizing ideas of functions such as moving point A and moving point B with time
iii. Compose a graph as a function of two or more graphs with different domains in a situation
iv. Introduce situations of step-functions with graph for generalization the idea of function which cannot be represented by equation

Sample Tasks for Understanding the Standards

Task 1: Domain, Range and Interval of a Quadratic Function

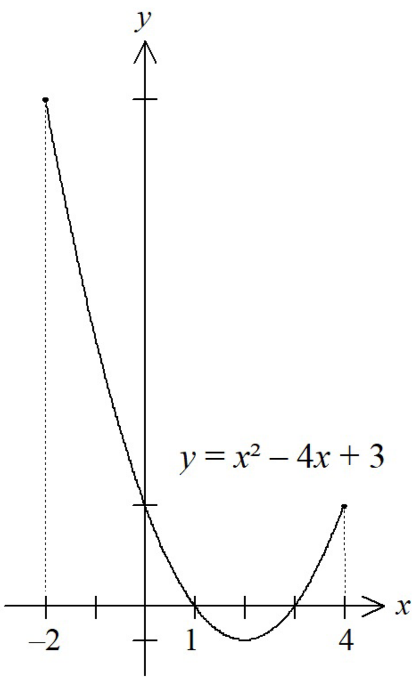


Diagram 1

Diagram 1 shows the graph of $y = x^2 - 4x + 3$ for $-2 \leq x \leq 4$.

i. What is the interval of x and range of y for each of the following domains?
   - $-2 \leq x \leq 4$
   - $-2 \leq x \leq 1$
   - $1 \leq x \leq 4$

ii. What is the minimum value of y for each of the following domains?
   - $-2 \leq x \leq 4$
   - $-2 \leq x \leq 1$
   - $1 \leq x \leq 4$

iii. What is the maximum value of y for each of the following domains?
   - $-2 \leq x \leq 4$
   - $-2 \leq x \leq 1$
   - $1 \leq x \leq 4$

Task 2: Domain, Range and Interval of a Direct Proportion Function

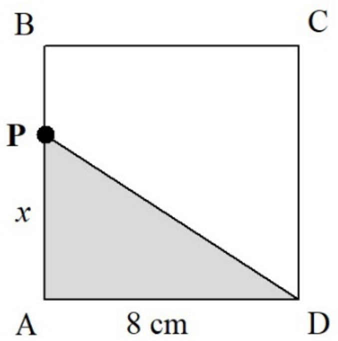


Diagram 2

Diagram 2 shows a square ABCD with side length 8 cm. A point P is moving from A to D through B and C along sides AB, BC and then CD.

Given y is the area of triangle $\triangle\mathrm{APD}$.

i. Express y as a function for the domain
   - $0 < x \leq 8$
   - $8 \leq x \leq 16$
   - $16 < x \leq 24$

ii. Sketch the graph of function y for the domain $0 < x \leq 16$?

Task 3: Domain and Range of an Inverse Proportion Function

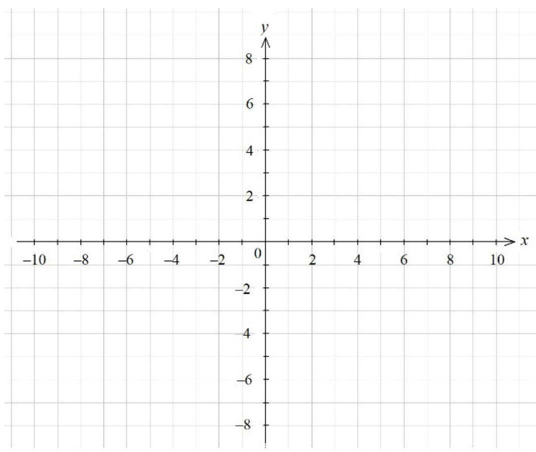


Diagram 3

Given an inverse proportion function $y = -\frac{2}{x}$.

i. Sketch the graph of function y on Diagram 3 for the domain $1 \leq x \leq 10$.

ii. Consider the domain $0 < x \leq 1$.
   - As x is approaching 0, what will happen to the value of y?

iii. As y is approaching 0, what will happen to the value of x?

iv. What is the range of y for the domain $0 < x \leq 10$?

v. Sketch the graph of function y on Diagram 3 for the domain $-10 \leq x \leq -1$.

vi. Consider the domain $-1 \leq x \leq 0$.
   - As x is approaching 0, what will happen to the value of y?
   - As y is approaching 0, what will happen to the value of x?

vii. What is the range of y for the domain $-10 \leq x < 0$?

Task 4: Step Functions

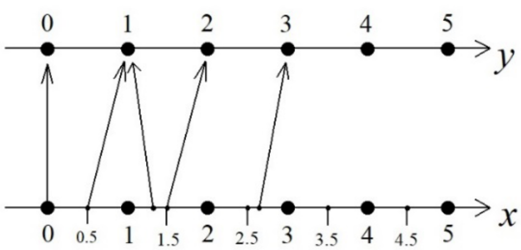


Diagram 5

Consider the domain $0 \leq x \leq 5$ where x is a real number and y is the value when x is rounded to the nearest whole number.

Diagram 5 shows a double number line mapping some values of x to the corresponding values of y. Examples shown are

$0 \rightarrow 0$ $\quad 0.5 \rightarrow 1$ $\quad 1.3 \rightarrow 1$
$1.5 \rightarrow 2$ $\quad 2.7 \rightarrow 3$

i. Map each of the following values of x to its correcponding value of y.

2,	3,	4,	5,
0.2,	0.8,	1.9,	2.3,
3.2,	3.8,	4.4,	4.6.

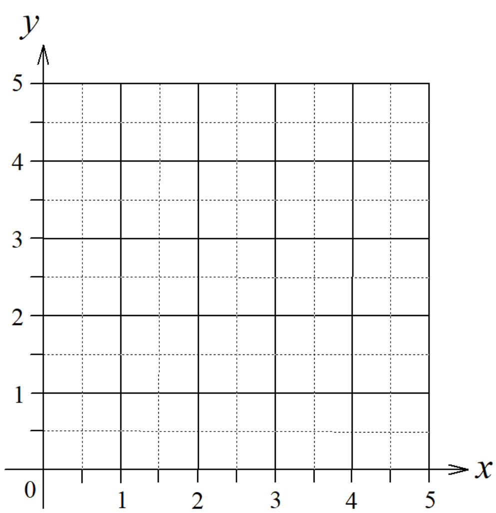


Diagram 6

ii. Draw the graph of y against x for the domain on Diagram 6.

iii. This type of function y is also known as a step function. Explain why y is a function of x.

iv. Another way to represent a step function is by listing out the values of y for different intervals of x. For example, for the domain $0 \leq x < 2.5$, y can be represented as follow:

$$y = \begin{cases} 0 & \text{if } 0 \leq x < 0.5 \\ 1 & \text{if } 0.5 \leq x < 1.5 \\ 2 & \text{if } 1.5 \leq x < 2.5 \end{cases}$$

- Complete this way of representing function y for the domain $0 \leq x \leq 5$.

Task 5: Functions in Real World

Car Parking Fee

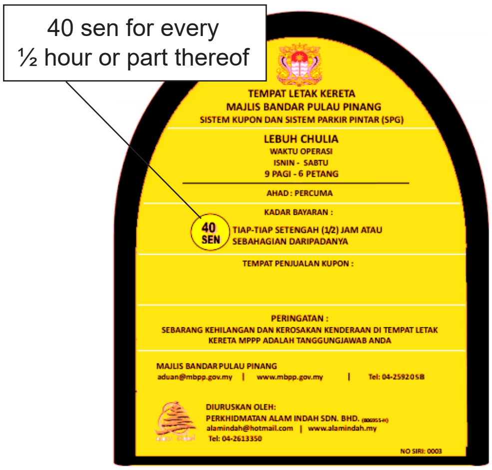


Diagram 7

Diagram 7 shows the rate of parking fee charged by Penang Island City Council.

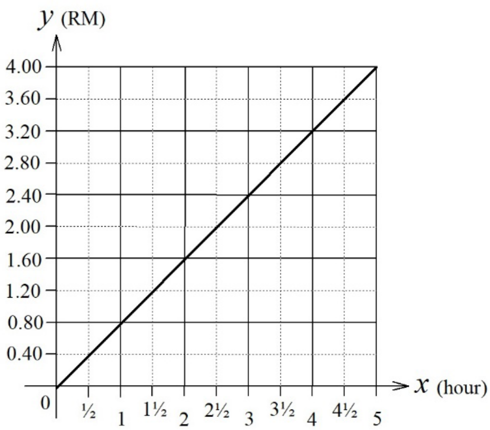


Diagram 8

i. A student think that since it charges 40 sen, which is RM0.40 for every ½ hour, so the parking fee is directly proportional to parking time. He then plotted the graph as shown in Diagram 8 to represent the relationship between parking fee, y and time x.

- What is wrong with the graph? Why is it wrong?

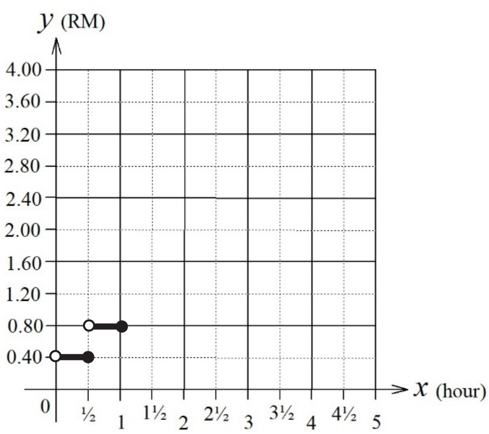


Diagram 9

ii. Diagram 9 shows the graph of parking fee, y in RM against parking time, x in hour plotted for the interval $0 < x \leq 1$.

- Complete the graph for the interval $1 < x \leq 5$.
- Is y a function of x? Explain your reasons.

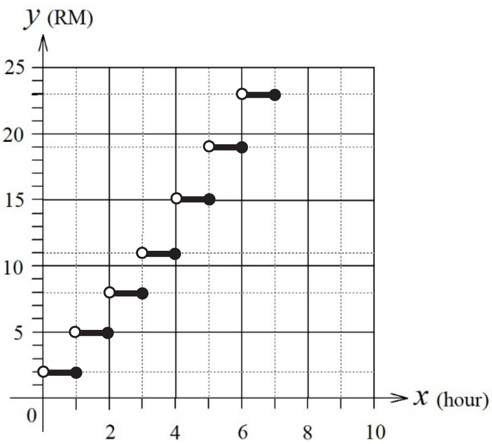


Diagram 10

iii. Diagram 10 shows the graph of parking fee, y against parking time, x of a private car park in Penang Island for the interval $0 < x \leq 7$.

- Based on the graph, explain the rate of parking charges at this private car park.
- Uncle Ben parked his car at this private car park for 4 hours 25 minutes. How much is the parking charge?
- Ms Aishah has RM20 only with her. How long can she park her car at this private car park?

Rotating Wheels

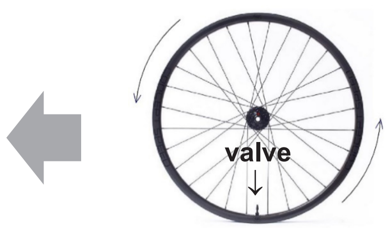


Diagram 11

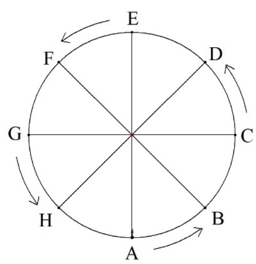


Diagram 12

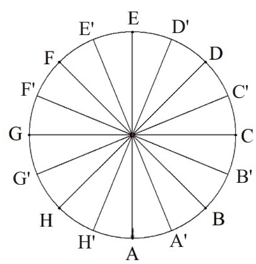


Diagram 13

Diagram 11 shows a bicycle wheel with its valve, A, at the ground level. The wheel is rotating counter-clockwise to move to the left as shown by the arrows. Given that the radius of the rim of the wheel is 50 cm and the valve rotates at a constant speed of 1 revolution in every 4 s.

i. What will happen to the location of the valve as the wheel move?

ii. Diagram 12 shows seven other positions of the valve, B, C, D, E, F, G, and H, as the wheel moves.
   - What is the time taken for the valve to move from
     - ❖ A to C?
     - ❖ A to E?
     - ❖ A to G?
     - ❖ A to B?
     - ❖ A to H?
   - What is the vertical distance of the valve from the ground level at
     - ❖ position C?
     - ❖ position E?
     - ❖ position G?

iii. Diagram 13 shows another eight positions of the valve, A’, B’, C’, D’, E’, F’, G’ and H’.
   - What is the time taken for the valve to move from
     - ❖ A to A’?
     - ❖ A to F’?

iv. Let the vertical height of the valve from the ground level be y cm after it starts to rotate x seconds from A. Diagram 14 shows part of the graph of y against x for the domain $0 \leq x \leq 4$.

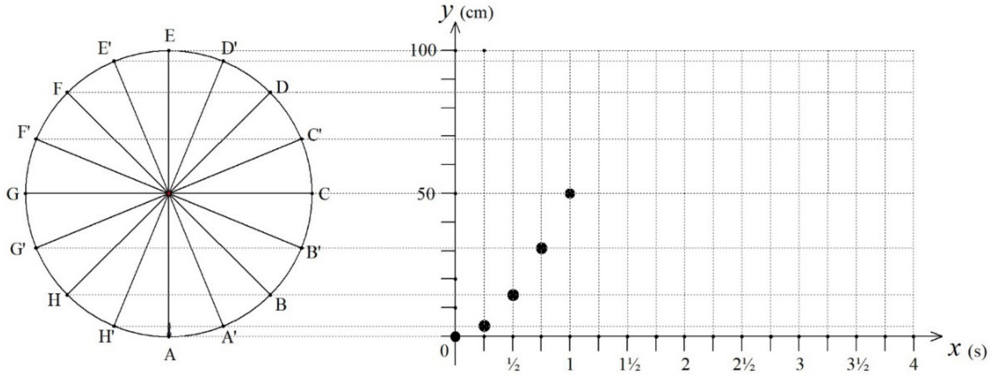


Diagram 14

In this graph, the coordinates (x, y) when the valve is at positions A, A', B, B' and C are already plotted. At A and C , y = 0 cm and y = 50 cm respectively. However, with the help of Diagram 14, the coordinates at A', B, and B' are plotted without the need to know the values of y.

- Explain how this can be done.
- Plot the coordinates for all other positions of the valve.
- Complete the graph by joining all the coordinates plotted with a smooth curve.

v. Explain why y is a function of x for the domain $0 \leq x \leq 4$.

- If we continue drawing the graph for the domain $4 < x \leq 8$, how would the graph look like?
- Is y still a function of x for the domain $0 \leq x \leq 8$? Explain your reasons.

CHAPTER 4

Space and Geometry [K3SG]

Topic 1: Exploring Angles, Construction and Designs in Geometry [K3SG1]

Standard 1.1: [K3SG1-1]

Exploring angles to explain simple properties on the plane geometry and do the simple geometrical Construction

i. Explain how to determine the value of angles using the geometrical properties of parallel lines, intersecting lines, and properties of figures
ii. Use ruler and compass to construct a simple figure such as perpendicular lines and bisectors
iii. Appreciate the process of reasoning that utilizes the properties of angles and their congruency in simple geometrical constructions

Sample Tasks for Understanding the Standards

Task 1: Lines and Angles

Lines, Rays and Line Segments

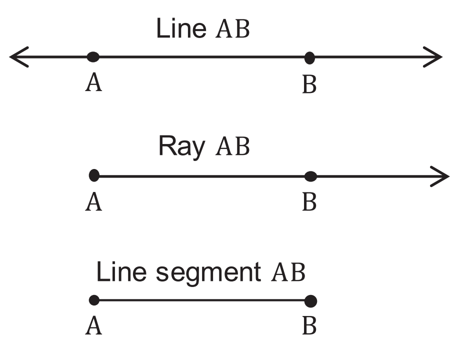


Diagram 1

Diagram 1 shows three basic figures that passes through points A and B, a line, a ray and a line segment.

i. Explain the similarities and differences between a line, a ray and a line segment.

Vertical Angles

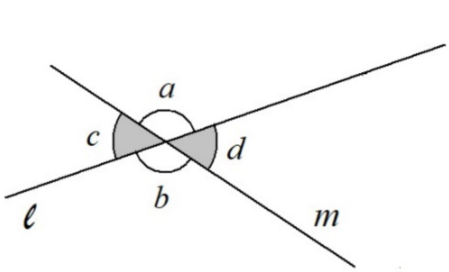


Diagram 2

Diagram 2 shows two intersecting lines, ℓ and m.

i. $\angle a$ and $\angle b$ are a pair of vertical angles.
   - What is the relationship between
     - $\angle a$ and $\angle c$
     - $\angle b$ and $\angle c$
   - Explain why $\angle a = \angle b$?

ii. $\angle c$ and $\angle d$ are another pair of vertical angles.
   - Prove that $\angle c = \angle d$.

Corresponding Angles and Alternate Interior Angles

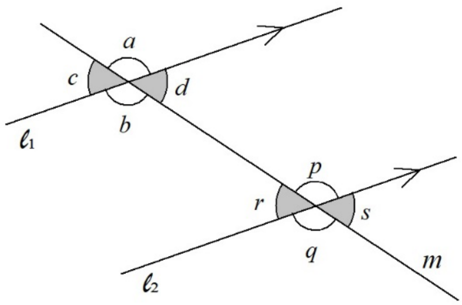


Diagram 3

Diagram 3 shows two parallel lines, ℓ_1, ℓ_2 and an intersecting line, m.

i. $\angle a$ and $\angle p$ are a pair of corresponding angles.

- What is the relationship between $\angle a$ and $\angle p$?
- There are three other pairs of corresponding angles in Diagram 3.
    - ❖ Name the three pairs of corresponding angles.
    - ❖ State the relationships among these pairs of corresponding angles.

ii. $\angle d$ and $\angle r$ are a pair of alternate interior angles.

- Explain why $\angle d = \angle r$.
- There is another pair of alternate interior angles in Diagram 3. Name the pair of alternate interior angles and state their relationship.

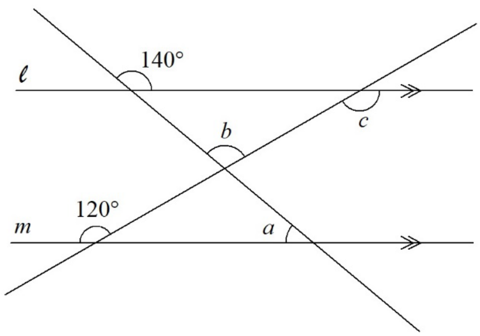


Diagram 4

iii. In Diagram 4, line ℓ is parallel to line m.

- Find $\angle a$, $\angle b$ and $\angle c$.
- Explain your reasoning in finding $\angle a$, $\angle b$ and $\angle c$ by using the properties of angle such as parallel lines and intersection.

Task 2: Sum of Interior Angles

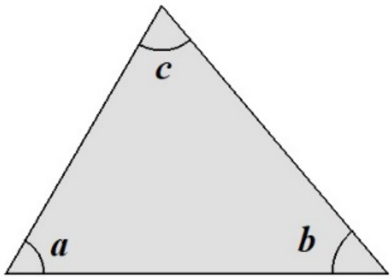


Diagram 5

i. Diagram 5 shows a triangle with interiors angles $\angle a, \angle b$ and $\angle c.$

- Prove that $\angle a + \angle b + \angle c = 180°$

  [Hint: Use the properties of lines intersecting with parallel lines.]

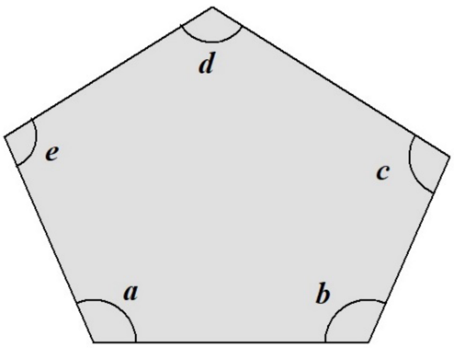


Diagram 6

ii. Diagram 6 shows a pentagon with interior angles $\angle a, \angle b, \angle c, \angle d$ and $\angle e$.

- Find the sum $\angle a + \angle b + \angle c + \angle d + \angle e$.

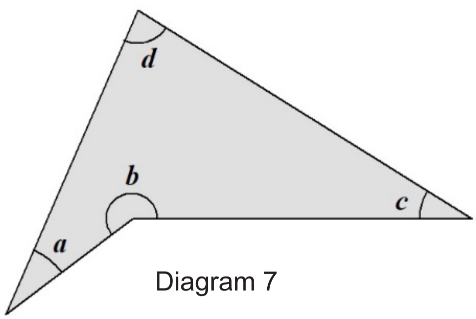


Diagram 7

iii. Diagram 7 shows a concave quadrilateral with interior angles $\angle a, \angle b \quad \angle c$ and $\angle d$.

- Prove that $\angle a + \angle b + \angle c + \angle d = 360°$.

  [Hint: Use sum of interior angles of a triangle.]

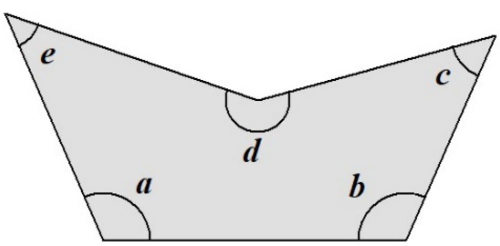


Diagram 8

iv. Diagram 8 shows a concave pentagon with interior angles $\angle a, \angle b, \angle c, \angle d,$ and $\angle e$.

- Find the sum $\angle a + \angle b + \angle c + \angle d + \angle e$.

Task 3: Determining a Location

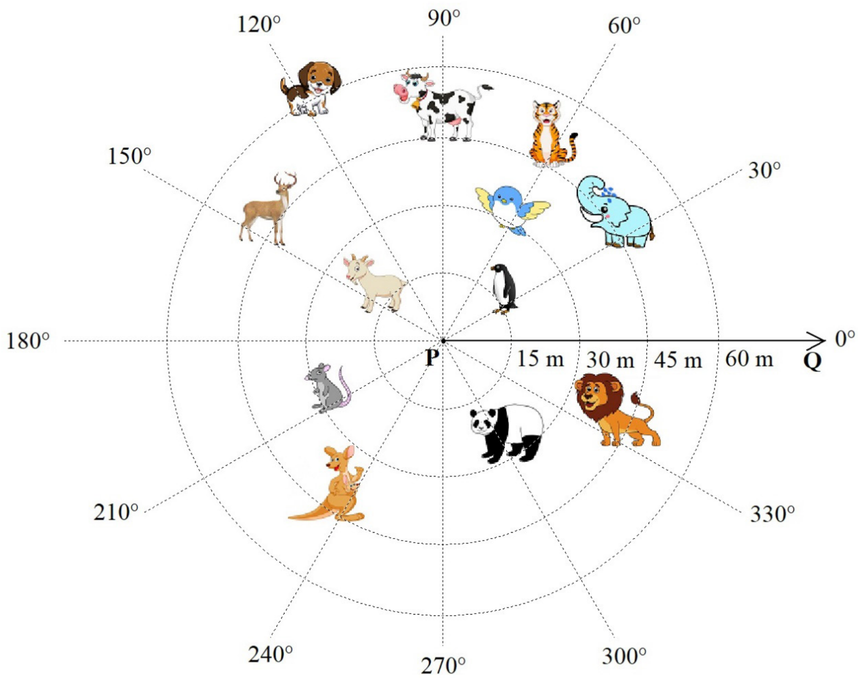


Diagram 9

Diagram 9 shows a system to determine location in a plane. P, the centre of the concentric circles is taken as the reference point. Two basic measurements used to determine a location are the distance from P and the angle from ray PQ.

i. Find the animals located at
   - 60 m from P
   - 45 m from P
   - 240° from ray PQ
   - 150° from ray PQ

ii. What animal is located at 45 m from P and 150° from ray PQ?

iii. State the location of the cow, the deer and the panda respectively.

iv. If there is an animal located at 180° from ray PQ, what can you say about the distance of the animal from P?

v. In a treasure hunt game, what would you do for each of the following instructions?
   - Stand at the tree. One treasure is 10 m from the tree.
   - Stand at the tree and look at the mountain top. Another treasure is 45° from this direction.

vi. In this system, why are both distance and angle important in determining a location?

Task 4: Geometrical Constructions

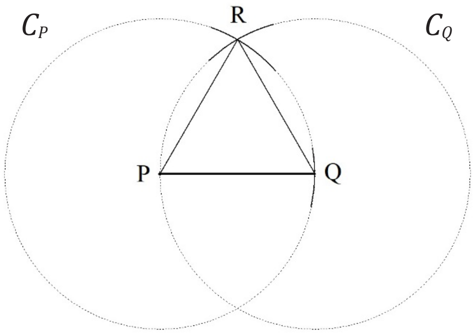


Diagram 10

i. Diagram 10 shows a line segment PQ. Two circles, C_P and C_Q are constructed with P and Q as the centre respectively. Both circles have equal radii, which is the length of PQ. R is an intersection point of C_P and C_Q.

- Why is $\mathrm{PQ} = \mathrm{PR} = \mathrm{QR}$?
- Why is $\angle\mathrm{PQR} = \angle\mathrm{PRQ} = \angle\mathrm{QPR}$?
- What is the value of $\angle\mathrm{PQR}$?

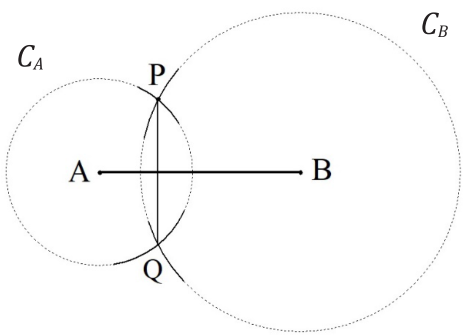


Diagram 11

ii. Diagram 11 shows a line segment AB and a point P not on the segment. Two circles C_A and C_B with centre A and B respectively are constructed. The radius of C_A is AP and the radius of C_B is BP. Q is an intersection point of C_A and C_B.

- Why is line segment PQ perpendicular to AB?

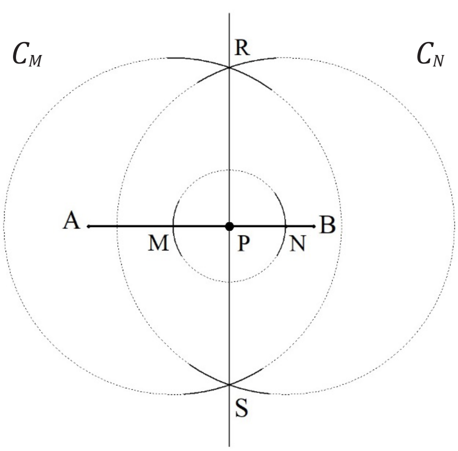


Diagram 12

iii. Diagram 12 shows a line segment AB and a point P on the segment. A circle, C_P is constructed with P as its centre. C_P intersects with AB at points M and N. Then, two other circles with equal radii, C_M and C_N with centre at M and N respectively are constructed. R and S are two intersection points of C_M and C_N.

- Why is line segment RS perpendicular to AB?

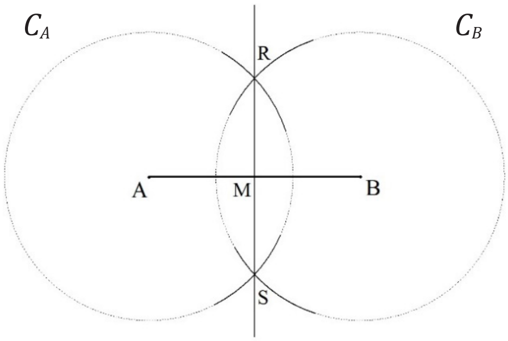


Diagram 13

iv. Diagram 13 shows a line segment AB. Two circles with equal radii, C_A and C_B are constructed with the centre A and B respectively. R and S are two intersection points of C_A and C_B. M is an intersection point of AB and RS.

- Why is $\mathrm{AM} = \mathrm{MB}$?
- Why is $\angle\mathrm{AMR} = 90°$?
- What does line RS do to line segment AB?

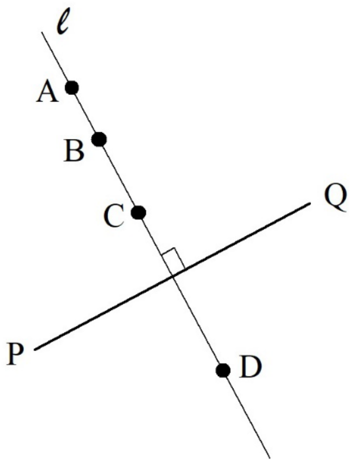


Diagram 14

v. Diagram 14 shows a line segment PQ and its perpendicular bisector ℓ. A, B, C and D are four points on ℓ.

- What relationship exists between line segments
  - ❖ AP and AQ?
  - ❖ BP and BQ?
  - ❖ CP and CQ?
  - ❖ DP and DQ?
- What can you conclude about the distance from any point on the bisector to P and Q respectively?

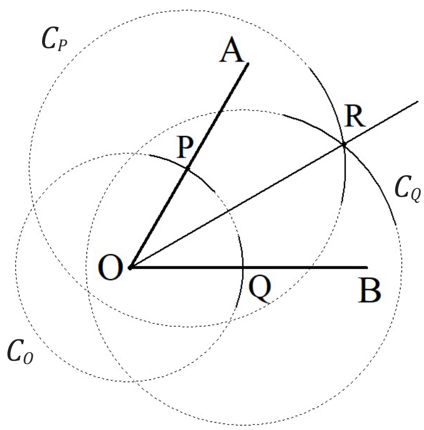


Diagram 15

vi. Diagram 15 shows an angle $\angle\mathrm{AOB}$. A circle, C_O, is first constructed with O as its centre. C_O intersects with line segments OA and OB at points P and Q respectively. Then, two other circles with equal radii, C_P and C_Q are constructed with P and Q as the centres. OR is a ray joining O to the intersection point of C_P and C_Q.

- Why is $\angle\mathrm{AOR} = \angle\mathrm{BOR}$?
- What does ray OR do to AOB?

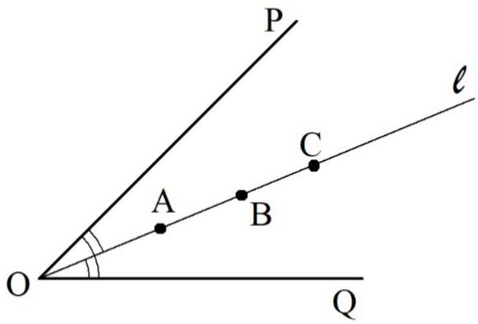


Diagram 16

vii. Diagram 16 shows an angle $\angle POQ$ and its angle bisector ℓ. A, B and C are three points on ℓ.

- What relationships exist between the distances from point A, B, C to the two sides of the angle, OP and OQ respectively?

Diagram 17

viii. Diagram 17 shows a line segment AB.

- Given C is a point such that $\angle CAB = 60°$ and $\angle CBA = 30°$. Use compass and ruler only to construct $\triangle ABC$.

- Given P is a point such that $\angle PAB = 45°$ and $\angle PBA = 90°$. Use compass and ruler only to construct $\triangle ABP$.

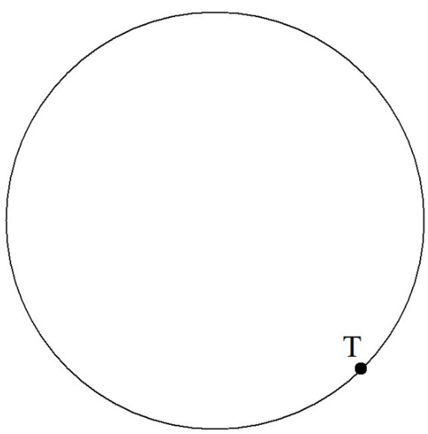


Diagram 18

ix. Diagram 18 shows a circle with its centre unmarked. T is a point on the circle.

- What properties of a circle can be used to locate the centre of the circle?

- Use compass and ruler only to locate the centre of the circle on Diagram 18. Mark and label the centre as O.

- The tangent of a circle is perpendicular to its radius that passes through the point of tangency. Use compass and ruler only to construct the tangent of the circle at point T on Diagram 18.

Task 5: Application of Geometrical Construction in Real World

Diagram 19

Diagram 19 shows a groundwater well pump and the locations of three villagers' houses in a rural village in Cambodia.

i. Mr Ratha and Mr Chantha want to construct a groundwater well together for their families use. They want it to be built at a location which will be the same distance from the two houses.
   - Help the two villagers to identify the possible locations to construct the well by doing some geometrical constructions on Diagram 19.
   - Which location will be the nearest to their houses?

ii. When Mr Vutha heard about his two neighbours' plan, he asks to join in the construction of the well. Now, the three of them are having a little problem in identifying the location for the well that will be the same distance from the three houses.
   - Help the three villagers to identify the best location for the well by doing some geometrical constructions on Diagram 19.

iii. If a fourth villager wants to join in the construction of the well, there will be a condition that need to be fulfilled in order to find a location that is the same distance from four houses.
   - What is the condition?
   - What if there are five houses or more?

Standard 1.2: [K3SG1-2]
Exploring the relationship of figures using congruency and enlargement for designs

i. Explore the congruence of figures through reflection, rotation and translation and explain the congruency using line of symmetry, point of symmetry and parallel lines
ii. Explore similarity of figures with enlargement using points, ratio, and correspondences
iii. Enjoy using transformations in creating designs

Sample Tasks for Understanding the Standards

Task 1: Transforming Geometric Figures

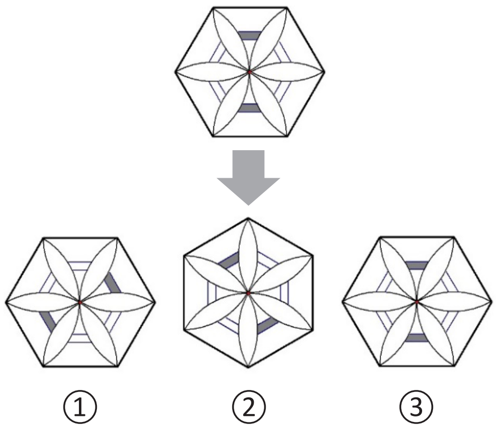


Diagram 1

i. Diagram 1 shows a geometric figure and its three images. Each image is the result of only one transformation.
   - Explain why enlargement cannot be involved in creating any of the three images.
   - Which image is the result of a translation? Justify your decision.
   - Which two images are the result of a reflection? Justify your decision. Then, describe each reflection clearly.
   - All three images are the results of a rotation. Describe each rotation.
     - ❖ A student claims that every image has more than one angle of rotation. How is this possible?
   - Each of the geometric figures has two lines of symmetry. Identify the lines of symmetry for each figure.
   - Is there any figure that has a point of symmetry? If any, where is the point?

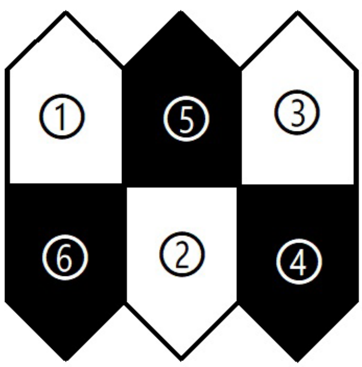


Diagram 2

ii. Three black pentagons and three white pentagons are tiled as shown in Diagram 2.
   - What is the minimum number of transformation to move pentagon ① to fit onto ②, ⑤, and ⑥, respectively?
     - ❖ Describe how it can be done.
   - What is the minimum number of transformation to move pentagon ④ to fit onto ⑥?
     - ❖ Describe how it can be done.
   - What is the minimum number of transformation to move pentagon ③ to fit onto ⑥?
     - ❖ Describe how it can be done.

Task 2: Isometric Transformations

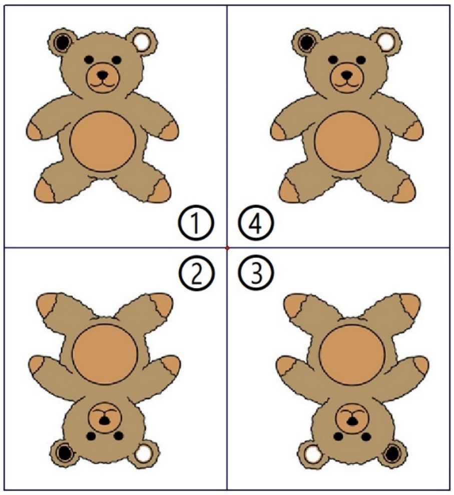


Diagram 3

Reflection, translation and rotation are isometric transformations.

i. In Diagram 3, teddy bear ① has undergone translation, reflection and rotation respectively. Study teddy bear ① and its three images.

- Identify the transformation that has moved teddy bear ① onto teddy bears ②, ③ and ④, respectively.
- What has changed and what remains unchanged between teddy bear ① and each of its images?
- What does it mean by all the teddy bears in Diagram 3 are congruent figures?
- What is an isometric transformation?

Diagram 4

ii. Diagram 4 shows a bird and its image after being rotated.

- Use compass and ruler to locate and mark the centre of rotation.
- Measure and state the angle of rotation.

Diagram 5

iii. Diagram 5 shows a bird and its image after being reflected.

- Use compass and ruler to draw and mark the axis of reflection.

Task 3: Non-Isometric Transformation

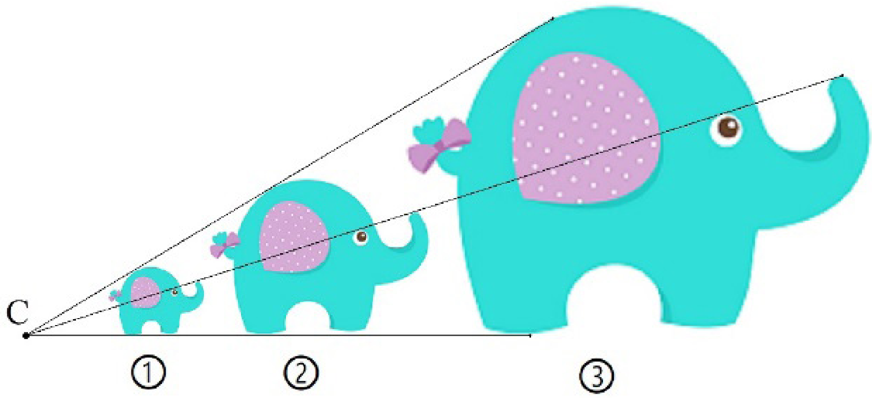


Diagram 6

The three elephants in Diagram 6 are similar figures.

i. Explain the meaning of similar figures.

ii. What transformation is involved here?

iii. What is point C? Why is it important?

iv. Given that the size of elephant ③ is 2 times the size of elephant ②, and the size of elephant ① is ½ the size of elephant ②.

- If the tail of elephant ① is 15 cm long, how long is the tail of elephant ③?
- What is the ratio between the size of elephant ③ and the size of elephant ①?
- If the ear of elephant ② has an area of 2500 cm^2, what is the area of the ear of elephant ③?

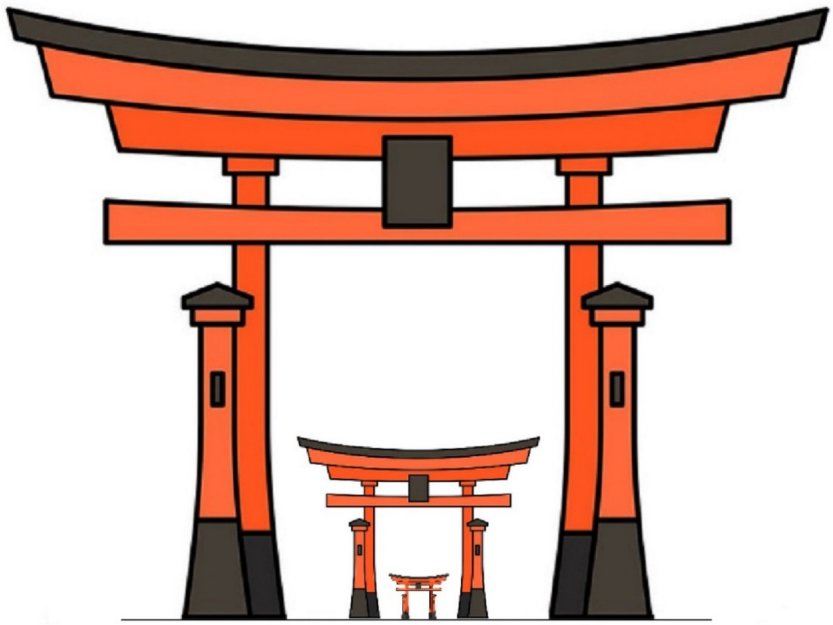


Diagram 7

v. Diagram 7 shows three Japanese torii gates created by enlarging the smallest gate.

- Locate and mark the centre of enlargement on Diagram 7.

Diagram 8

vi. Diagram 8 shows another three torri gates as a results of enlarging the smallest gate.

- Locate and mark the centre of enlargement on Diagram 8.

Task 4: Creative Designs Using Transformations

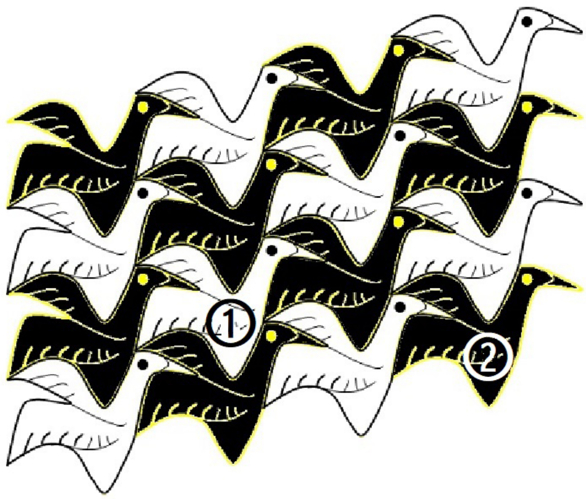


Diagram 9

Transformation is often found in the creation of creative visual art.

i. The tessellation of the white and black birds in Diagram 9 is a creative design produced using transformation. As we can see, both the black and white birds are congruent figures.

- What transformation takes bird ① onto bird ② ?

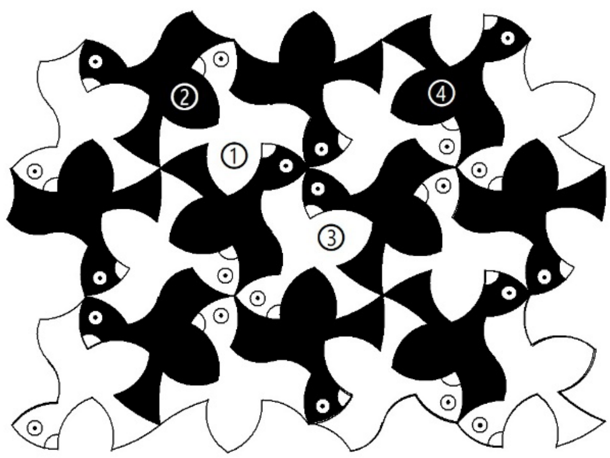


Diagram 10

ii. Diagram 10 shows another tessellation of white and black birds.

- What one transformation takes bird ① onto bird ②?
- What two transformations take bird ③ onto bird ④?

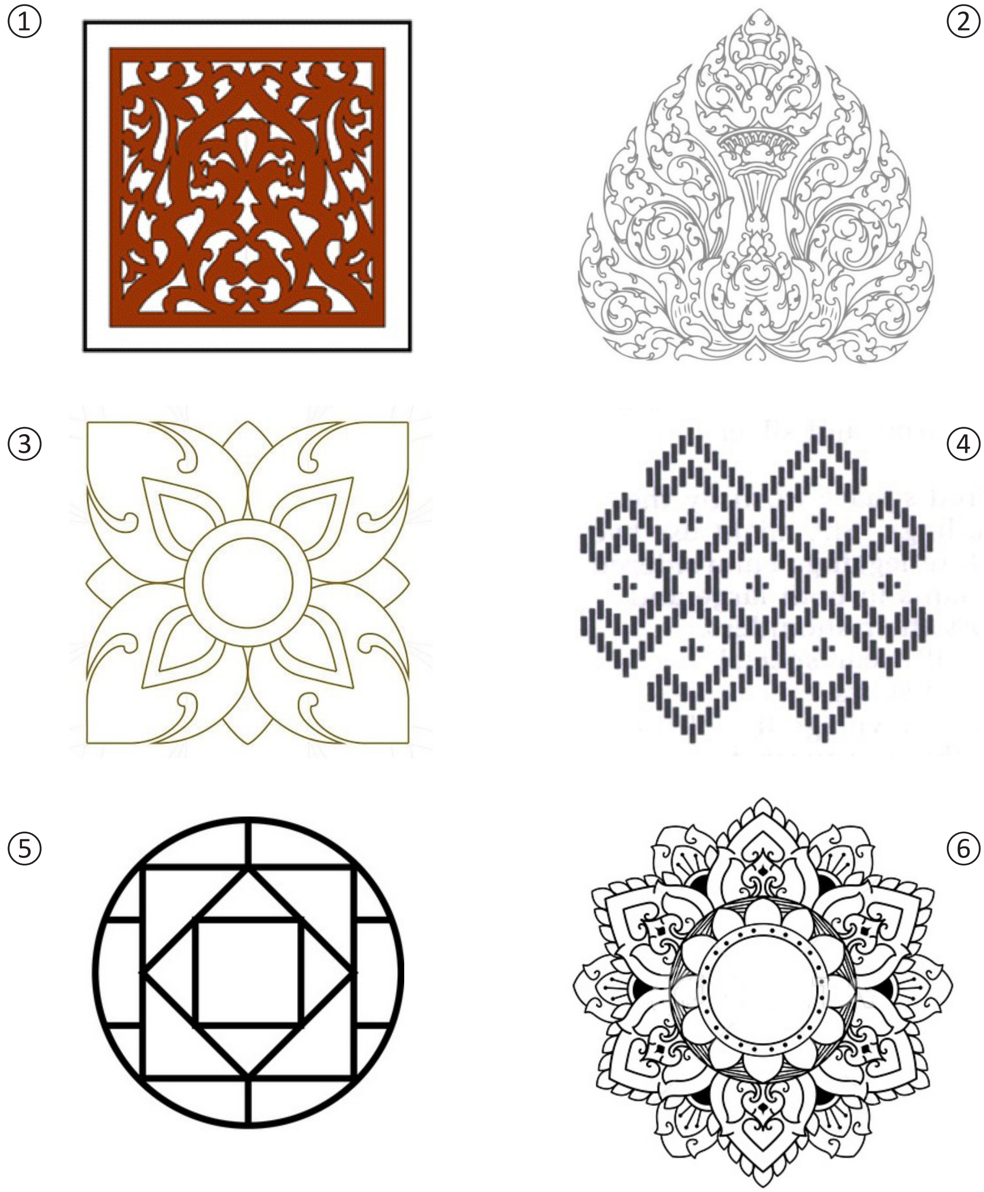


Diagram 11

iii. Symmetry is commonly seen in traditional native designs found in different cultural items such as textiles and sculptures. Diagram 11 shows six native designs from the ASEAN countries.

- Draw all the lines of symmetry for each of the designs.
- Identify all the rotational symmetry for each of the designs, if any.
- Identify the point of symmetry for each of the designs, if any.

Topic 2: Exploring the Space With Its Components [K3SG2]

Standard 2.1: [K3SG2-1]
Exploring space by using the properties of planes, lines and their combinations to form solids

i. Explore the properties produced by planes, lines and their combinations, such as parallel lines produced by intersection of parallel planes with another plane
ii. Produce solids by combining planes such as nets and motion such as rotation, reflection and translation
iii. Recognise the space of an object based on its properties and projection in life

Sample Tasks for Understanding the Standards

Task 1: Lines and Planes in Space

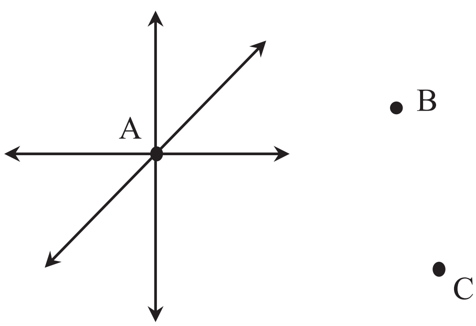


Diagram 1

i. Diagram 1 shows three points A, B, C and three lines passing through point A on a same plane.

- How many other lines passing through point A are there altogether?
- Of all the lines passing through point A, how many line will also pass through point B?
- Is there a line that passes through all three points A, B and C?
  - ❖ Why or why not?
- How many minimum points are required to determine a line?

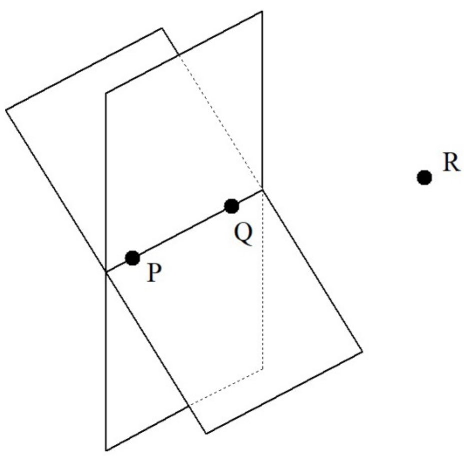


Diagram 2

ii. Diagram 2 shows three points P, Q and R in space. Two planes passing through both points P and Q are shown.

- How many planes passing through both points P and Q are there altogether?
- Of all the planes that passing through points P and Q, how many plane will also pass through point R?
- A minimum of three points in space are required to determine a plane. What is the condition of the three points?

Diagram 3

iii. Diagram 3 shows a camera tripod.

- Why does a camera tripod have 3 legs?

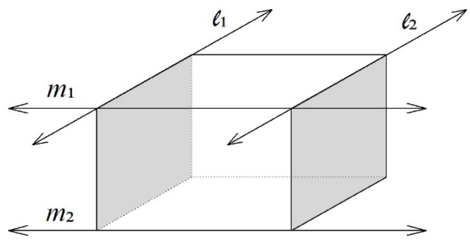


Diagram 4

iv. Diagram 4 shows a cuboid and four lines, ℓ_1, ℓ_2, m_1 and m_2 that pass through four of its edges.

- Why are ℓ_1 and ℓ_2 called parallel lines?
- Lines ℓ_1 and m_1 intersect with each other at a point because they are not parallel lines. However, lines ℓ_2 and m_2 are not parallel lines but they will not intersect with each other. Explain why ℓ_2 and m_2 will not intersect with each other.
  [*Note*: ℓ_2 and m_2 are known as a pair of skew lines.]
- What is the condition for two non-parallel lines in space to intersect with each other?

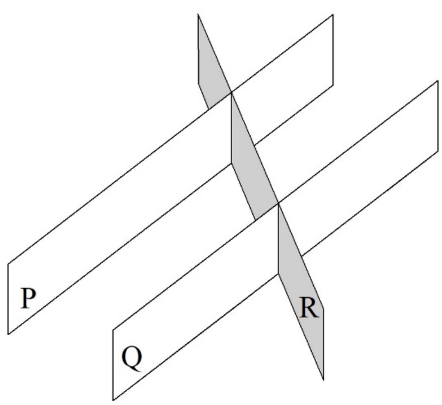


Diagram 5

v. Diagram 5 shows three planes, P, Q and R. Planes P and Q are parallel, whereas plane R intersects with planes P and Q.

- Why are planes P and Q called parallel planes?
- When plane R intersects with planes P and Q, two intersecting lines are formed.
  - ❖ What is the positional relationship between the two intersecting lines?
  - ❖ What if planes P and Q are not parallel?
- How can we define the distance between two parallel planes such as planes P and Q?
- How can we define the angle between two intersecting planes such as planes P and R?

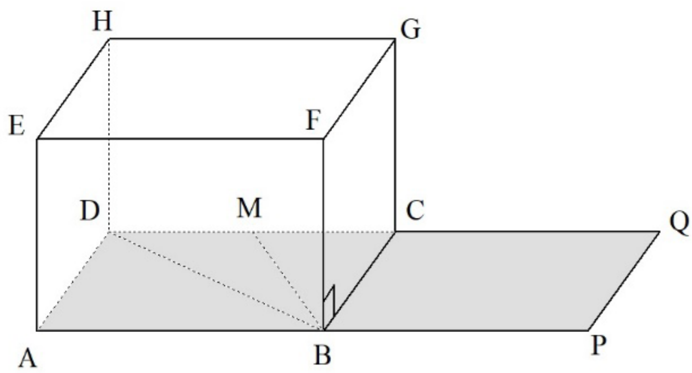
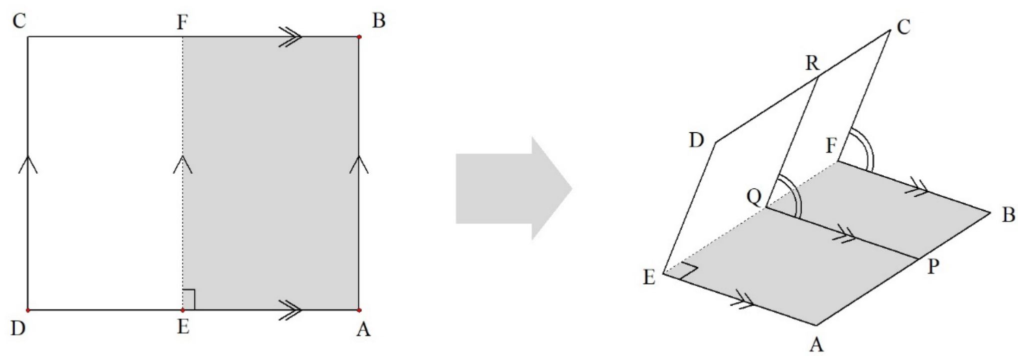


Diagram 6

vi. Diagram 6 shows a cuboid ABCDEFGH placed on plane APQD. M is a point on edge CD. Given that line FB is perpendicular to plane ABCD and $\angle FBC = 90°$.

- What is $\angle FBP$, $\angle FBA$, $\angle FBD$ and $\angle FBM$, respectively?
- What is the positional relationship between lines GC, HD and EA with line FB?
- What is the positional relationships between lines GC, HD and EA with plane APQD, respectively?



(a) (b)

Diagram 7

vii. In Diagram 7(a), a rectangular card ABCD is folded along EF such that EF ∥ AB ∥ CD. Then, part of the card, EDCF is turned to form two planes intersecting at EF as shown in Diagram 7(b). P, Q and R are three points on lines AB, EF and DC, respectively such that PQ ∥ AE ∥ BF and QR ∥ ED ∥ CF.

- What is the measure of $\angle DEF$?
- What is the positional relationships between $\angle PQR$, $\angle BFC$ and $\angle AED$?
- What is the measures of $\angle PQE$ and $\angle RQF$ respectively?

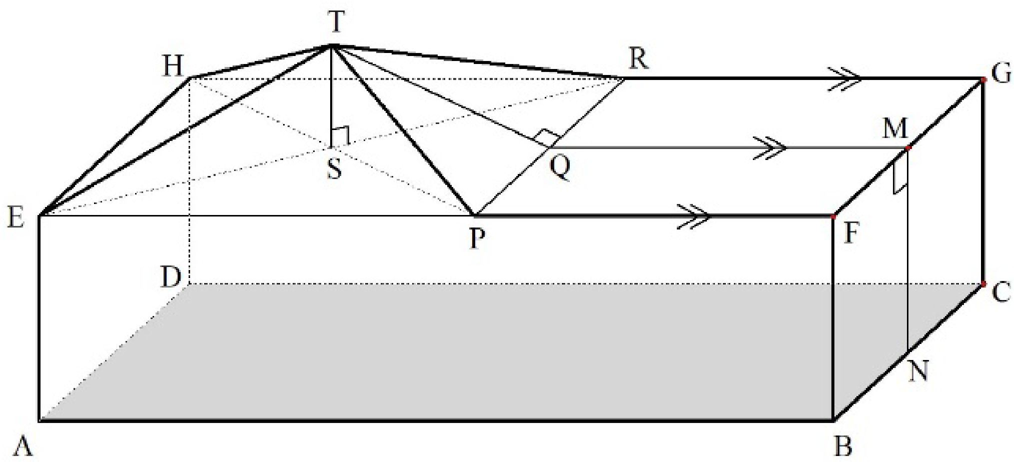


Diagram 8

viii. Diagram 8 shows a solid made up of a cuboid $ABCDEFGH$ as the base with a rectangular pyramid $EPRHT$ on top of the cuboid. TS is the perpendicular line from the apex of the pyramid to its base. Line TQ is perpendicular to line PR. M is a point on edge FG such that $QM \parallel PF \parallel RG$ and N is a point on edge BC such that $NM \perp FG$.

- Which edges are parallel lines with PR?
- Which edges are intersecting lines with PR?
- Which edges are skew lines with PR?
- State the distance from point E to (a) point A, (b) line FB, and (c) face $BCGF$, respectively.
- State the distance from point T to (a) point P, (b) line PR, and (c) base $EPRH$ of the pyramid, respectively.
- State the angle between base $EPRH$ of the pyramid and (a) line TE, (b) line TQ, (c) face PRT, respectively.

Task 2: Solids of One-Directional Movement

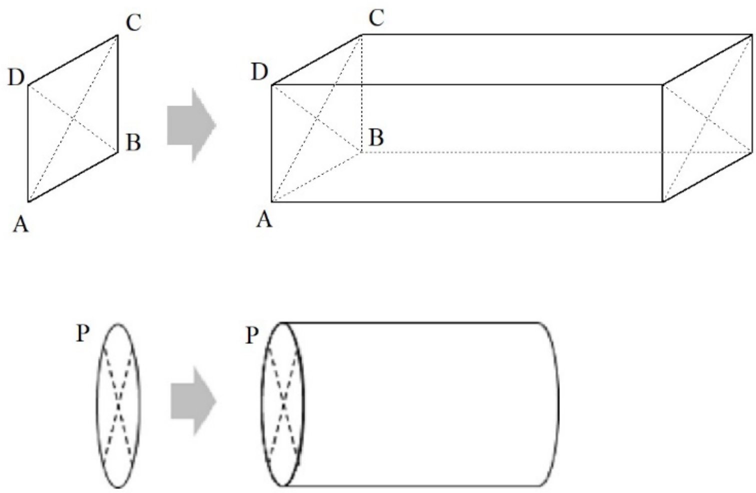


Diagram 9

i. Diagram 9 shows a cuboid generated from square ABCD and a cylinder generated from circle P.

- Describe how square ABCD could be moved to generate the cuboid.
- Describe how circle P could be moved to generate the cylinder.
- What transformation is involved in generating the cuboid and the cylinder?

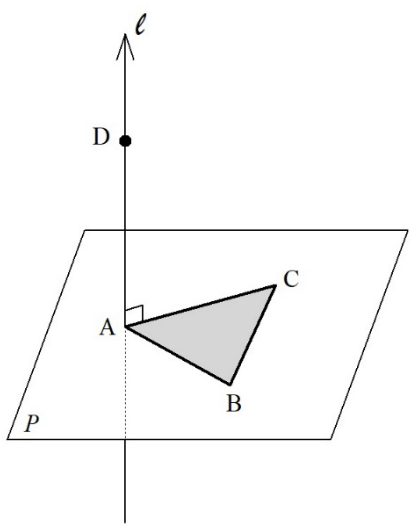


Diagram 10

ii. Diagram 10 shows a triangle ABC on a plane P and line ℓ is perpendicular to plane P.

- If $\triangle ABC$ moves vertically up along line ℓ from A to D, what solid will be created by the movement of $\triangle ABC$?
- Sketch the solid.

Task 3: Solids of Revolution

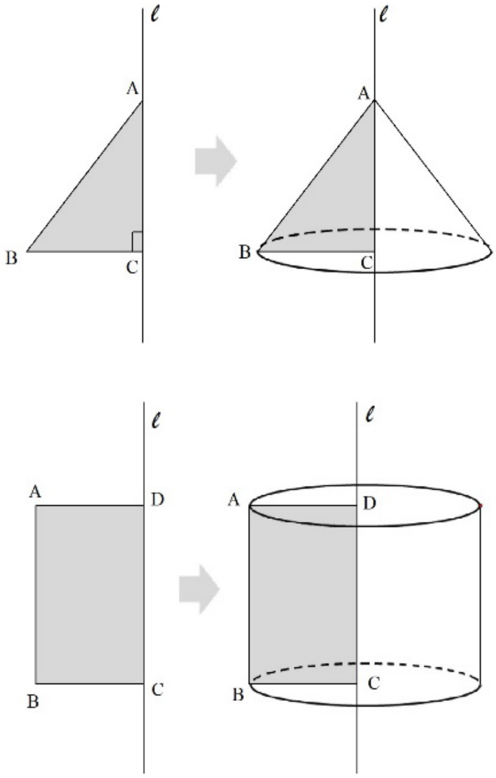


Diagram 11

i. Diagram 11 shows a cone generated from $\triangle ABC$ and a cylinder generated from rectangle $ABCD$.

- Describe how $\triangle ABC$ could be moved to generate the cone.
- Describe how rectangle $ABCD$ could be moved to generate the cylinder.

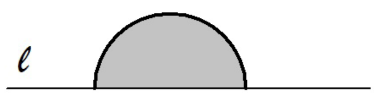


Diagram 12

ii. Diagram 12 shows a semicircle and a line ℓ.

- What solid will be generated if the semicircle is revolved once about axis ℓ ?

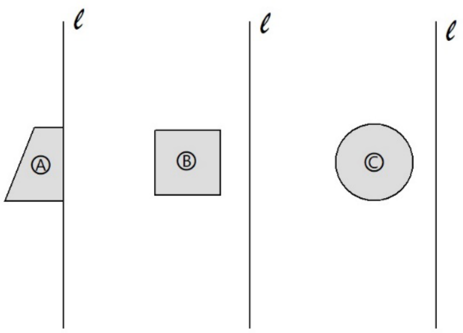


Diagram 13

iii. Diagram 13 shows a trapezium, a square and a circle with a line ℓ.

- Sketch the solid that will be created by the following revolutions.
  - ❖ Revolving trapezium Ⓐ once about axis ℓ.
  - ❖ Revolving square Ⓑ once about axis ℓ.
  - ❖ Revolving circle Ⓒ once about axis ℓ.

Task 4: Projections of Points, Line Segments and Faces

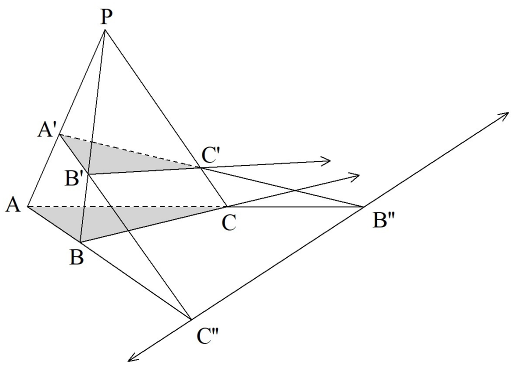


Diagram 14

Diagram 14 shows a triangular pyramid with base ABC and apex P. △A'B'C' is a cross section of the pyramid. The projections of AB and A'B' intersect at point C''. Likewise, the projections of AC and A'C' intersect at point B''. This makes B''C'' the line of intersection of plane AC''B'' and plane A' C''B''.

i. Given the projections of BC and B'C intersect at A''.
   - Where will A'' be?
   - Verify your answer on Diagram 14.

ii. How can we define the angle between the two intersecting planes, plane AC''B'' and plane A' C''B''?

iii. What will points A, B' and C become, respectively?
   - Taking the view from the perspective of point P.
   - Taking the view from the perspective of point C''.
   - Taking the view from the perspective of point B''.

iv. What will line segments AB, BB', A'C' and B'C' become, respectively?
   - Taking the view from the perspective of point P.
   - Taking the view from the perspective of point C''.
   - Taking the view from the perspective of point B''.

v. What will faces ABC, A'B'C', ABB'A', and BCC'B' become, respectively?
   - Taking the view from the perspective of point P.
   - Taking the view from the perspective of point C''.
   - Taking the view from the perspective of point B''.

Task 5: Projection of Solids

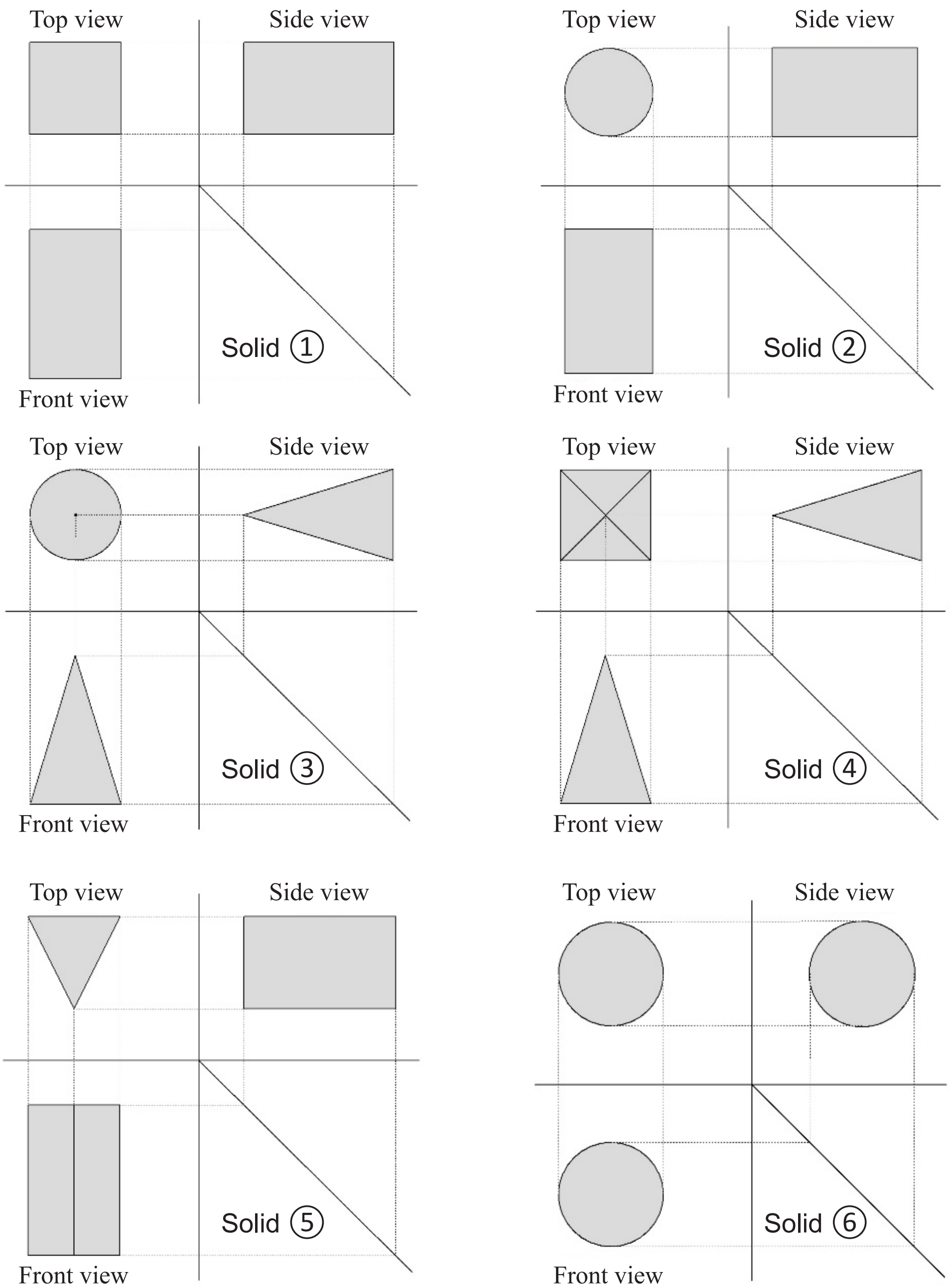


Diagram 15

Diagram 15 shows the projections of 6 solids.

i. What solid does each projection represent?

ii. Sketch the solid for each projection.

Task 6: Nets of Solids

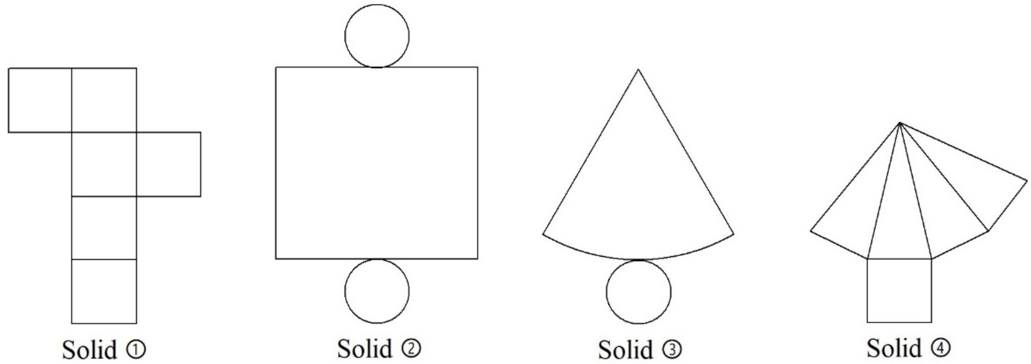


Diagram 16

i. Diagram 16 shows the nets of 4 solids. Identify and sketch each solid.

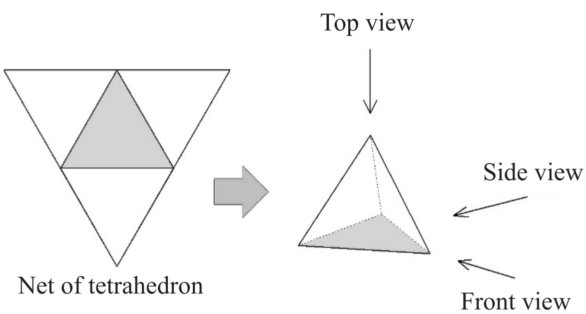
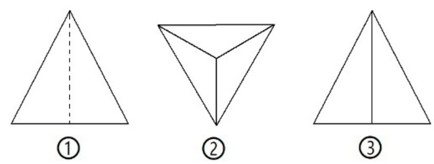


Diagram 17

ii. Diagram 17 shows a net folds up to be a solid tetrahedron.

- Match the top, front and side views with the following drawings.



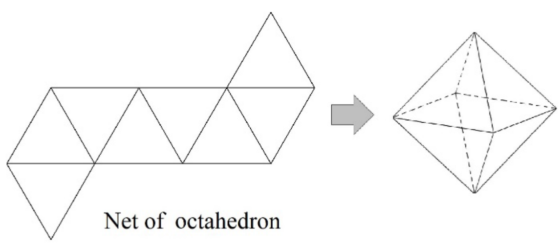


Diagram 18

iii. Diagram 18 shows a net folds up to be a solid octahedron.

- Draw the top, front and side views of the octahedron.

iv. Diagrams 19 and 20 show the nets of two regular polyhedrons.

- Make a copy of the nets.
- Cut out the nets and then build the regular polyhedrons.

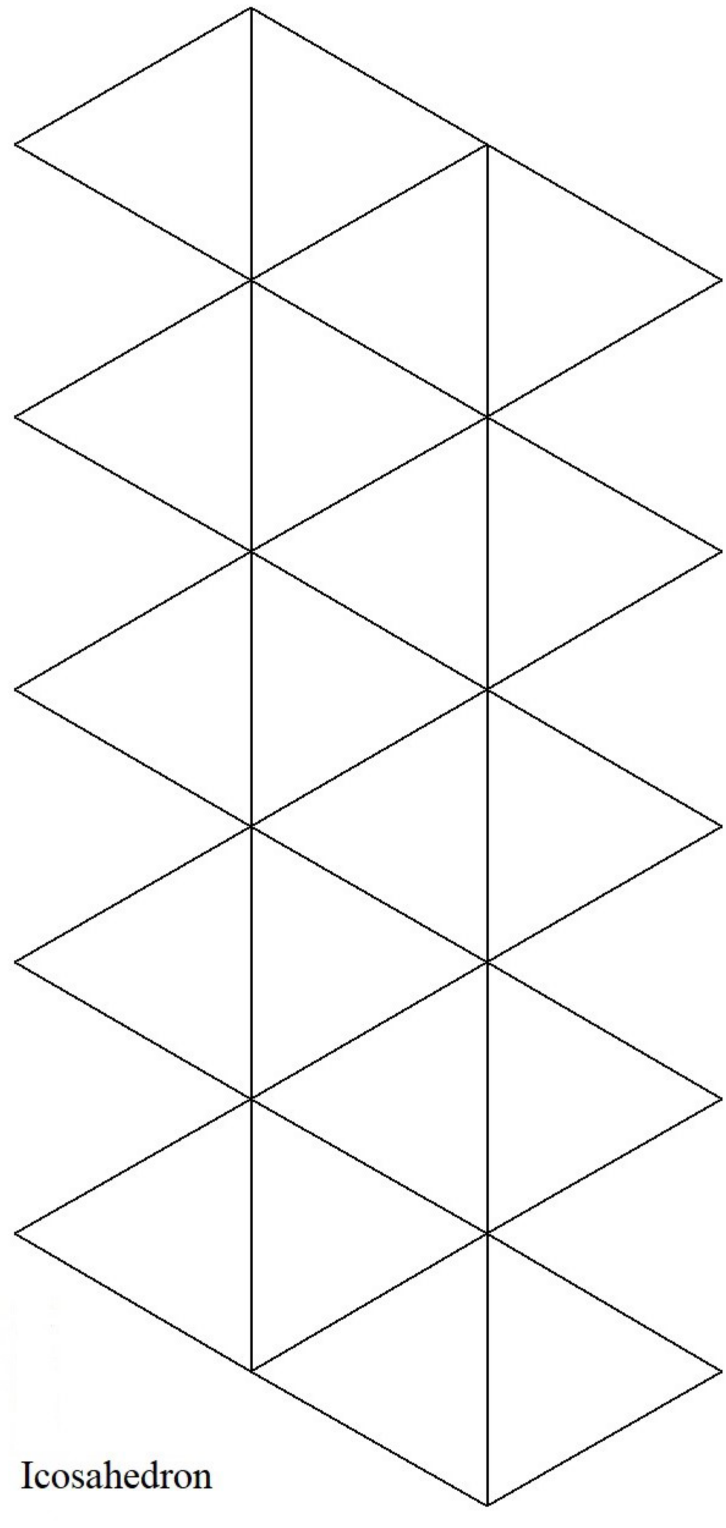


Diagram 19

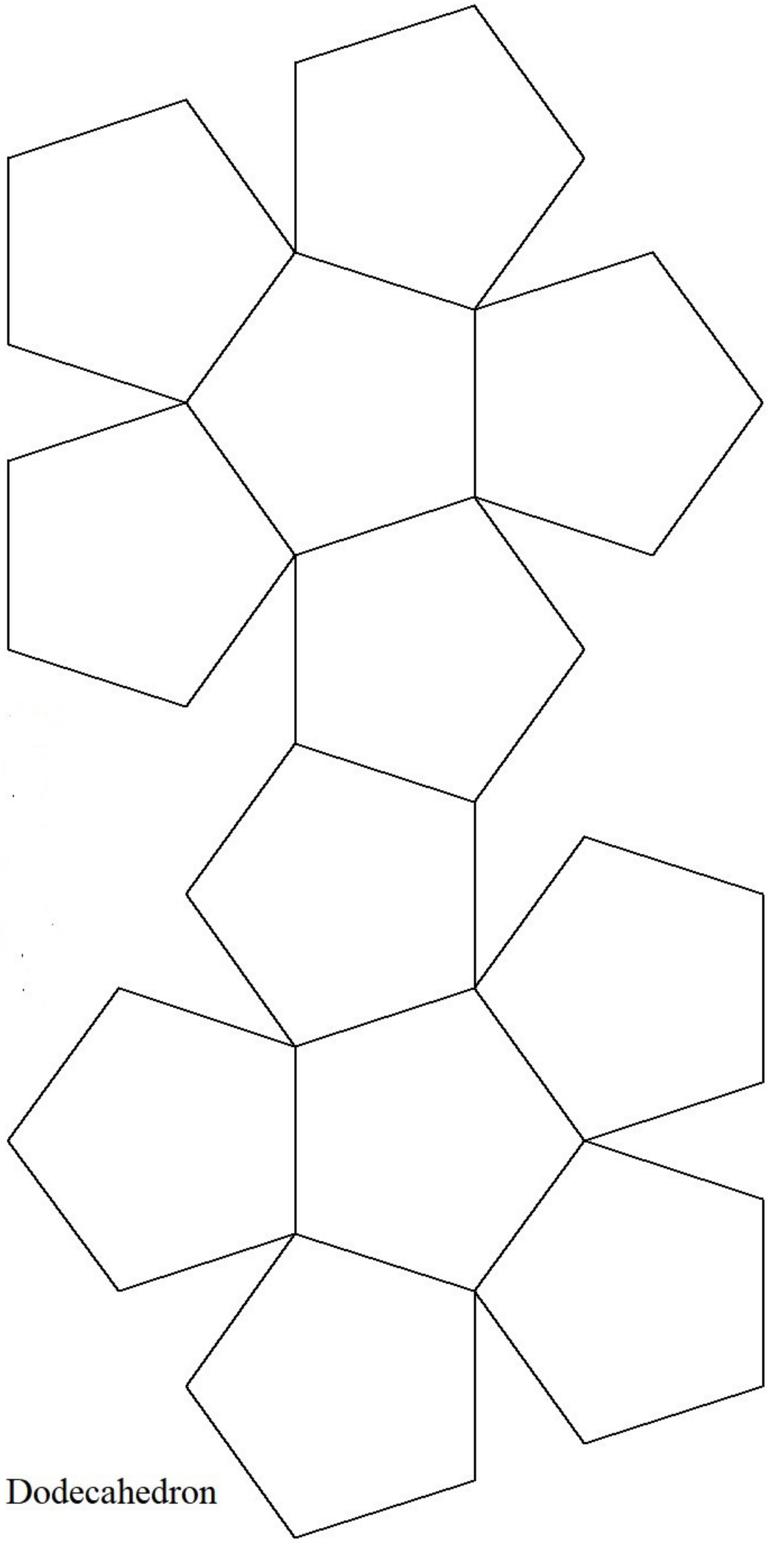


Diagram 20

Task 7: Surface Area and Volume of Solids

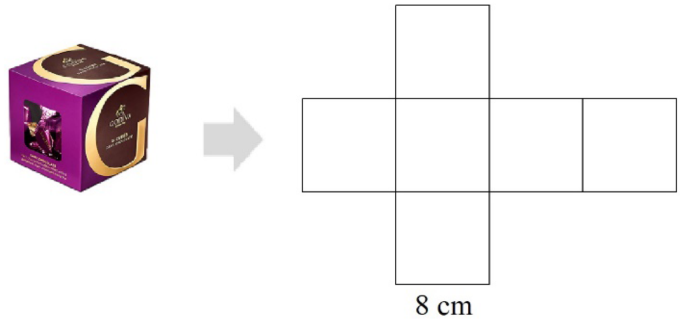


Diagram 21

i. Diagram 21 shows a candy box in the shape of a cube and its net. Each edge of the square base is 8 cm.

- Calculate the surface area and volume of the candy box.

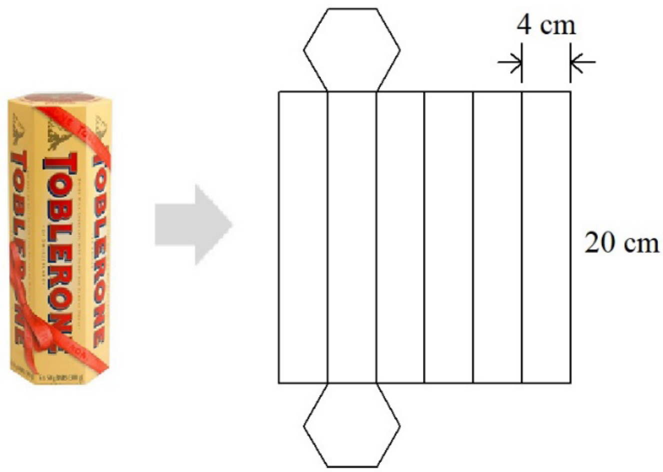


Diagram 22

ii. Diagram 22 shows another candy box in the shape of a prism and its net. The base of the prism is a regular hexagon. Each edge of the hexagon is 4 cm and the height of the prism is 20 cm.

- Calculate the surface area and volume of the candy box.

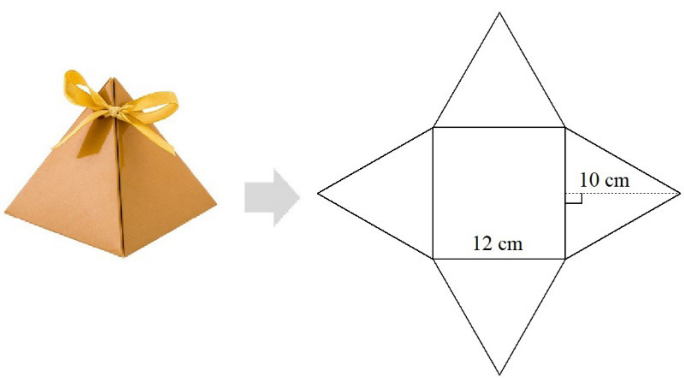


Diagram 23

iii. Diagram 23 shows a gift box in the shape of a pyramid and its net. The base of the pyramid is a square with 12 cm edge. The distance from its apex to the edge of the square base is 10 cm.

- Calculate the surface area and volume of the gift box.

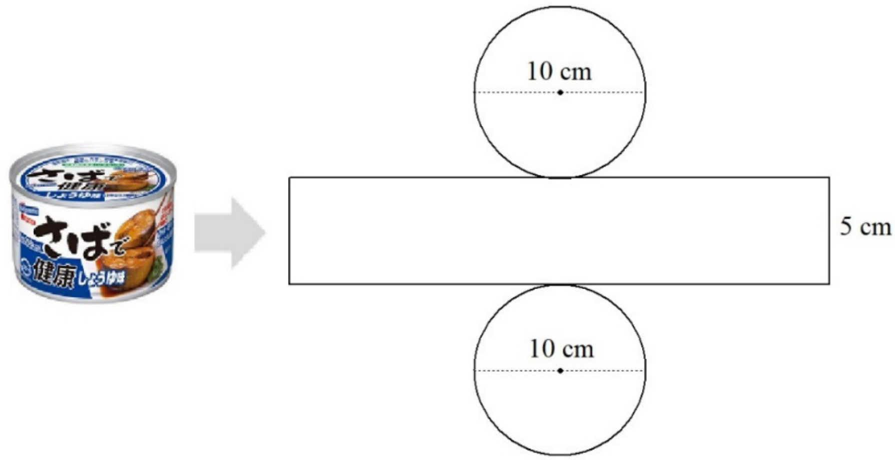


Diagram 24

iv. Diagram 24 shows a tin of canned food and its nets. The circular base of the tin has a diameter of 10 cm and its height is 5 cm.

- Calculate the surface area and volume of the tin.

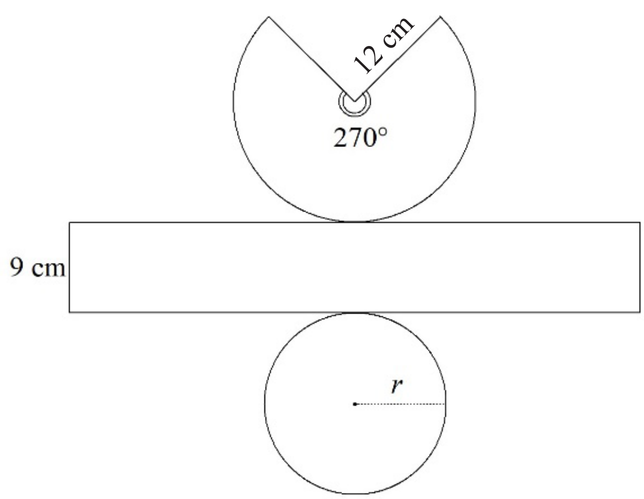


Diagram 25

v. Diagram 25 shows the net of a gift box.

- Sketch the shape of the gift box.
- Find r, the radius of the circular base.
- Calculate the surface area and volume of the gift box.

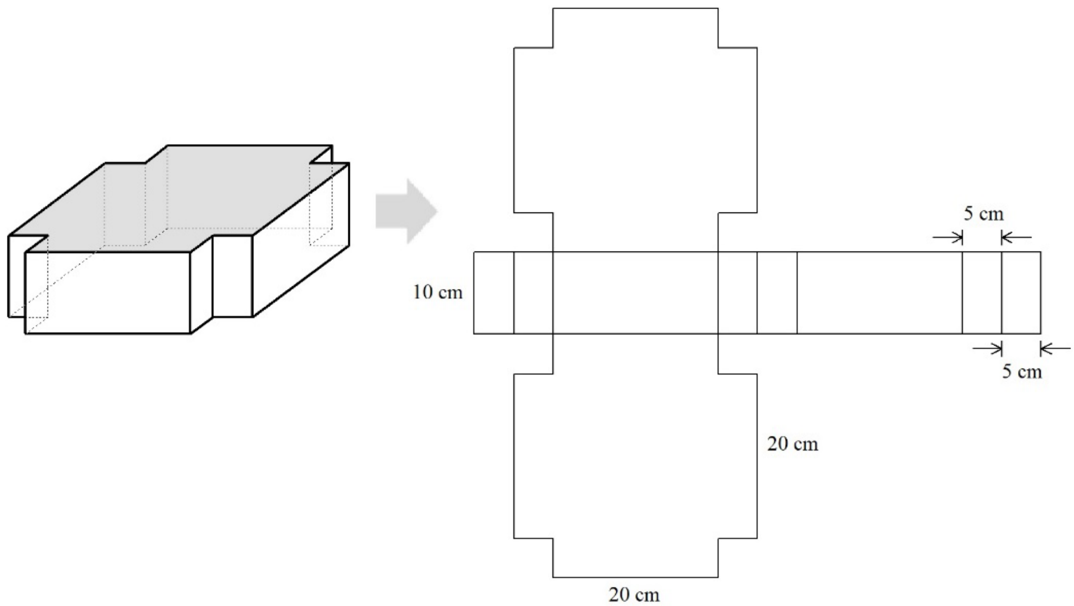


Diagram 26

vi. A gift box has the shape shown in Diagram 26. The height of the gift box is 10 cm, the longer edges are 20 cm long, and the short edges of the square corner cut-outs are each 5 cm long. Part of the net of the gift box is also shown in Diagram 26.

- Complete the net of the gift box.
- Calculate the surface area and volume of the gift box.

Topic 3: Exploring the Ways of Argument for Proving and Its Application in Geometry [K3SG1]

Standard 3.1: [K3SG3-1]
Exploring properties of congruency and similarity on plane geometry

i. Explore ways of arguments using the congruence of two triangles and appreciate the logic of argument in simple proving
ii. Explore ways of arguments using the similarity of two triangles based on ratio and angles and appreciate the logic of arguments in simple proving
iii. Explore the proof of the properties of circles such as inscribed angles, intercepted arcs
iv. Appreciate proving through making the order of proven propositions to find new propositions

Sample Tasks for understanding the standards

Task 1: Congruent and Similar Figures

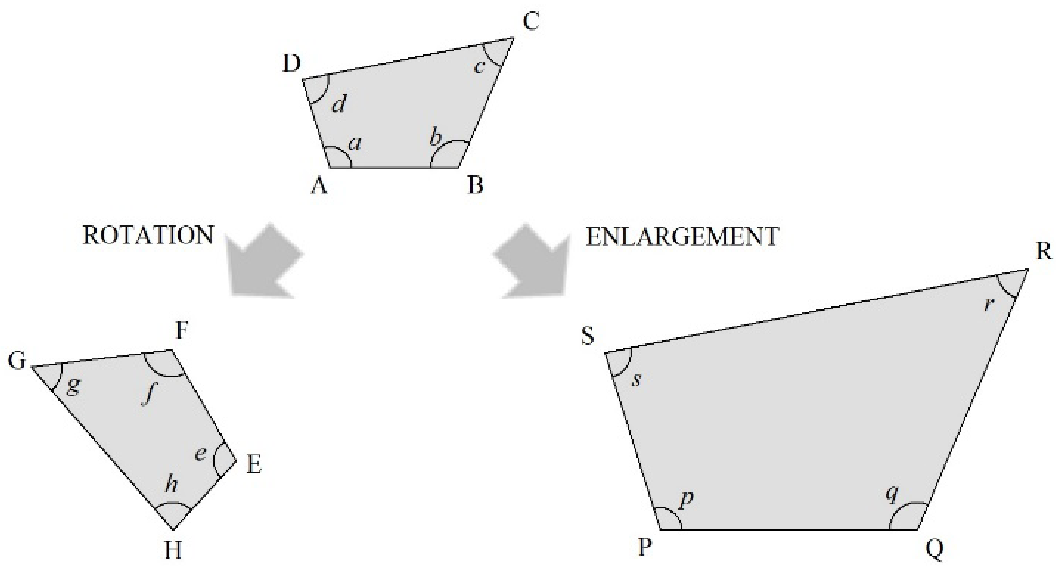


Diagram 1

Diagram 1 shows a quadrilateral ABCD being rotated to become quadrilateral EFGH and enlarged to become quadrilateral PQRS.

ABCD is said to be congruent to EFGH, but similar to PQRS.

i. Explain the similarities and differences between congruent figures and similar figures.

Task 2: Proof Using Congruence Conditions for Triangles

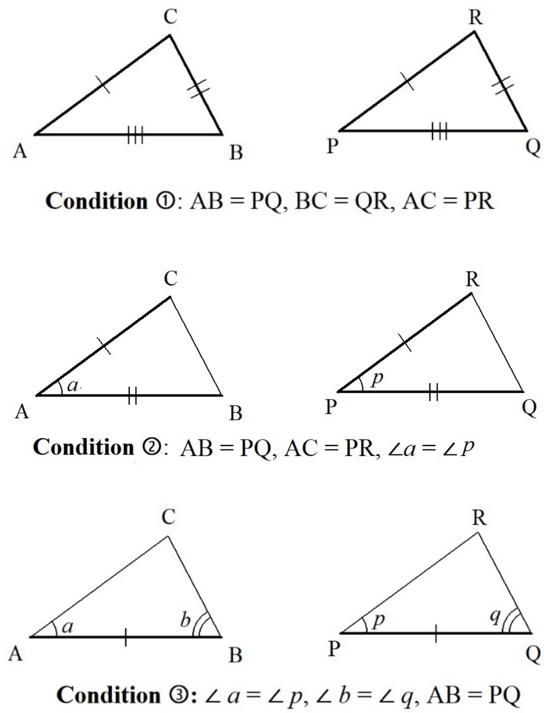


Diagram 2

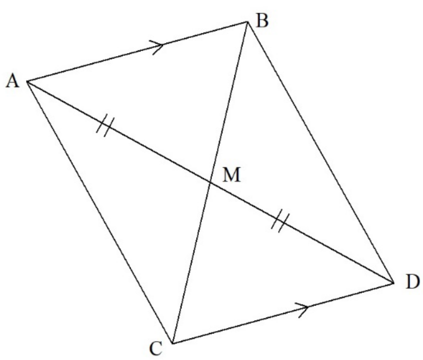


Diagram 3

Diagram 2 shows the three conditions for congruence of triangles:

① All three pairs of corresponding sides are equal (Side-Side-Side).
② Two pairs of corresponding sides and the angle between them are equal (Side-Angle-Side).
③ Two pairs of corresponding angles and the side between them are equal (Angle-Side-Angle).

Triangles $\triangle ABC$ and $\triangle PQR$ are congruent if any of the conditions is fulfilled.

i. In the figure shown in Diagram 3, $AB \parallel CD$ and $AM = DM$. Line segments AD and BC intersects at point M. Two students, Antonio and Buwan, tried to write the proof for $CM = MB$ using a congruence condition for triangles.

<u>Antonio's Written Proof</u>
Since $\angle BAM = \angle CDM$,
$\angle AMB = \angle CMD$, and $AM = DM$,
therefore, $\triangle AMB$ $\triangle CMD$
So, $CM = MB$.

<u>Buwan's Written Proof</u>
Given that $AB \parallel CD$ and alternate interior angles formed by parallel lines are equal, so
$\angle BAM = \angle CDM$ ... Supposition ①

Since vertical angles are equal, so
$\angle AMB = \angle CMD$ … Supposition ②

Also given $AM = DM$ … Supposition ③
From suppositions ①, ②, and ③, because of the Angle-Side-Angle congruence condition,
$\triangle AMB$ $\triangle CMD$.

Since the corresponding sides in congruent triangles are equal, therefore $CM = MB$.

- Which proof is easier to understand? Why?
- Prove that $AC \parallel BD$.
  (State all the suppositions clearly in your proof.)

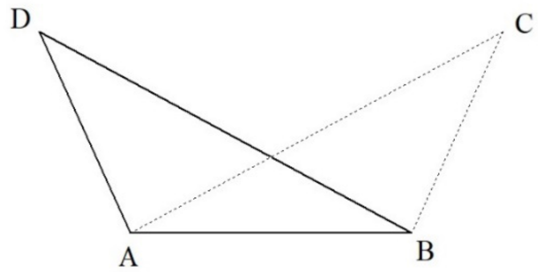


Diagram 4

ii. In the figure shown in Diagram 4, $AD = BC$ and $AC = BD$.

- Using any of the congruence conditions for triangles, prove that $\angle ADB = \angle ACB$.
- Prove that $\angle DAC = \angle DBC$.

(State all the suppositions clearly in your proof.)

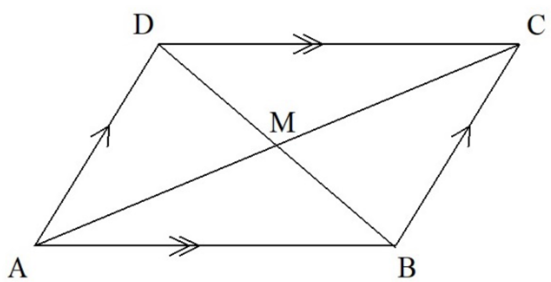


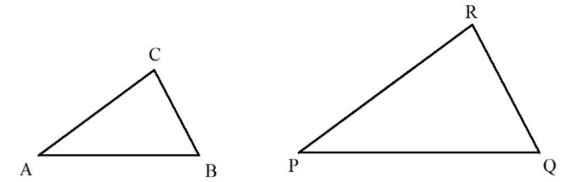
Diagram 5

iii. Diagram 5 shows a quadrilateral $ABCD$. Given that $AB \parallel DC$ and $AD \parallel BC$. The diagonals AC and BD intersect at point M.

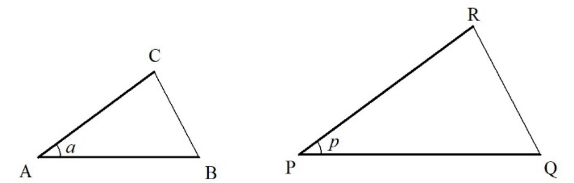
- Prove that $\angle ABC = \angle ADC$.
- Prove that $AB = DC$ and $AD = BC$.
- Prove that M is the mid-point of both diagonals.

(State all the suppositions clearly in your proof.)

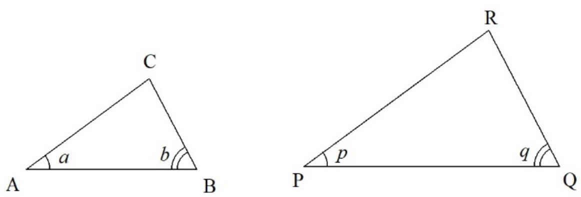
Task 3: Proof Using Conditions for Similarity of Triangles



Condition ① : AB:PQ = AC:PR= BC:QR



Condition ② : AB:PQ = AC:PR and $\angle a = \angle p$



Condition ③ : $\angle a = \angle p$ and $\angle b = \angle q$

Diagram 6

Diagram 6 shows the three conditions for similarity of triangles:

① The ratios of all three pairs of corresponding sides are equal.
② The ratios of two pairs of corresponding sides and the angle between them are equal.
③ Any two pairs of corresponding angles are equal.

Triangles $\triangle ABC$ and $\triangle PQR$ are similar if any of the conditions is fulfilled.

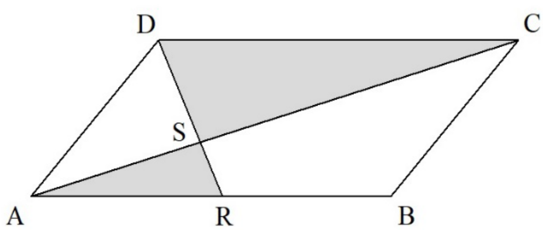


Diagram 7

i. Diagram 7 shows a parallelogram ABCD. R is a point on AB and S is the intersection point of DR and AC. Prove that $\triangle ARS$ is similar to $\triangle CDS$

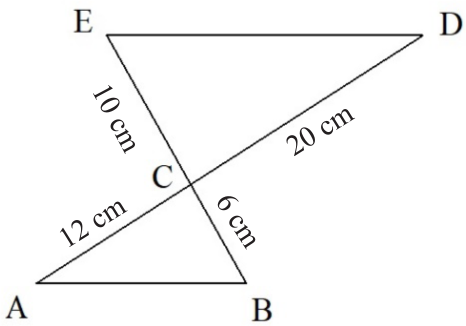


Diagram 8

i. In the figure shown in Diagram 8, C is the intersection point of line segments AD and BE. Given BC = 6 cm, AC = 12 cm, CE = 10 cm and CD = 20 cm.

- Prove that $\triangle ABC$ is similar to $\triangle DEC$.
- Prove that line segment AB is parallel to line segment ED.

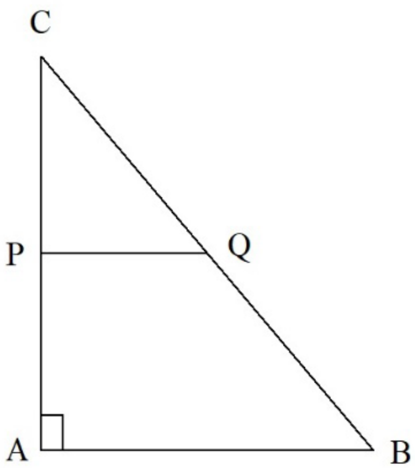


Diagram 9

ii. Diagram 9 shows a right-angled triangle $\triangle ABC$. Given that P and Q are the mid-points of AC and BC, respectively. You are required to prove that (a) $AB = 2PQ$, (b) $\triangle ABC$ is similar to $\triangle PQC$, and (c) area of $\triangle PQC$ = ¼ × area of $\triangle ABC$.

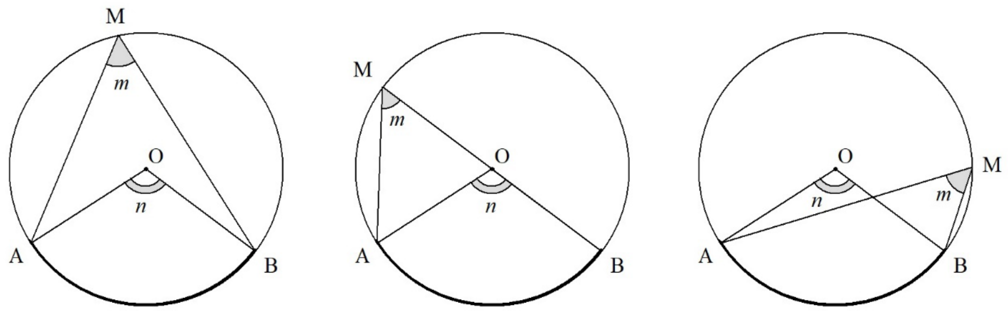
- Which of the three proofs should you work out first? Explain your reasons.
- Work out the three proofs.

Diagram 10

i. Diagram 10 shows a tree and a man standing under the sun at a certain time of a day. The shadow of the tree was 16.8 m and the shadow of the man was 1.2 m.

- Explain how and why the idea of similar triangles can be used to find the height of the tree.
- If the height of the man is 1.7 m, what is the height of the tree?

Task 4: Properties of a Circle



Case ①: Centre O is inside $\angle AMB$

Case ②: Centre O is on the side of $\angle AMB$

Case ③: Centre O is outside $\angle AMB$

Diagram 11

i. Diagram 11 shows three cases of the positional relationships between the inscribed angle $\angle AMB$ subtended from minor arc AB to point M and the corresponding central angle $\angle AOB$ of a circle with centre O. In all three cases we can prove that $\angle AMB$ = ½ $\angle AOB$.

- Which case will you prove first? Explain your reasons.
- State clearly your order of proving the three cases.
- Work out the three proofs according to your sequence.

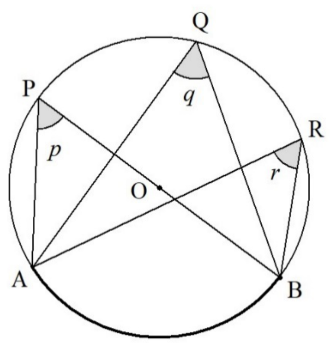


Diagram 12

ii. Diagram 12 shows a circle with centre O. $\angle p$, $\angle q$ and $\angle r$ are three inscribed angles subtended from minor arc AB to points P, Q and R, respectively.

- Using the results from the proofs in (i), prove that $\angle p = \angle q = \angle r$.

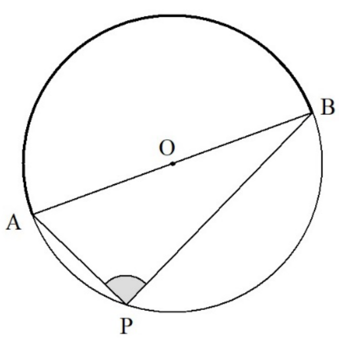


Diagram 13

iii. Diagram 13 shows a circle with centre O and diameter AB. Using the results from the proofs in (i), prove that $\angle APB$ = 90°.

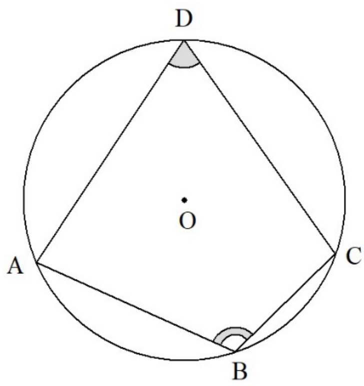


Diagram 14

iv. Diagram 14 shows a circle with centre O and A, B, C, D are four points on the circle.

- Prove that $\angle\mathrm{ADC} = 180° - \angle\mathrm{ABC}$.

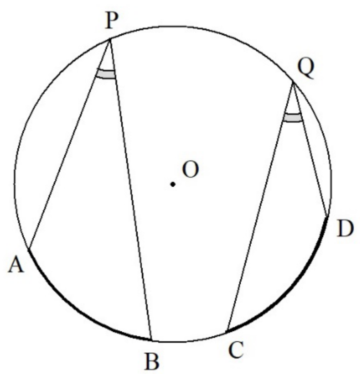


Diagram 15

v. Diagram 15 shows a circle with centre O. AB and CD are two minor arcs with equal lengths.

- Prove that $\angle\mathrm{APB} = \angle\mathrm{CQD}$.

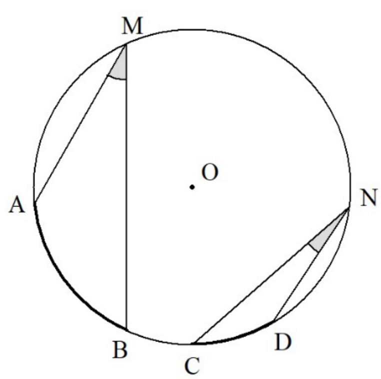


Diagram 16

vi. Diagram 16 shows a circle with centre O. AB and CD are two minor arcs where the relationship of the arc lengths is $\mathrm{AB} = 2\mathrm{CD}$.

- Prove that $\angle\mathrm{AMB} = 2\angle\mathrm{CND}$.
- If $\mathrm{AB} = 3\mathrm{CD}$, make a conjecture about the relationship between $\angle\mathrm{AMB}$ and $\angle\mathrm{CND}$.
- Prove your conjecture.

Standard 3.2: [K3SG3-2]

Exploring Pythagorean theorem in solving problems in plane geometry and spaces

i. Explore the proving of Pythagorean theorem using diagram and use it in solving problems involving plane figures
ii. Apply Pythagorean theorem on prism by viewing the figures through faces
iii. Explore the situations for simple trigonometry using special angles in relation to the Pythagorean theorem
iv. Appreciate the use of Pythagorean theorem in life

Sample Tasks for Understanding the Standards

Task 1: Proving Pythagorean Theorem

Prove by Tiling Pattern

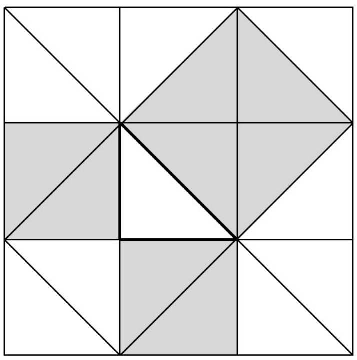


Diagram 1

Diagram 1 shows a tiling pattern with three darken squares.

i. Explain how this tiling pattern could be used to prove the Pythagorean Theorem.

Algebraic Proof

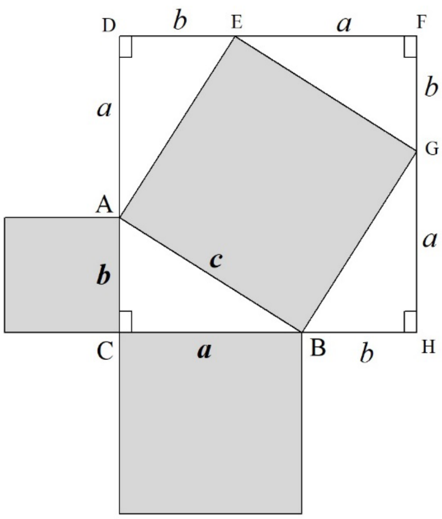


Diagram 2

Diagram 2 shows a right-angle triangle $\triangle ABC$ with sides a, b and c. Three other right-angle triangles congruent to $\triangle ABC$ are added as shown to form a bigger square CHFD.

i. Use the figure in Diagram 2 to prove that $a^2 + b^2 = c^2$.

(*Hint*: Consider the relationship between the area of ABGE and the area of CDFH.)

Geometrical Proof by Euclid

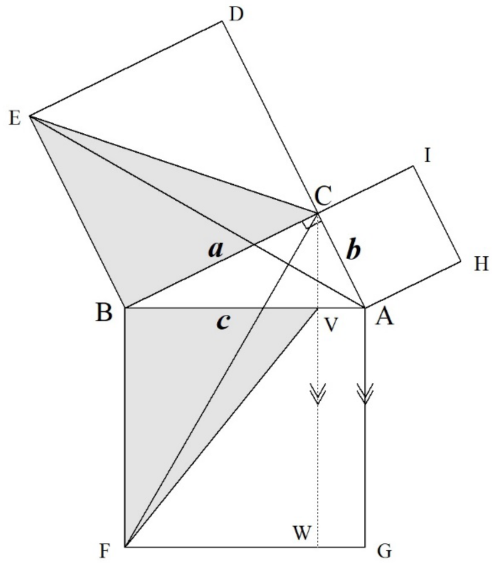


Diagram 3

Diagram 3 shows a figure used by ancient mathematician Euclid to prove the Pythagorean Theorem. In the diagram, CVW is a straight line parallel to AG.

i. Valid arguments involving a^2 can be built up based on the figure. Euclid argued that the area of $\triangle\mathrm{BEC} = \triangle\mathrm{BFV}$ based on the answers to the following questions:

- Why is ACD a straight line?
- Why is the area of $\triangle\mathrm{BEC} = \triangle\mathrm{BEA}$?
- Why is the area of $\triangle\mathrm{BEA} = \triangle\mathrm{BFC}$?
- Why is the area of $\triangle\mathrm{BFC} = \triangle\mathrm{BFV}$?
- Why is the area of rectangle $\mathrm{BFWV} = a^2$?

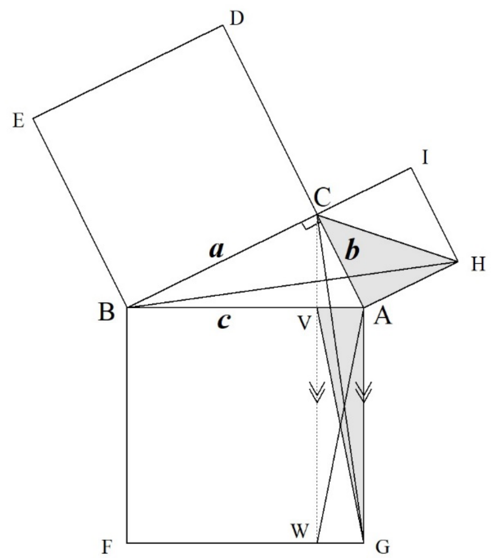


Diagram 4

ii. In a similar manner, Diagram 4 can be used to build up the arguments involving b^2.

- Explain why $\triangle\mathrm{ACH} = \triangle\mathrm{AGV}$.
- Explain why the area of rectangle $\mathrm{AGWV} = b^2$.

iii. Explain why $c^2 = a^2 + b^2$.

Task 2: Application of Pythagorean Theorem

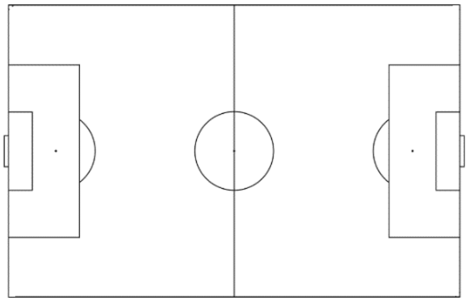


Diagram 5

i. A teacher was given the task to construct a football field as shown in Diagram 5. In order to ensure that all corners of the field are 90°, he used the 3-4-5 method to construct the corners. He first made a 12-m loop with three knots A, B and C to make intervals of 3 m, 4m and 5 m on the loop. He then stretched the loop to form a triangle with sides 3 m, 4 m and 5 m as shown in Diagram 6.

- Prove that $\angle BAC = 90°$.

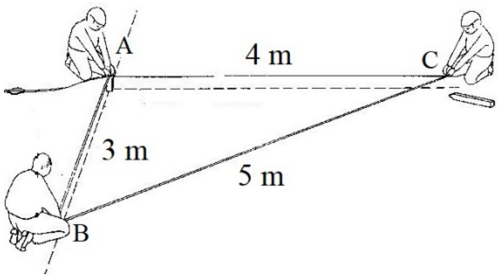


Diagram 6

Diagram 7

ii. A timber log with a diameter of 40 cm is to be cut into a timber block with a square cross section as shown in Diagram 7.

- What is the maximum side length of the square in order to cut out the thickest timber block?

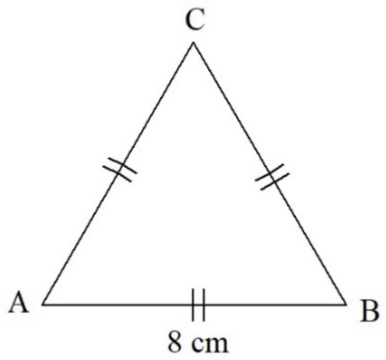


Diagram 8

iii. Diagram 8 shows an equilateral triangle $\triangle ABC$ with each side 8 cm.

- Find the area of $\triangle ABC$.

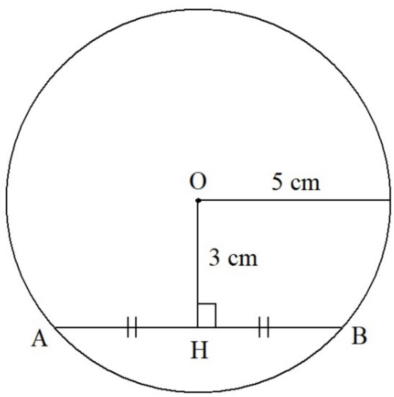


Diagram 9

iv. Diagram 9 shows a circle centre O with radius 5 cm. AB is a chord 2 cm from the centre of the circle.

- Find the length of chord AB.

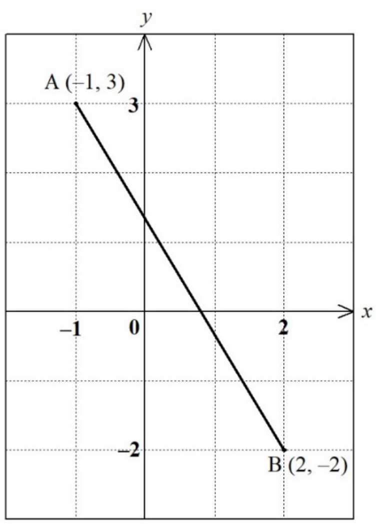


Diagram 10

v. Diagram 10 shows the locations of point A(–1, 3) and point B(2, –2)

- Find the distance between A and B.

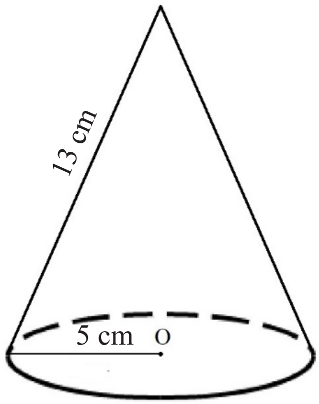


Diagram 11

vi. Diagram 11 shows a circular cone with a base radius of 5 cm and a generatrix of length 13 cm.

- Find the height and the volume of the cone.

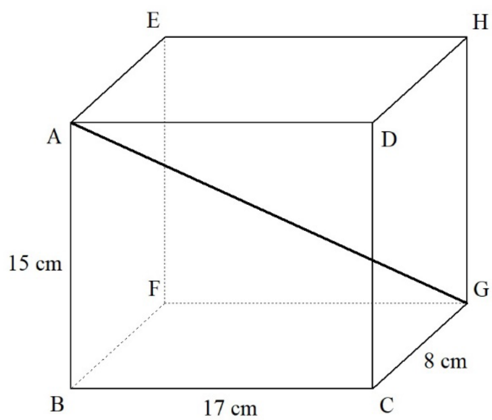


Diagram 12

vii. Diagram 12 shows a rectangular prism with a dimension of 8 cm × 15 cm × 17 cm as shown. AG is a diagonal of the prism.

- Find the length of AG.

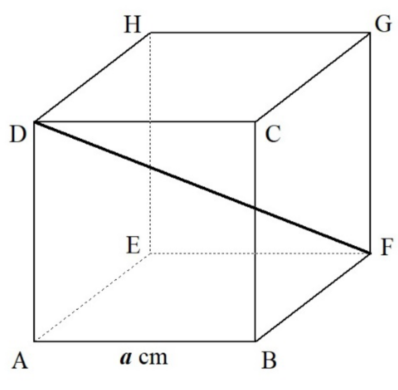


Diagram 13

viii. Diagram 13 shows a cube with side length a cm. DF is a diagonal of the cube.

- Prove that the ratio of its diagonal to its side, $\frac{\mathrm{DF}}{\mathrm{AB}} = \sqrt{3}$.

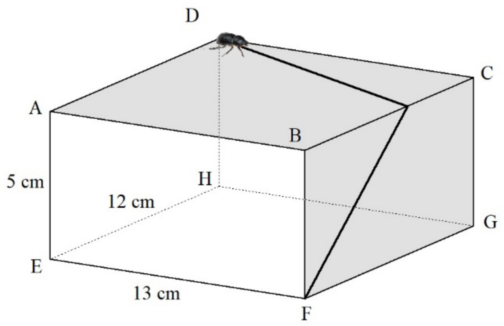


Diagram 14

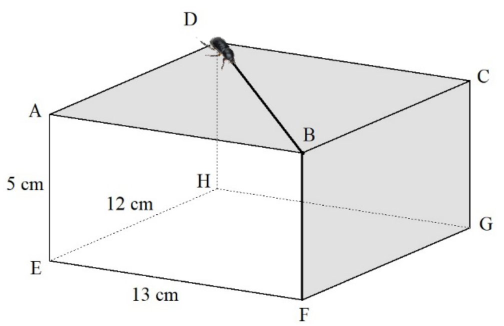


Diagram 15

ix. Diagram 14 shows a rectangular prism box with a beetle at vertex D. The beetle starts to crawl in a straight line towards edge BC and it then turns to crawl in a straight line until it reaches vertex F as shown.

A student argues that if the beetle crawls towards vertex B in a straight and then crawls along edge BF as shown in Diagram 15, the distance travelled will be the shortest.

- Prove that the student's argument is wrong.

  [*Hint*: Open up the box at face BCGF so that ADGF becomes a rectangle.]

- Find the shortest distance travelled by the beetle from D to F.

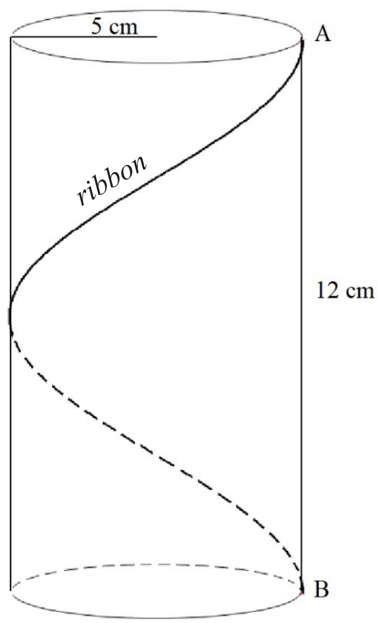


Diagram 16

x. Diagram 16 shows a cylinder with a base radius of 5 cm, height 12 cm and generatrix AB.

A ribbon ties around the cylinder goes around once from point A to point B as shown in the diagram.

- Find the shortest length of ribbon, rounding your answer to the nearest tenth.
  (Let $\pi = 3.14$)

Task 3: Simple Trigonometry with Special Angles

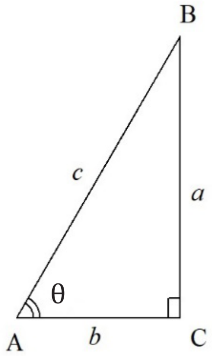


Diagram 17

i. Diagram 17 shows a right-angle triangle $\triangle ABC$ with an angle θ. Given that $0° < \theta < 90°$. the length of sides BC, AC and AB are a, b and c, respectively.

- Why are $\sin\theta$, $\cos\theta$ and $\tan\theta$ called the trigonometrical ratios with respect to $\triangle ABC$?

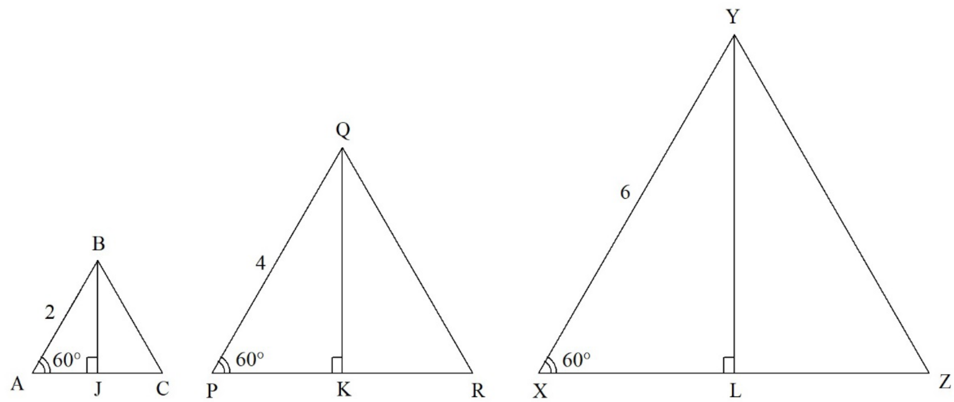


Diagram 18

ii. Diagram 18 shows three equilateral triangles, $\triangle ABC$, $\triangle PQR$ and $\triangle XYZ$ with sides 2 units, 4 units, 6 units, respectively. Given that BJ, QK and YL are perpendicular bisectors to AC, PR and XZ respectively.

- Find the length of sides AJ, PK and XL, respectively.
- Find the length of sides BJ, QK and YL, respectively.
- Find each of the following trigonometrical ratios using the measures in $\triangle ABC$, $\triangle PQR$ and $\triangle XYZ$ respectively.
  - ❖ $\sin 60°$
  - ❖ $\cos 60°$
  - ❖ $\tan 60°$
- Show that the each of the trigonometrical ratios $\sin 60°$, $\cos 60°$, $\tan 60°$ and found from the three right-angle triangles with different sizes is a constant.
- Calculate values of $\sin 60°$, $\cos 60°$, and $\tan 60°$, rounded to four decimal places.

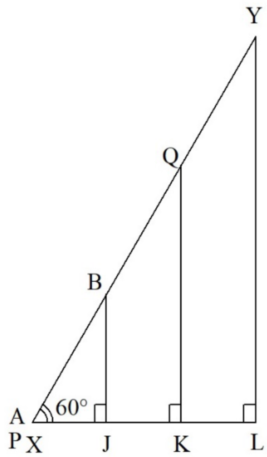


Diagram 19

iii. Diagram 19 shows the situation when the three right-angle triangles are overlapped.

- Based on this diagram, explain why each of the trigonometrical ratios $\sin 60°$, $\cos 60°$, and $\tan 60°$ is a constant ratio?
- Will each of the trigonometrical ratios for angles other than $60°$ also a constant? Explain your reasoning.

iv. Diagram 20 shows the same three triangles in Diagram 18.

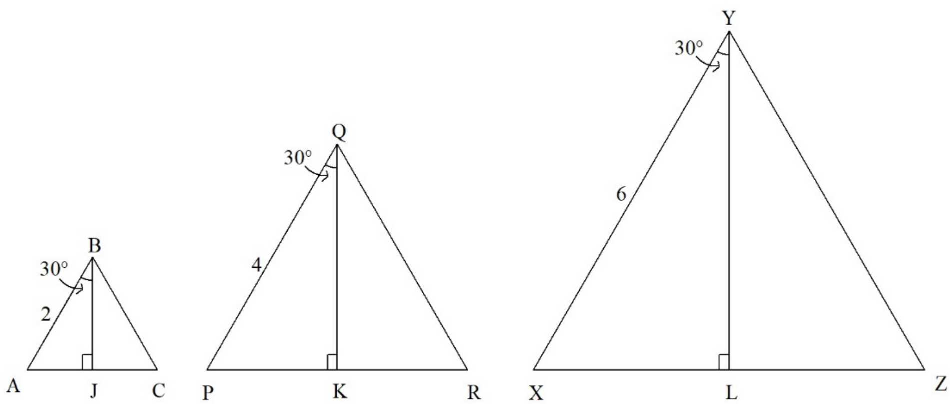


Diagram 20

- Explain why $\angle ABJ = \angle PQK = \angle XYL = 30°$.
- Find each of the following trigonometrical ratios using the measurements in $\triangle ABC$, $\triangle PQR$ and $\triangle XYZ$ respectively.
  - ❖ $\sin 60°$
  - ❖ $\cos 60°$
  - ❖ $\tan 60°$
- Explain why each of the trigonometrical ratios $\sin 30°$, $\cos 30°$ and $\tan 30°$ is also a constant.

v. What conclusion can you make about the ratios $\sin\theta$, $\cos\theta$ and $\tan\theta$ for any angle θ, where $0 < \theta < 90°$? What if $0 = \theta$? What if $0 = 90°$?

vi. What relationship exists between the following trigonometrical ratios?

- $\sin 30°$ and $\cos 60°$
- $\sin 60°$ and $\cos 30°$
- $\tan 30°$ and $\tan 60°$

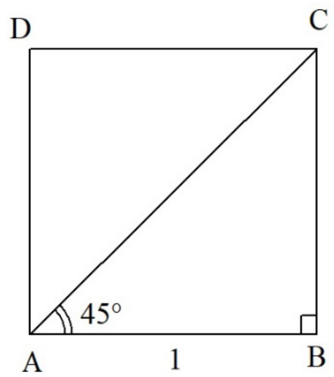


Diagram 21

vii. Diagram 21 shows a square ABCD with sides 1 unit and AC is its diagonal.

- Explain why $\angle BAC = \angle BCA = 45°$.
- Find the length of side AC.
- Show that $\sin 45° = \cos 45° = \frac{\sqrt{2}}{2}$.
- Find $\tan 45°$.

viii. From your answers in (ii), (iv) and (vii), verify each of the following trigonometric identities.

- $\sin^2 30° + \cos^2 30° = 1$
- $\sin^2 45° + \cos^2 45° = 1$
- $\sin^2 60° + \cos^2 60° = 1$

ix. Using the Pythagorean Theorem, prove that for the case of a right-angled triangle with an acute angle θ, $\sin^2\theta + \cos^2\theta = 1$.

CHAPTER 5

Statistics and Probability [K3SP]

Topic 1: Exploring Distribution with the Understanding of Variability [K3SP1]

Standard 1.1: [K3SP1-1]

Exploring distribution with histograms, central tendency and variability

i. Use histogram with different class intervals to show different distribution of the same set of data.
ii. Identify alternative ways to show a distribution such as dot plot, box plot and frequency polygon
iii. Investigate central tendencies such as mean, median, mode and their relationships in a distribution
iv. Investigate dispersion such as range and inter-quartile range in a distribution
v. Appreciate the analysis of variability through the finding of the hidden structure of distribution on situations using the measure of central tendency and dispersion

Sample Tasks for Understanding the Standards

Task 1: Describing a Set of Data Using Central Tendency and Dispersion

Representative Value of a Set of Data

Two classes of students, Class A and Class B, took a Mathematics test.

i. Table 1 shows the test scores of 23 students from Class A.

- Draw a dot plot to represent the test scores of this class of students on Diagram 1.
- At a glance, what important information about the distribution of the test scores could you get from the dot plot?

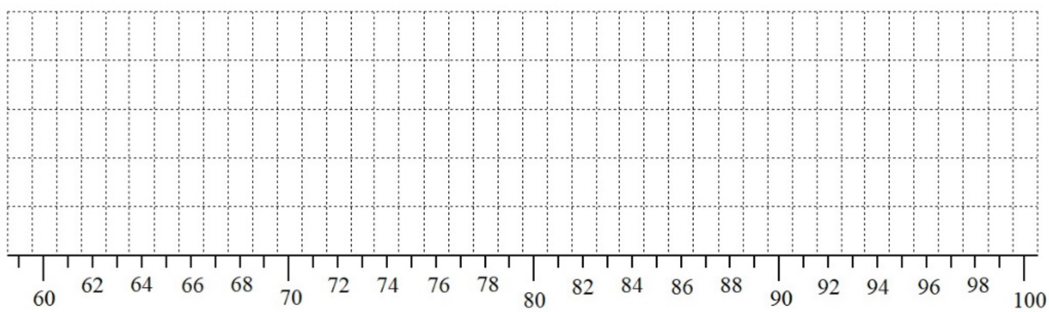


Diagram 1

Table 1
Test Scores of Class A

Student No.	Score
A1	69
A2	98
A3	64
A4	78
A5	98
A6	73
A7	62
A8	69
A9	98
A10	68
A11	68
A12	74
A13	72
A14	65
A15	70
A16	69
A17	72
A18	65
A19	74
A20	70
A21	98
A22	66
A23	72

- Find the following measures of central tendency of the test scores.
  - ❖ Mean
  - ❖ Mode
  - ❖ Median
- Find the following measures of dispersion for the test scores.
  - ❖ Range
  - ❖ Inter-quartile range
- In reporting the performance of Class A, the teacher decided to pick 98 to be the representative score of the class since it is the highest score. Is the score 98 appropriate to be chosen as the representative score of the class? Explain your reasons.
- In your opinion, what value of score is most appropriate to represent the scores of this class of students? Justify your choice.

Table 2
Test Scores of Class B

Student No.	Score
B1	60
B2	61
B3	61
B4	64
B5	64
B6	65
B7	65
B8	66
B9	67
B10	67
B11	68
B12	68
B13	68
B14	68
B15	68
B16	69
B17	69
B18	69
B19	70
B20	70
B21	71
B22	71
B23	72
B24	73
B25	73
B26	74
B27	76
B28	78

ii. Table 2 shows the test scores of another 28 students from Class B in ascending order.

- Find the following measures of central tendency and dispersion for the test scores.
  - ❖ Mean, mode and median
  - ❖ Range and inter-quartile range
  - ❖ In your opinion, which of these measures of central tendency is most appropriate to represent the scores of Class B? Justify your choice.
  - ❖ What do these measures of dispersion tell you about the distribution of Class B's test scores?

iii. Compare the mean scores of Class A and Class B.

- Based on this comparison, which class performed better in the test? Justify your decision.
- How confident are you with the decision? Explain your reasons.
- What can you conclude on the use of measures of central tendency and dispersion to represent a set of data? Explain briefly.

Comparing Two Sets of Data Using Box Plot

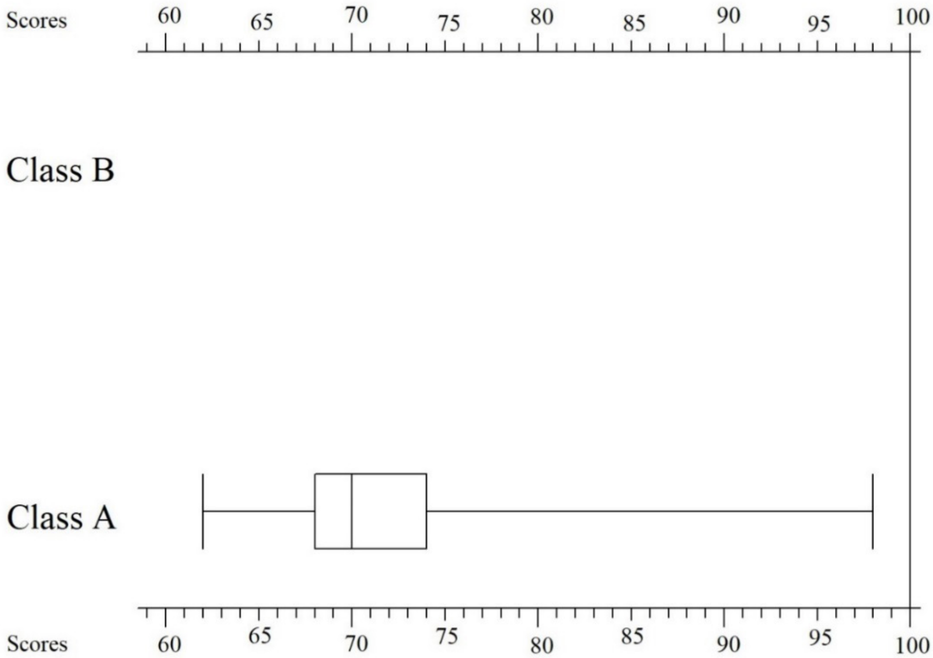


Diagram 2

i. Diagram 2 shows the box plot for the test scores of Class A from Table 1.

- Draw a box plot for the test scores of Class B from Table 2 on Diagram 2.
- Compare the two box plots.
  - ❖ What can you say about the two classes' test scores from the aspect of central tendency and dispersion?
  - ❖ Based on this comparison, which class performed better in the test? Justify your decision.
  - ❖ Is this decision same as the decision based on comparison of mean scores in the previous sub-task? Explain briefly.
- Which set of data has a greater variability? How does the variability affect the choice of representative value for the set of data?

Task 2: Using Histogram to Investigate Trends in Data

Table 3
Hand Circumference of Lower Secondary Male (M) and Female (F) Students

Student No.	Hand (cm) / Gender	Student No.	Hand (cm) / Gender
1	17.2 / F	39	21.0 / M
2	17.5 / F	40	21.1 / M
3	18.1 / F	41	21.1 / M
4	18.2 / F	42	21.3 / F
5	18.3 / F	43	21.3 / M
6	18.5 / F	44	21.4 / F
7	18.5 / F	45	21.4 / M
8	19.0 / F	46	21.7 / F
9	19.1 / F	47	21.8 / F
10	19.2 / F	48	22.0 / M
11	19.2 / M	49	22.3 / F
12	19.3 / F	50	22.3 / M
13	19.3 / F	51	22.3 / M
14	19.3 / F	52	22.3 / M
15	19.4 / F	53	22.4 / M
16	19.4 / F	54	22.5 / M
17	19.5 / F	55	22.5 / M
18	19.5 / F	56	22.6 / F
19	19.5 / M	57	22.6 / F
20	19.6 / F	58	22.7 / M
21	19.6 / F	59	22.7 / M
22	19.6 / F	60	22.8 / M
23	19.7 / F	61	22.8 / M
24	19.8 / F	62	22.9 / M
25	19.8 / M	63	22.9 / M
26	20.0 / F	64	23.1 / M
27	20.0 / F	65	23.1 / M
28	20.2 / M	66	23.5 / M
29	20.3 / F	67	23.6 / F
30	20.3 / F	68	23.6 / M
31	20.3 / M	69	23.6 / M
32	20.5 / F	70	23.7 / M
33	20.5 / M	71	23.8 / F
34	20.5 / M	72	23.8 / M
35	20.6 / M	73	23.8 / M
36	20.8 / F	74	24.5 / M
37	20.9 / F	75	24.9 / M
38	21.0 / M		-

Diagram 3

The circumference of a hand as shown in Diagram 3 is one of the basic measurements to determine glove size. A manufacturer investigated some data on hand circumference in order to produce hand gloves with appropriate sizes for school students.

i. Table 3 shows the raw data of hand circumference measured in cm, in ascending order, obtained from 75 lower secondary male (M) and female (F) students.

   - Find the following measures of central tendency and dispersion of hand circumference for this group of students.

     ❖ Mean, median and mode

     ❖ Range and inter-quartile range

Table 4
Frequency Distribution of Hand Circumference of Lower Secondary Students

Hand Circumference, x cm	Frequency	
	1 cm interval	2 cm interval
$17 \leq x < 18$	2	7
$18 \leq x < 19$	5	
$19 \leq x < 20$	18	30
$20 \leq x < 21$	12	
$21 \leq x < 22$	10	26
$22 \leq x < 23$	16	
$23 \leq x < 24$	10	12
$24 \leq x < 25$	2	

ii. The raw data from Table 3 are grouped into classes with 1 cm and 2 cm of interval respectively. The frequency distributions for the students' hand circumference with the two different class intervals are shown in Table 4.

- Which class will 20.6 cm belong to?
- Which class will 22.0 cm belong to?
- Based on Table 4, find the (a) mean, (b) median, (c) mode, (d) range and (d) inter-quartile range of the students' hand circumference based on the data for the two different class intervals, respectively.
- Compare these values of measures of central tendency and dispersion from those obtained from Table 3.
  - ❖ What do you observe about each of the values? Explain briefly.
  - ❖ If the values are different, which values do you think are more appropriate to represent the set of data?

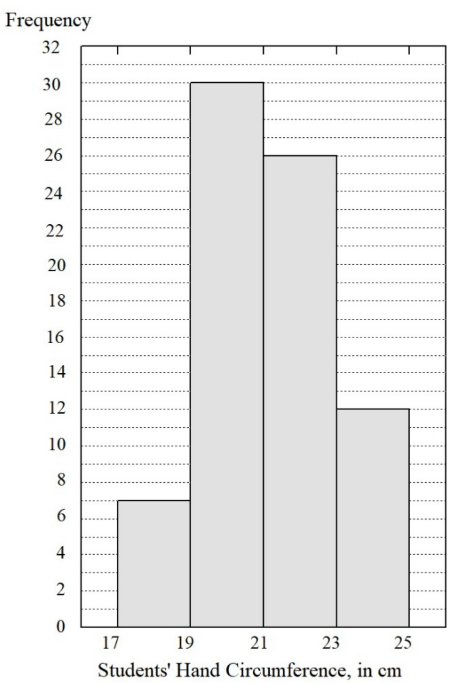


Diagram 4

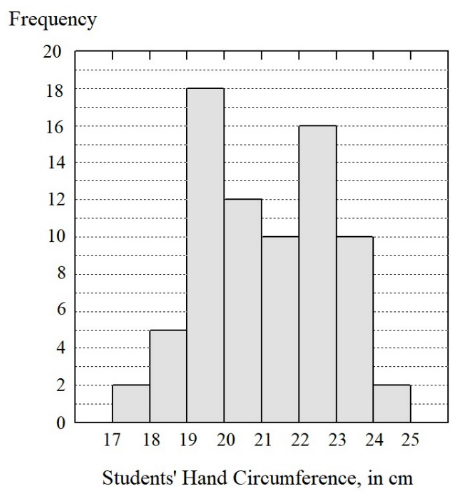


Diagram 5

iii. The two frequency distributions of students' hand circumference are displayed using histogram with class interval 2 cm and 1 cm, respectively as shown in Diagram 4 and Diagram 5.

- Compare the two histograms.
  - ❖ What is the same and what is different about the information on the students' hand circumference that you can read from the two histograms?
- What characteristic of the data shown in Diagram 5 indicates that the glove sizes of the male and female students may need to be different in term of measurement?
- What can you say about the effect of changing the size of class interval on the characteristics of the data shown by the histograms?

Table 5
Frequency Distribution of Hand Circumference for Male and Female Students

Hand Circumference, x cm	Frequency	
	Male	Female
$17 \leq x < 18$	-	
$18 \leq x < 19$	-	
$19 \leq x < 20$	3	
$20 \leq x < 21$	5	
$21 \leq x < 22$	6	
$22 \leq x < 23$	13	
$23 \leq x < 24$	8	
$24 \leq x < 25$	2	

iv. Based on the information from Diagram 5, the manufacturer decides to have two different sets of measurement for the male and female students glove sizes. Hence, the data for the male and female students were separated as shown in Table 5.

- Complete the frequency distribution for the female students in Table 5.
- Based on Table 5, find the (a) mean, (b) median, (c) mode, (d) range and (e) inter-quartile range of the male and female students' hand circumference, respectively.

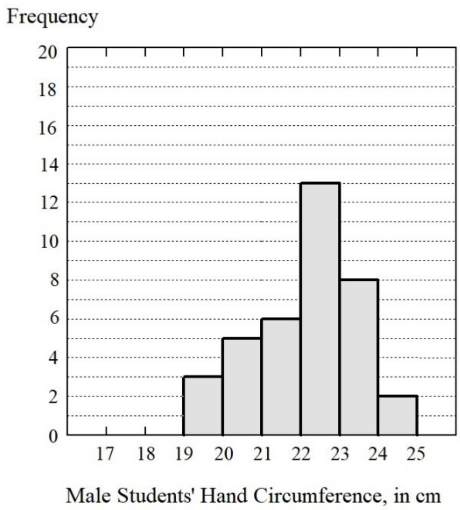


Diagram 6

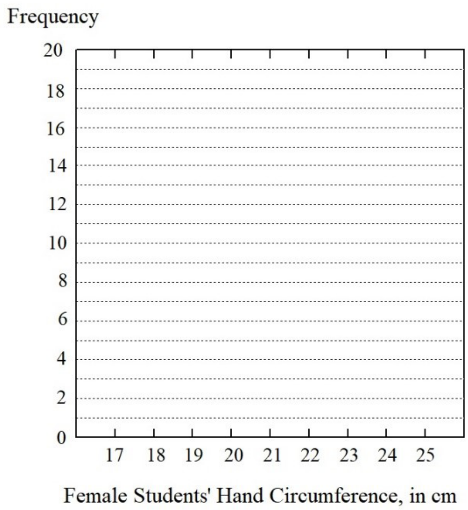


Diagram 7

v. Diagram 6 shows the histogram of the male students' hand circumference.

- Draw the histogram of the females' hand circumference on Diagram 7.
- Compare the two histograms.
- Based on the comparison, what can you say about the distributions of the male and female students' hand circumference?

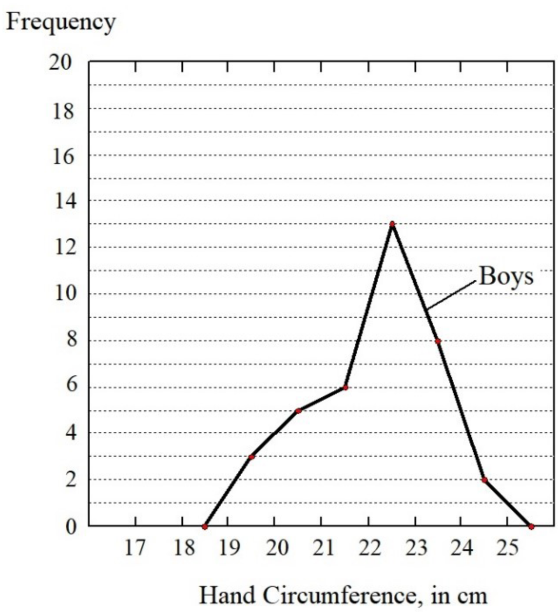


Diagram 8

vi. Diagram 8 shows the frequency polygon of the male students' hand circumference.

- Draw the frequency polygon for the girls' hand circumference on Diagram 8.
- Compare the two frequency polygons. What can you read from them?
- What is the advantage of frequency polygon in analysing trend as compare to histogram?

vii. The manufacturer decided to make three different sizes, Small, Medium, and Large, for the hand gloves. However, two different sets of measurement for each size will be determined for the male and female students.

- What measurement of each size would you recommend for the male and female students? Justify your decision.
- What proportions of the different sizes would you recommend the manufacturer to produce? Justify your decision.

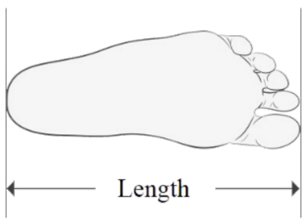


Diagram 9

viii. Diagram 9 shows the measurement of foot length used to determine shoe size. In a project to determine appropriate shoe sizes for school students, the foot lengths of 170 male and female students were measured. Table 6 shows the frequency distribution of the foot lengths.

- Draw the histogram for the foot lengths on Diagram 10.
- Analyse the histogram.
  - ❖ What characteristic of the data indicates that there may be a need to analyse further on the frequency distribution?
- What do you think is the appropriate next step of analysing the data? Explain your reasons.

Table 6
Frequency Distribution of Foot Length of School Students

Foot Length, x cm	Frequency
$20 \leq x < 21$	5
$21 \leq x < 22$	17
$22 \leq x < 23$	25
$23 \leq x < 24$	15
$24 \leq x < 25$	14
$25 \leq x < 26$	22
$26 \leq x < 27$	29
$27 \leq x < 28$	18
$28 \leq x < 29$	14
$29 \leq x < 30$	4
$30 \leq x < 31$	5
$31 \leq x < 32$	2

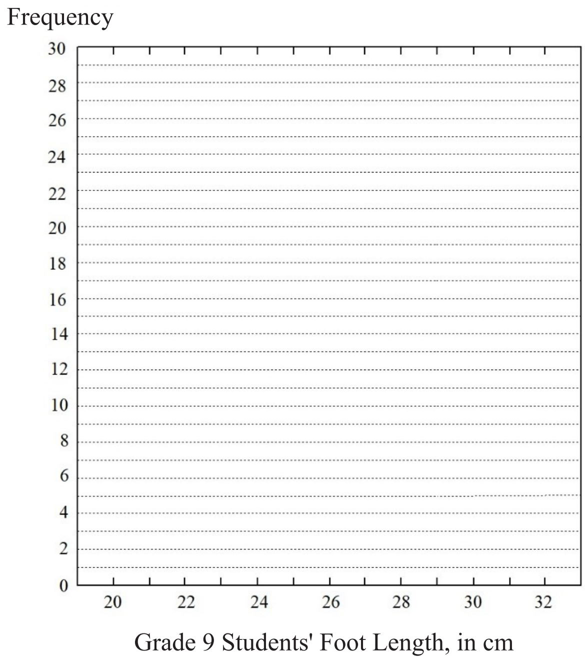


Diagram 10

Task 3: Relationships Among Mean, Median and Mode

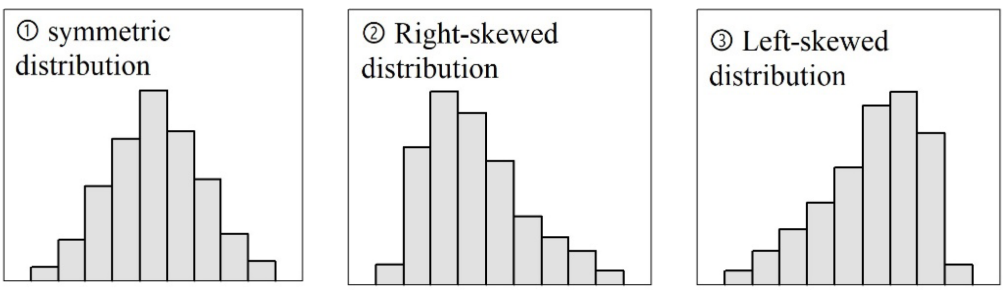


Diagram 11

i. Diagram 11 shows three histograms used to represent one symmetric distribution and two skewed distributions of three sets of data.

- Match the histogram with the following relationships.
  - ❖ Mode < Median < Mean
  - ❖ Mode = Median = Mean
  - ❖ Mean < Median < Mode

- Justify your choice for each relationship.

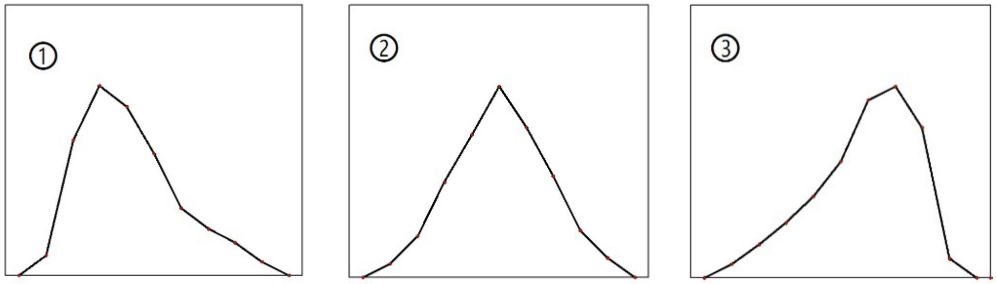


Diagram 12

ii. Diagram 12 shows the frequency polygons of the distributions of three set of data.
- Match the frequency polygons with the following distributions.
  - Symmetric
  - Right skewed
  - Left skewed

- In what ways will skewed distribution affect the measures of central tendency and dispersion for a set of data?

- What will you suggest to reduce the effects?

Task 4: Variability Arises From Data Distribution and Informal Statistical Inferences

i. Variability is an essential element of statistical thinking due to the fact that it arises everywhere in statistics. It represents the conception of variation, which may not be defined mathematically at a certain stage of learning, before it is defined formally as a specific measurement. For this standard, it is considered as an observable characteristic of an entity which describes how much variation is present in data and how spread out the data are. It is disccussed at every stage of learning statistics, depending on what is already learnt and what is not known yet by the learners at each stage. In addition, through the use of informal statistical inference, which is trying to make inference without using formal knowledge and procedures of statistics, any related discussion becomes open ended until variation is defined as a formal concept mathematically at a later stage. As such, informal statistical inference plays an important role in school curriculum toward developing statistical reasoning among students. Furthermore, the informal nature of discussion can also bring fun and interest to the learning of statistics in school curriculum. Thus, in this era of artificial intellegence and data science, discussion on variability and informal statistical inference is inevitable in school statistical education curriculum.

As such, one implicit aim of this topic is to help students develop understanding of variability arises from variation in data distribution through the formation of informal statistical inferences.

- In your opinion, to what extend are the tasks in this topic able to promote a better understanding of **variability arises from variation in data distribution** through informal statistical inferences among lower secondary school students.

Topic 2: Exploring Probability With Law of Large Numbers and Sample Space [K3SP2]

Standard 2.1: [K3SP2-1]
Exploring probability with descriptive statistics, law of large numbers and sample space

i. Experiment with tossing coins and dice to explore the distribution of the relative frequency and understand the law of large numbers
ii. Use the idea of equally likely outcomes to infer the value of a probability
iii. Analyse sample space of situations represented by a table to determine the probability and use it for predicting occurrence.
iv. Use various representations such as table, tree diagram, histogram and frequency polygon for finding probability
v. Analyse data related to issues on sustainable development and use probability to infer and predict future events

Sample Tasks for Understanding the Standards

Task 1: Probability From Empirical Trials

Empirical Probability

Heads  Tails

Diagram 1

Diagram 1 shows the two sides of a new 10-piso coin of the Philippines.

i. In an experiment of tossing the coin, the frequencies of getting heads for different number of total trials are recorded as shown in Table 1. The difference in frequencies between two consecutive total number of trials, $f_n - f_{n-1}$, and its ratio to the difference in total number of trials, $N_n - N_{n-1}$, are calculated.

Table 1
Ratio of Difference in Frequency to Difference in Total Tosses

n	Total Number of Tosses, N	Frequency of Tossing Heads, f	$f_n - f_{n-1}$	$N_n - N_{n-1}$	$\frac{f_n - f_{n-1}}{N_n - N_{n-1}}$
1	50	23	-	-	-
2	100	45	22	50	0.440
3	200	104	59	100	0.590
4	400	201	97	200	0.485
5	600	307	106	200	0.530
6	800	412	105	200	0.525
7	1000	522	110	200	0.550
8	1200	628	106	200	0.530
9	1400	718			
10	1600	810			
11	1800	896			
12	2000	1002			

- Complete the table.
- Do you notice any convergence in the value of the ratio , as the total trials increases?

i. The relative frequencies of tossing heads for different number of total trials are also calculated as shown in Table 2.

Table 2
Relative Frequency of Tossing Heads

Total Number of Trials, N	Frequency of Tossing Heads, f	Relative Frequency of Tossing Heads,
50	23	0.460
100	45	0.450
200	104	0.520
400	201	0.503
600	307	0.512
800	412	0.515
1000	522	0.522
1200	628	0.523
1400	718	
1600	810	
1800	896	
2000	1002	

- Complete the table.

ii. Diagram 2 shows a part of the line graph plotted for the results in Table 2.

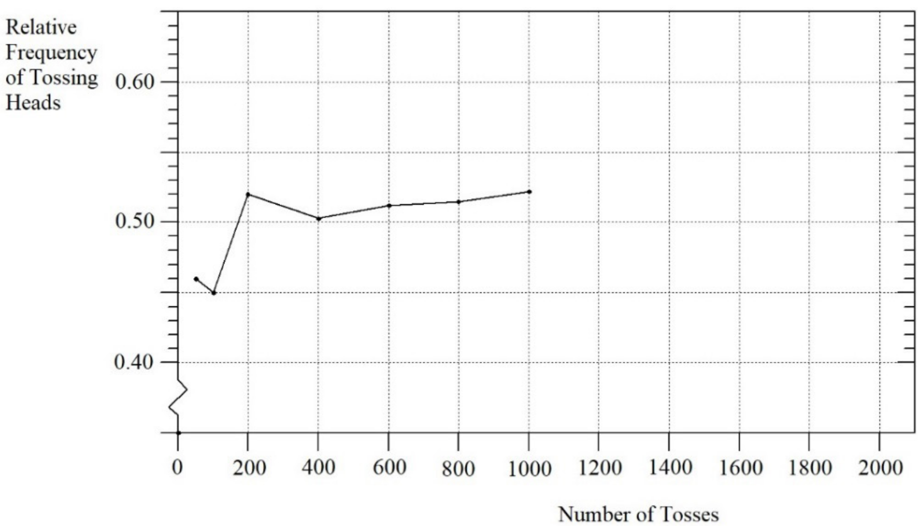


Diagram 2

- Complete the line graph.
- What happen to the relative frequency as the number of trials increases?
- Compare what happen to the relative frequency and the ratio $\frac{f_n - f_{n-1}}{N_n - N_{n-1}}$, as the total trials increases.
- Explain what happen to the relative frequency as the number of trials increases with the Law of Large Numbers.

iv. Table 3 shows the results of three students each tossed a fair coin 100 times.

Table 3
Results of Tossing a Fair Coin 100 Times

Student	Heads	Tails
A	61	39
B	42	58
C	50	50

- When asked to comment on these results, Chee Seng says:
  "*Student C gets the most accurate result because in 100 trials, we expect to get 50 heads and 50 tails. Student A's result is very not accurate and since he gets much more heads than tails, if he continue to toss the coin for another 10 times, he should get more tails than heads in order to balance out his result. Similarly, Students B should also get more heads in his next 10 trials.*

  ❖ What is wrong with Chee Seng's comment? Why is it wrong?

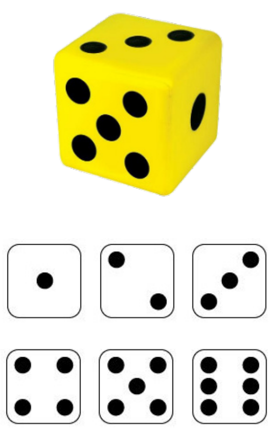


Diagram 3

v. Diagram 3 shows a die and its six faces.

- Carry out an experiment of throwing a die.
- Record your result of getting ⚀ in Table 4.

Table 4
Result of Throwing a Die

Total Number of Trials	Frequency of Throwing ⚀	Relative Frequency of Throwing ⚀
50		
100		
200		
300		
400		
500		

- What is the probability of throwing ⚀ ?

Empirical Probability in Real World

i. Clinical trials have found a drug to have a success rate of 92% in curing patients infected with COVID-19. In the coming year, 14 290 COVID-19 patients will be treated with the drug.

- How many patients would be expected to be cured?
- How certain are you with the expected number of cured patients? Explain briefly.
- Based on your answer, make a more reasonable estimate of the number of cured patients. Explain your reasoning.

ii. Table 5 shows the number of live births in Malaysia from years 2013 to 2018.

Table 5
Birth Statistics of Malaysia, 2013 to 2018

Year	Total	Male	Female
2018	501 945	259 582	242 363
2017	508 685	262 575	246 110
2016	508 203	262 755	245 448
2015	521 136	269 255	251 881
2014	511 865	264 396	247 469
2013	503 914	260 725	243 189

Source: Department of Statistics Malaysia Official Portal.
Available online at: https://www.dosm.gov.my/

- What is the relative frequency of giving birth to (a) a baby boy, and (b) a baby girl in Malaysia from years 2013 to 2018?
- In this period of time, if a mother in Malaysia gives birth to 2 babies, what is the probability that the babies will be a boy and a girl?
- If another mother gives birth to 3 babies, what is the probability that the babies will be all girls?

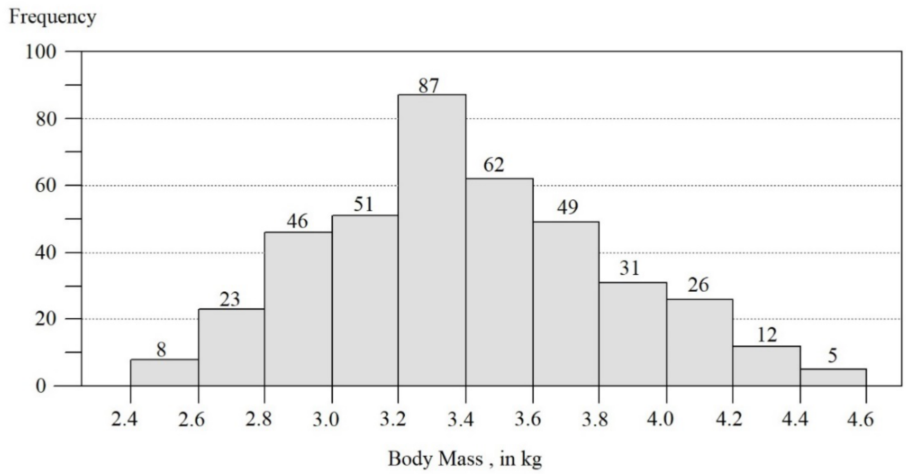


Diagram 4

iii. The histogram in Diagram 4 shows the distribution of body mass of 400 babies at birth.

- Find the probability that a baby is born with a body mass of less than 3.0 kg.
- Find the probability that a baby is born with a body mass of 3.0 kg or more, but less than 4.0 kg.

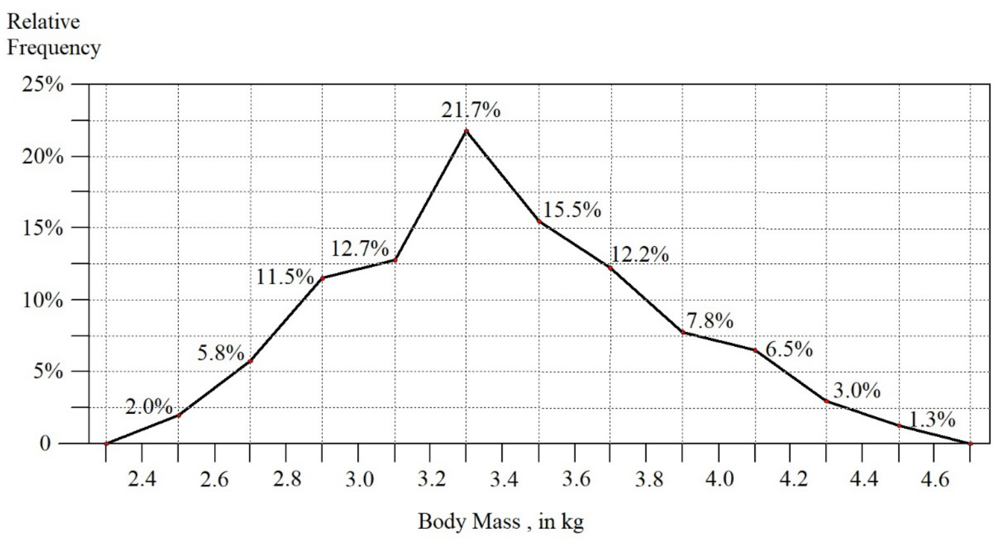


Diagram 5

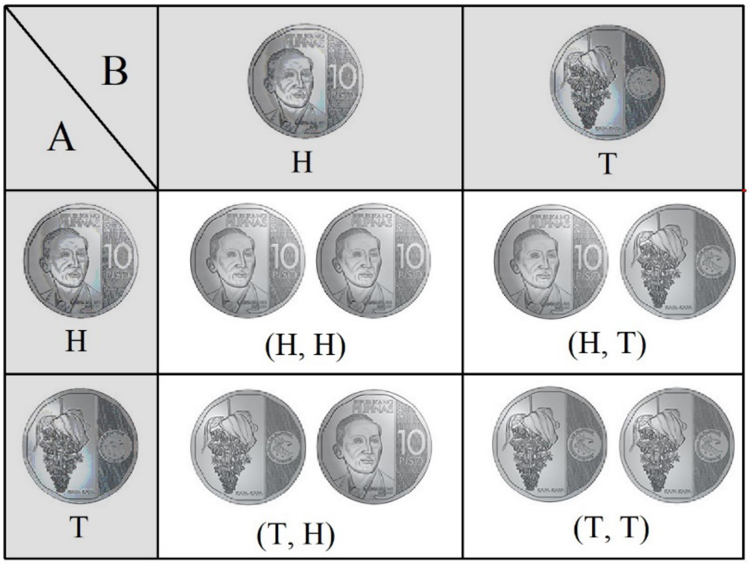
iv. The frequency polygon in Diagram 5 shows the distribution of relative frequencies of the 400 newly born babies' body mass from the histogram in Diagram 4.

- Find the probability that a baby born with a body mass of less than 3.0 kg.
- Find the probability that a baby is born with a body mass of 4.0 kg or more, but less than 4.4 kg.

Task 2: Probability From Theoretical Perspective

Tossing Two Fair Coins

Table 6
Sample Space of Tossing Two Coin, A and B



A \ B	H	T
H	(H, H)	(H, T)
T	(T, H)	(T, T)

Table 6 shows the sample space of tossing two coins, A and B. Assuming each side of the coin is **equally likely** to occur.

i. Explain the meaning of sample space.

ii. What does it mean by “equally likely to occur”?

iii. What is the probability of each of the following events?
- Two heads
- One heads and one tails, in any order
- **Not** two tails

iv. The two coins are tossed 1000 times.
- How many times do you expect to get two heads?
- How certain are you with your answer? Explain briefly.

v. A fair coins is tossed 4 times and the sequences of the outcomes are recorded. Compare the probability of obtaining the following sequences: HHHH and HTTH.

- In a discussion about the question, a student explain that:

  “The probability of getting HTTH is higher than the probability of getting HHHH because in real life, it is very difficult for the same thing to happen 4 times continuously.”

  ❖ What is wrong with the student’s reasoning? Explain briefly.

Tossing Two Fair Dice

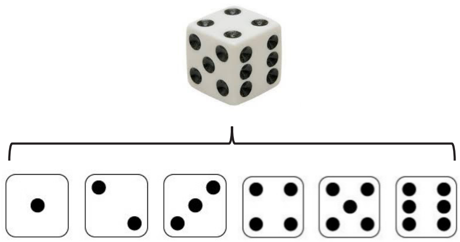


Diagram 6

Diagram 6 shows a fair die and its six faces.

Table 7
Sample Space of Tossing Two Dice

#1 \ #2	⚀	⚁	⚂	⚃	⚄	⚅
⚀	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$	$(1, 5)$	$(1, 6)$
⚁	$(2, 1)$	$(2, 2)$	$(2, 3)$	$(2, 4)$	$(2, 5)$	$(2, 6)$
⚂	$(3, 1)$	$(3, 2)$	$(3, 3)$	$(3, 4)$	$(3, 5)$	$(3, 6)$
⚃	$(4, 1)$	$(4, 2)$	$(4, 3)$	$(4, 4)$	$(4, 5)$	$(4, 6)$
⚄	$(5, 1)$	$(5, 2)$	$(5, 3)$	$(5, 4)$	$(5, 5)$	$(5, 6)$
⚅	$(6, 1)$	$(6, 2)$	$(6, 3)$	$(6, 4)$	$(6, 5)$	$(6, 6)$

Table 7 shows the sample space of tossing two **fair** dice together.

i. What does it mean by the dice are “fair”?

ii. Find the probability of the following.
- Both the tossed numbers are the same
- The sum of the tossed numbers is 10
- The sum of the tossed numbers is less than 6

iii. The sum of the two tossed numbers could be 2 to 12, inclusively.
- Which sum has the greatest probability?

iii. Three fair dice are rolled. What is the probability of each of the following events?
- Three event numbers
- Two even and one odd numbers
- No even number at all

Tossing a Fair Die and a Fair Coin

Diagram 7

Diagram 7 shows a fair die and a fair coin. The die and coin are tossed together.

i. Show the sample space using
   - a table
   - a tree diagram

ii. Find the probability of getting
   - number '6' and 'heads'
   - an even number and 'tails'

Picking Marbles Randomly

Diagram 8

i. Diagram 8 shows a bag with 10 black marbles. You pick one marble **at random** from the bag.

   - What does it mean by picking "at random"?

   - What is the probability of you getting a black marble?

   - The marble picked is then put back to the bag and you repeat picking a marble following the same procedure for a total of 10 times.
     - ❖ How many black marbles do you expect to get?
     - ❖ How certain are you with your answer? Explain your reasons.

Diagram 9

ii. Diagram 9 shows a bag with 7 black marbles and 3 white marbles. You pick one marble at random from the bag.

- What is the probability of you getting a black marble?

- The marble picked is then put back to the bag and you repeat picking a marble following the same procedure for a total of 10 times.
  - ❖ How many black marbles do you expect to get?
  - ❖ How certain are you that you will get exactly your expected number of black marbles? Explain your reasons.

iii. You pick two marbles together from the bag in Diagram 9. This situation is the same as picking one marble first and then pick a second marble **without** putting the first marble back to the box.

- If the first marble picked is white, what is the probability of getting a black marble in the second pick?

- If the first marble picked is black, what is the probability of getting a white marble in the second pick?

- What is the probability that you will pick one white and one black marbles in any order?

- What is the probability that both marbles will be white?

- What is the probability that both marbles will be black?

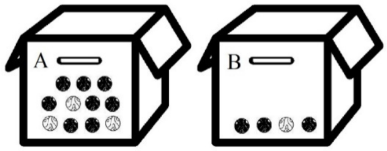


Diagram 10

iv. Diagram 10 shows two boxes, A and B, with black and white balls. Agus is to pick a ball at random from one of the boxes. If he pick a black ball, he will win a prize. So, he has to decide which box to pick the ball from. He thinks to himself:

"There are more black balls in box A. So, I will be more likely to pick a black ball from A than from B."

- What is wrong with Agus's reasoning?

Spinning Two Spinners

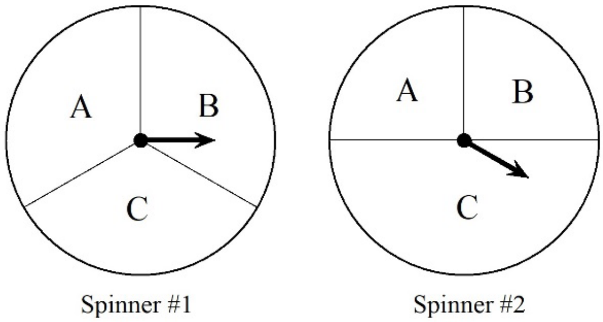


Diagram 11

Diagram 11 shows two spinners. Each spinner board is divided into three regions, A, B and C. Assuming each spinner moves freely when it is spun.

i. Spinner #1 is spun and stopped.

- What is the probability that it will fall in region A, B and C, respectively?

ii. Spinner #2 is spun and stopped.

- A student claims: "*Since there are* 3 *possible outcomes, so the probability for the spinner to fall in region A is* $\frac{1}{3}$."

  ❖ What is wrong with the student's reasoning? Explain briefly.

iii. Both spinners are spun one after another.

- Draw a tree diagram to show the sample space.
- What is the probability of each of the following events?

  ❖ One spinner falls in region A and one spinner falls in region C.

  ❖ Both spinners fall in region B.

  ❖ Both spinners fall in region C.

  ❖ None of the spinners fall in region A.

Task 3: Not Equally Likely Outcomes

Tossing a Biased Coin

Heads Tails

Diagram 12

Diagram 12 shows an ancient coin used in India around the end of the 5^{th} century. The coin is biased and each side of the coin is **not** equally likely to occur when it is tossed. Given the probability of tossing heads is $\frac{3}{5}$.

i. The coin is tossed once.
   - What is the probability of tossing tails?

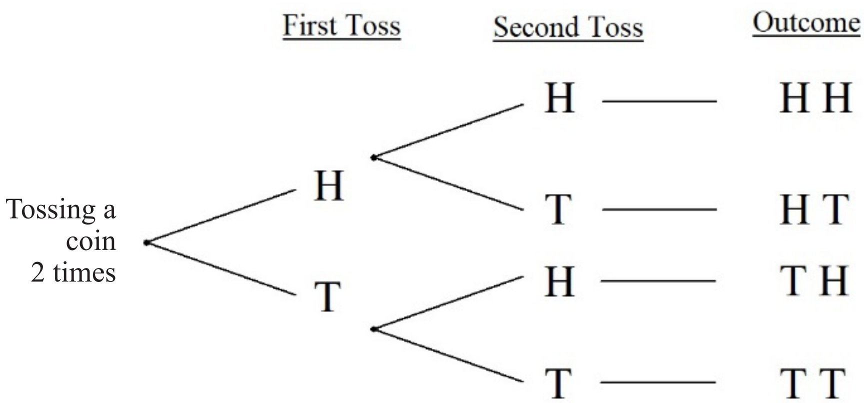


Diagram 13

ii. The tree diagram in Diagram 13 shows the sample space of tossing the coin twice in succession. What is the probability of tossing each of the following events?
   - Two heads
   - One heads and one tails, in any order
   - **Not** two tails

iv. The coin is tossed three times in succession. What is the probability of tossing each of the following events?
   - Three tails
   - One head and two tails in any order

Task 4: Misconception About Probability

i. On the eve of a test, Alejandro tells his parent that he has a 50% chance of passing the test. He goes on to explain that:

"There are only two possible outcomes. Either I pass the test, or I fail. So, the probability of me passing the test is $\frac{1}{2}$*, which is 50%!"*

- What is wrong with Alejandro's reasoning?

ii. Ahmed is playing a board game. He needs to toss a die and get "6" to start moving his counter. However, after 8 times of tossing, he is still unable to get a "6" to start the game. Based on this experience, Ahmed concludes that:

"It is very hard to get "6". I think the probability of getting a "6" is lower than other numbers."

- What is wrong with Ahmed's reasoning?

iii. Aroon tosses a fair coin 8 times and gets 8 heads in a row. Before the next toss, he thinks to himself:

"Since I have got 8 heads in a row, I will have a better chance to get a tails than a heads this time."

- What is wrong with Aroon's reasoning?

iv. Analu takes a quiz with 10 'true or false' questions. After the quiz, he tells his friends:

"I just guess the answers for all the questions. Since the probability that my guess is right is $\frac{1}{2}$ *for each question, I am certain that I will get 5 right answers."*

- What is wrong with Analu's reasoning?

v. Liverpool FC is to play against Manchester United in a charity football match. Adriel says to his friends:

"Liverpool FC can either win, lose or draw, so the probability that Liverpool will win is $\frac{1}{3}$*."*

- What is wrong with Adriel's reasoning?

Task 4: Application of Probability in Real World

Draw Lots

Three brothers want to go for a movie but they only have enough money to buy one ticket. After discussion, they agree to draw lots in order to decide who will be the lucky person to see the movie. So, they prepare three pieces of paper as shown in Diagram 14.

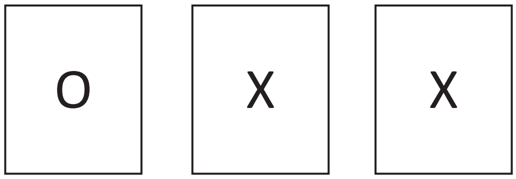


Diagram 14

The papers are folded and put into a box. The three brothers then take turn to draw a piece of paper from the box without looking into the box. The person who draw the paper with the O mark will win the ticket and get to see the movie.

i. The three brothers think to themselves.

First Brother:

"*The first person to pick will have two pieces of paper with the "X" and the chances to pick "X" is higher. So, I better let my two brothers pick first.*"

Second Brother:

"*The first person to pick will have the chance to pick "O" first. So, I better pick first before the "O" is picked by my two brothers.*"

Third Brother:

"*Whether I pick first or last, my chance of winning the ticket will still be the same. So, there is no need to rush. I better let my two brothers to draw the lots first.*"

- Which brother reason correctly? Support your answer with some calculation of probability.
- What if they have enough money to buy two tickets?

A Game of Choices

Diagram 15

You are playing a game with your friend to win a prize. There are three boxes in front of you as shown in Diagram 15. Your friend will put the prize into one of the boxes without you seeing it. Then, you will make an initial choice by choosing one box that you think has the prize in it, without opening it. At that point, your friend will open a box with no prize from among the two boxes that you did not choose. You will then have to decide whether to stick with your original choice, or switch your choice to the remaining unopened box.

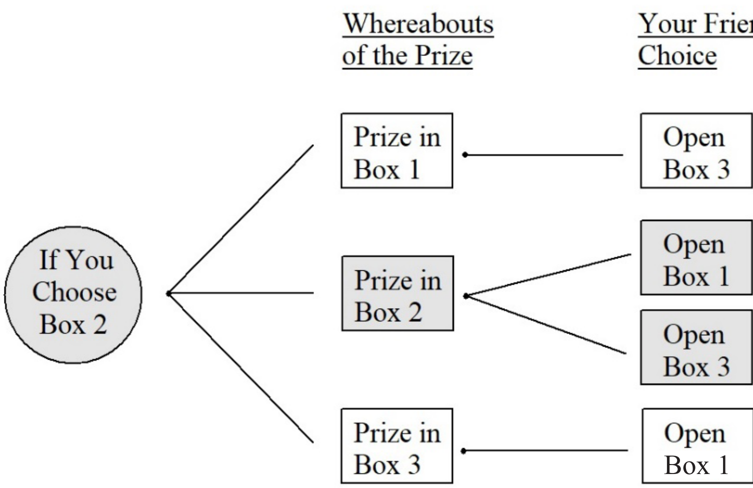


Diagram 16

The tree diagram as shown in Diagram 16 illustrates the case of you choosing Box 2 initially.

i. Let’s say you choose Box 2 initially.

- If the prize is in Box 1, then your friend is left with only one choice. Explain why.
- If the prize is in Box 2, then your friend is left with two choices. Explain why.
- If the prize is in Box 3, then your friend is also left with only one choice. Explain why.

ii. Should you switch to change your choice? Explain your reasons.

False Positives and False Negatives in Medical Test

Statistical data shows that 2% of senior citizen from a region are infected with COVID-19.

i. A particular COVId-19 test gives a correct positive result with a probability of 0.95 when the virus is present, but gives an incorrect positive result with a probability of 0.15 when the virus is not present (false positive).

- What is the probability of the test to give an incorrect negative result when the virus is present (false negative)?

- What is the probability of the test to give a correct negative result when the virus is not present?

ii. A senior citizen from that region, Uncle Sam is tested positive by the test.

- What is the probability that Uncle Sam is really infected by the virus?

iii. Another senior citizen from the same region, Auntie Pamela is tested negative to the test.

- What is the probability that Auntie Pamela is really **not** infected by the virus?

iv. If 1000 senior citizen from the region are tested positive by the test, how many of them do you expect to really have **not** infected by the virus?

Note. You may refer to readings on Bayes' Theorem in order to solve this problem. Although this theorem may not be found in your country lower secondary school mathematics curriculum, knowledge on it is a worthwhile enhancement for teachers in order to understand the decision making process in artificial intelligence and data science.

Predicting the Weather

You plan to go outing with your family members in one non-rainy day next week. The TV station has just broadcast the weather forecast for next week as shown in Table 8. In addition, the odds in favour of raining on Monday is also shown.

Table 8
Whether Forecast for Next Week

Day	Probability of Raining	Odds in favour of Raining
Monday	30%	3 : 7
Tuesday	60%	
Wednesday	75%	
Thursday	60%	
Friday	85%	
Saturday	90%	
Sunday	50%	

i. Find the odds in favour of raining on each of the remaining day for next week and complete Table 8.

ii. Based on the data in Table 8, at a glance, should you cancel your plan? Justify your decision.

iii. Find the odds in favour of raining for 7 successive days in next week.
   - Should you revise your decision in (ii)? Explain your reasons.

Task 4: Variability and Uncertainty in Sampling

i. For this standard, variability is considered from the perspectives of uncertainty due to sampling in determining measures of likelihood. A good understanding of how this uncertainty could affect the measures of likelihood is crucial for a better understanding of probability.

   As such, one implicit aim of this topic is to help students develop understanding of variability arises from uncertainty in sampling through the formation of informal statistical inferences.

   - In your opinion, to what extend are the tasks in this topic able to promote a better understanding of **variability arises from uncertainty in sampling** through informal statistical inferences among lower secondary school students.

Topic 3: Exploring Statistics with Sampling [K3SP3]

Standard 3.1: [K3SP3-1]
Exploring sampling with the understanding of randomness

i. Discuss the hidden hypothesis behind sample and population
ii. Use randomness to explain sampling
iii. Analyse the data exploratory such as dividing the original into two for knowing better data representations and discuss appropriateness such as regrouping
iv. Appreciate data sampling in a situation with sustainable development

Sample Tasks for Understanding the Standards

Task 1: Population and Sampling Variability

Census Survey and Sample Survey

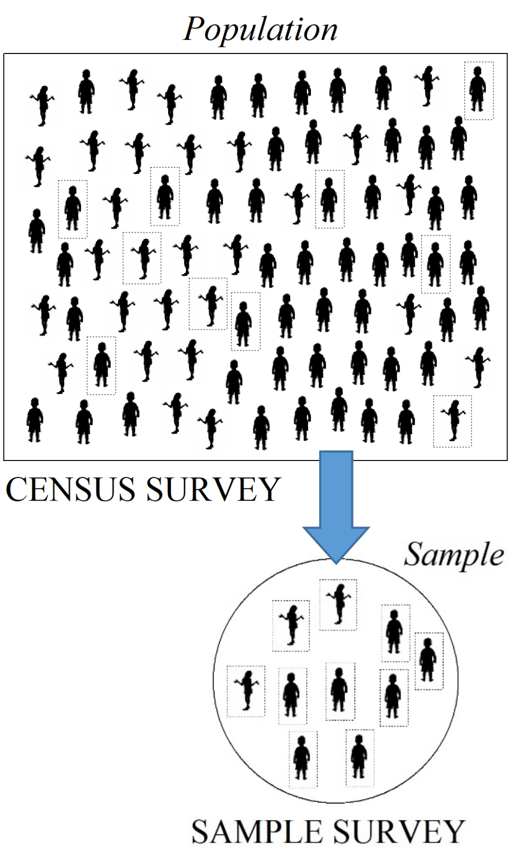


Diagram 1

Diagram 1 shows the meaning of census survey and sample survey.

i. Compare the strengths and weaknesses of census survey and sample survey.

ii. Would a census or sample survey be more appropriate for each of the following investigations? Explain your reasons.

- The length of time an electric light bulb produced by a factory will last
- The causes of car accidents in a district
- Public opinions on a newly installed facility in a township

iii. The purpose of a sample is to provide an estimate of a particular characteristic of the total population.

- What is a good sample?
- How to choose a good sample?

Estimating Population Characteristic From Sample Characteristic

A school intends to organise a mathematics seminar for all its 200 lower secondary students. The school conducted a census survey where all students from the population were asked whether or not they are in favour of the seminar and their replies were shown in Diagram 2, where Y indicates a “yes’ and N indicates a “no”.

1st student →	Y Y Y N Y N Y N N N	Y N N Y Y N Y Y Y N	
	N Y N Y Y Y N Y Y Y	Y N Y N Y Y Y Y Y Y	← 40th student
41st student →	N N Y N N N N Y Y N	Y N Y N Y N N N Y N	
	N N N Y N Y Y Y N Y	Y Y Y Y Y N Y Y Y Y	← 80th student
81st student →	N Y Y Y Y Y Y Y Y Y	N N N Y N Y Y N Y Y	
	N Y Y Y Y N Y N Y Y	Y N N N Y N Y Y Y Y	← 120th student
121st student →	N Y N Y Y Y N Y Y Y	Y Y Y Y N Y Y N Y Y	
	Y N N Y N Y N Y Y Y	N Y Y Y Y Y Y Y Y Y	← 160th student
161st student →	N N N Y Y Y Y Y Y Y	N Y N Y N N Y Y N N	
	N Y Y Y Y N N Y Y N	Y N Y Y N N N N Y Y	← 200th student

Diagram 2

i. Find the proportion of students from the population who are in favour of the seminar.

ii. Find the proportion of the (a) first 10, and (b) last 10 students who are in favour of the seminar.
   - Are these two good samples? Explain your reasons.

iii. Generate a set of 20 random numbers between 1 and 200 inclusively. Use this set of random numbers to select a random sample of 20 students from the population.
   - Find the proportion of students of this sample who are in favour of the seminar.
   - Generate another two sets of 20 random numbers to select two other random samples of 20 students.
   - Find the proportion of students who are in favour of the seminar in the two cases above.
   - Compare the results of the three samples. Comment briefly on the results of your comparison.

[**Note**. 1. Random numbers can be generated easily by the lucky-draw method. Write the numbers 1 – 200 separately on 200 pieces of cards. Place all the cards face down, mix them well, and then draw 20 cards to generate a set of 20 random numbers between 1 – 200 inclusively.

2. Random numbers can also be generated from a random number table, a calculator or an electronic spreadsheet.

Task 2: The Effect of Sample Size

Diagram 3 shows the heights, in cm, of a population of 120 lower secondary students from a school.

1st student →	152	146	162	185	169	158	149	183	161	173	
	173	180	184	167	166	183	181	156	151	180	← 20th student
21st student →	170	176	175	184	173	167	177	156	162	167	
	178	175	164	182	147	161	177	147	152	154	← 40th student
41st student →	180	153	161	152	166	152	180	183	168	184	
	166	165	158	161	156	176	179	179	159	170	← 60th student
61st student →	183	180	166	172	181	169	175	158	183	155	
	157	173	184	174	181	157	163	164	163	157	← 80th student
81st student →	158	148	155	145	178	161	158	168	164	156	
	166	185	156	157	160	171	175	167	160	170	← 100th student
101st student →	180	175	155	153	147	174	173	156	168	184	
	147	181	179	183	154	160	165	175	167	170	← 120th student

Diagram 3

i. Find the mean height of the population.

ii. Randomly select a sample of 10 students.
- Find the mean height of the sample.
- How close is the sample mean to the population mean? Explain your answer.

iii. Table 1 shows the data obtained from 16 samples each with 10 students' heights randomly selected from the population. The mean heights for sample #1 to sample #10 have been calculated.

Table 1
Sample Mean of Students' Height in cm With Sample Size n = 10

Sample	Student Height	Sample Mean
#1	148, 161, 173, 152, 174, 175, 155, 170, 166, 175	164.9
#2	173, 180, 153, 168, 170, 163, 168, 184, 183, 170	171.2
#3	158, 177, 161, 152, 161, 179, 180, 153, 175, 156	165.2
#4	184, 165, 154, 179, 160, 178, 158, 157, 147, 147	162.9
#5	173, 157, 184, 181, 158, 183, 185, 175, 157, 156	170.9
#6	168, 151, 156, 173, 178, 157, 180, 154, 158, 152	162.7
#7	183, 180, 173, 157, 175, 178, 158, 146, 183, 161	169.4
#8	173, 175, 154, 184, 160, 169, 164, 167, 180, 155	168.1
#9	156, 177, 157, 179, 146, 180, 157, 184, 183, 151	165.0
#10	170, 166, 174, 152, 163, 161, 155, 179, 162, 147	162.9
#11	164, 156, 165, 164, 160, 158, 174, 167, 171, 146	
#12	181, 182, 145, 185, 160, 184, 185, 152, 147, 166	
#13	155, 149, 176, 183, 168, 158, 147, 164, 175, 152	
#14	158, 180, 152, 148, 158, 160, 174, 155, 173, 167	
#15	175, 166, 156, 156, 157, 166, 161, 163, 180, 171	
#16	175, 147, 173, 166, 183, 173, 147, 181, 156, 164	
#17		
#18		

- Calculate the mean heights for sample #11 to sample #16 and fill in the appropriate columns in Table 1.
- Select two more random samples (#17 and #18) each with 10 students and fill in the data in Table 1.
- Calculate the mean heights for these two random samples and complete Table 1.

iv. Table 2 shows the mean heights obtained from 16 samples each with 20 students' heights randomly selected from the population.

Table 2
Sample Mean of Students' Height in cm With Sample Size n = 20

Sample	Mean Height (cm)	Sample	Mean Height (cm)	Sample	Mean Height (cm)
#1	163.7	#7	165.2	#13	168.9
#2	165.8	#8	168.4	#14	168.6
#3	167.7	#9	163.9	#15	167.2
#4	163.7	#10	164.7	#16	169.2
#5	170.4	#11	167.8	#17	
#6	162.6	#12	163.8	#18	

- Select two more random samples (#17 and #18) each with 20 students' heights from the population.
- Find the sample means for samples #17 and #18 to complete Table 2.

v. Complete the frequency distribution table for the sample means with sample size n = 10 from Table 1 and sample size n = 20 from Table 2.

Table 3
Frequency Distribution Table for Sample Means With Sample Sizes n = 10 and n = 20

Sample Means, x cm	Frequency	
	Sample Size n = 10	Sample Size n = 20
$145 \le x < 150$		
$150 \le x < 155$		
$155 \le x < 160$		
$160 \le x < 165$		
$165 \le x < 170$		
$170 \le x < 175$		
$175 \le x < 180$		
$180 \le x < 185$		

- Draw the histograms of sample size n = 10 and n = 20 respectively,
- Compare the two histograms. What do you observe about the distribution of sample means for the two different sample sizes?
- What effect does the sample size have on estimating the mean of the population?

vi. Generally, how can we increase the reliability of an estimate of a population characteristic based on sampling survey?

Diagram 4

Table 4
Mean Mass in g of 10 Samples of Eggs With Sample Size n = 20

58.6 g	60.9 g	61.3 g	62.8 g
63.2 g	63.3 g	63.4 g	63.5 g
64.0 g	64.2 g		

vii. Diagram 4 shows the eggs produced by a farm in one day. Ten random samples each with 20 eggs were selected and the mean mass, in g, for each sample of eggs was calculated. Table 4 shows the sample means arranged from smallest to largest.

- Based on the data, state whether each of the following statements is true or false.

  ❖ The mean for the population is at least 58.6 g.
  ❖ The mean for the population is **not** more than 64.2 g.
  ❖ Among the sample means, there is a value which is exactly the same as the mean of the population.
  ❖ The mean of the population is close to 63 g.
  ❖ If we increase the sample size to 40, the sample mean would be more reliable as an estimate for the population mean.

Task 3: Application of Sample Survey

Capture-Recapture Method

Diagram 5

i. Diagram 5 shows a fish pond belongs to a farmer Ko Aung Myint. He wants to estimate the number of fish in his pond. At 10 different locations of the pond, he caught 60 fish and each fish was "tagged" and then released back to the pond. A week later, 80 fish was caught randomly from the pond and it was found that 12 of them had tags.

- What is the population?

- What is the sample?

- Estimate how many fish there are in the pond.

Quality Control of Factory Products

Diagram 6

Diagram 6 shows your favorite dark chocolate bar. The weight on the wrapping indicates 100 g. To avoid customer complaints and lawsuits, the manufacturer has to make sure that 98% of all chocolate bars weigh 100 g or more.

i. As a quality control check , 84 chocolate bars were randomly selected from a batch of chocolate bars produced by a factory and 2 of them were found to weigh less than 100 g.

- Estimate how many percent of the chocolate bars are up to the standard specified on the wrapping?

Task 4: Variability and Random Sampling

i. For this standard, variability is considered from the perspective of random nature of sample survey. As such, one implicit aim of this topic is to help students develop understanding of variability arising from randomness of samples through the formation of informal statistical inferences.

- In your opinion, to what extend are the tasks in this topic able to promote a better understanding of **variability arising from randomness of sample** through informal statistical inferences among lower secondary school students?

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APPENDIX A

Framework for CCRLS in Mathematics

Nature of Mathematics

Mathematics has been recognised as a necessary literacy for citizenship, not only for living economically but also for establishing a society with fruitful arguments and creations for better living.

In Ancient Greek, 'Mathema' meant the subject of learning, a learning theme which developed the ways of thinking by using their mother language, Greek. Currently, it means a natural language for using Generative AI. Even Ethics is a subject of Mathema in Ancients. Since that era, mathematics has been taught as the training of reasoning by using language for all academic subjects with visual and logical-symbolic representations.

This is especially significant after Descartes' proposal, where algebraic representations became sophisticated and spread as a universal language for any mathematical sciences, which is necessary for learning science and engineering. The narrow image of mathematics in the school curriculum is deeply influenced by his proposed universal mathematics, which tries to represent any subject in algebraic form that cannot be managed without learning algebraic operation as an artificial language. Even programming language is influenced by Algebraic representations.

In the current era, school mathematics has expanded its role to establish 21st-century skills through reviewing mathematics as the science of patterns for future prediction and designing with big data, producing innovation for technological advancement and business models. Mathematics is an essential subject for establishing common reasoning for the sustainable development of society through viable arguments in understanding each other and developing critical reasoning as a habit of mind. This means mathematics should be learned as the basis for all subjects.

For clarifying the framework in CCRLS on mathematics and by knowing the role of mathematics education, the humanistic and philosophical natures of mathematics are confirmed as follows.

Humanistic nature of mathematics

The humanistic nature of mathematics is explained by the attitudes of competitiveness and understanding of others with sympathetic minds on the episodes of challenging mathematicians such as Blaise Pascal, Rene Descartes, Isaac Newton and Gottfried Wilhelm Leibniz.

For example, suppose you read the letter from Blaise Pascal to Pierre de Fermat. In that case, you will recognise Pascal's **competitive attitude** toward Fermat's intelligence and his **sympathetic attitude** toward Fermat: Pascal demanded that Fermat understand his excellence in finding Pascal's Triangles beyond Fermat's achievement.

On the other hand, if we read Pascal's Pensées, we recognise how Pascal denied Descartes's geometry, which used an algebraic form from the aspect of Ancient Greek geometry, which used natural language. Even Descartes himself tried to overcome the difficulties of ancient geometry through algebra. If you read the letter from Descartes to the Royal Highness Elisabeth, you will recognise how Descartes appreciated and felt happy that Elisabeth used his algebra ideas in geometry. Despite being a princess, Elisabeth has continuously been learning mathematics in her life.

The contrast between Pascal and Descartes as mathematicians implies that even mathematicians can have different identified mathematics. Pascal established his identity in mathematics based on geometry from Ancient Greece, whereas Descartes established his identity in mathematics based on Algebra originated from Ancient Arab.

There were discussions on who developed calculus between Britain and Continental mathematicians. In that context, Johann Bernoulli, a continental mathematician, posed a question in the journal about the Brachistochrone problem. The answer is the locus of the point on the circumference of the circle when it rotates on the line. No one replied, and Bernoulli extended the deadline for the answer and asked Newton to reply. Newton answered it within a day. Finally, six contributions of the appropriate answer, including Newton and other Continental mathematicians, were accepted. However, even in Calculus, Newton, well-known for

using algebraic representation, preferred physical representation for his Calculus in his Philosophiæ Naturalis Principia Mathematica. However, Continental mathematicians progressed from algebraic form to functional form. Those differences implied the differences between their identities in mathematics.
All those episodes show that mathematics embraces the humanistic nature of proficiency for competitiveness and understanding others to share ideas. Those Mathematicians' ideas are deeply related to their identity in mathematics.

Philosophical nature of mathematics

The philosophical nature of mathematics can be explained from ontological and epistemological perspectives. From the ontological perspective, mathematics can be seen as a system for universal understanding and a common scientific language. Plato and Aristotle are usually compared on this perspective. In Platonism, Plato believes that the existence of the world of "idea" and mathematics existed in the world of "idea". In this context, mathematical creation is usually explained by the word "discover", which means taking out the cover from which it already exists. At the moment of discovery, the reasonableness, harmony and beautifulness of a mathematical system are usually felt.

Aristotle tried to explain how to reach an idea from the "material" to the "form". This explains that abstract mathematics can be understood with concrete materials using terms such as "modelling", "instruments", "embodiment", "metaphor", and "change representation". Mathematical inventions such as creating definitions are usually based on the embodiment or metaphor of notions. For example, the triangle as a figure is usually defined based on the triangular images of a shape. From an ontological perspective, anyone can understand and acquire mathematics. If acquired, it is a common scientific language used to express any subject. Once ideas are represented using the shared common language, the world can be autonomously perceived in the same view. From the ontological perspective, 'I understand what you are saying in mathematics' means to **re-**present others' ideas in one's mind. The ontological meaning of mathematics is a unique subject that can **re-**present others' ideas precisely in one's mind as long as we try to understand them.

As mentioned, a computer programming language can be seen as a mathematical language. Ontologically, the Computer programme has also a similar nature: It exists like an idea mathematically and logically, but it was invented as an artificial language that runs electronically with human actions on the physical causality in time and the action using a computer is understood like the extension of the human body.

From the epistemological perspective, mathematics can be developed through processes which are necessary to acquire mathematical values and ways of thinking. From this perspective, idealism and materialism are compared. In the context of Hegel, a member of German idealism, Imre Lakatos explained the development of mathematics through proof and refutation using counter-examples. In other words, beyond contradiction is the nature of the mathematical activity, and it provides the opportunity to think mathematically to overcome contradictions. In this context, mathematics is not a fixed system but an expandable system that can be restructured through a dialectic process to construct viable arguments. Plato also used dialectic to reach ideas with counter-examples in mathematics. The origin of dialectic in Philosophy is known as the origin of indirect proof in Mathematics. Dialectics have been the content of teaching in mathematics since Ancient Greece.

In education today, dialectics are a part of critical thinking for creation. George Pólya and Hans Freudenthal give parallel perspectives for mathematical developments. For the discovery of mathematics, Pólya explained mathematical problem-solving processes with mathematical ideas and ways of thinking in general. Freudenthal enhanced the activity to re-organise mathematics with the term mathematisation on the principle of reinvention. Genetic epistemologist Jean Piaget established his theory for operations based on various theories, including the discussion of Freudenthal and explained the mathematical development of operations by the term reflective abstraction. Reflection is also a necessary activity for mathematisation by Freudenthal. These are perspectives of constructivism.

On materialism, under the Vygotskyian perspective, intermediate tools such as language become the basis for reasoning in the mind. Under his theory, high-quality mathematical thinking can be developed depending on high-quality communication in mathematics classrooms. This is the reason why Dialectical-critical discussion has been enhanced in mathematics classes.

From the epistemological perspective, mathematics can be developed through the processes of (dialectic) communication, problem-solving and mathematisation, which include the re-organisation of mathematics. Those processes are necessary to acquire mathematical values and ways of thinking through reflection.

Aims of Mathematics in CCRLS

The aims of mathematics in CCRLS for developing basic human characters, creative human capital, and well-qualified citizens in ASEAN for a harmonious society are as follows:

- Develop mathematical values, attitudes and habits of mind for human character,
- Develop mathematical thinking and be able to engage in appropriate processes,
- Acquire proficiency in mathematics contents and apply mathematics in appropriate situations.

The Framework for CCRLS in Mathematics was developed based on the three components with discussions of mathematics' humanistic and philosophical nature. Due to the current necessity of the Era of Generative AI, the framework was revised, as shown in Figure 3. This framework also depicts the concrete ideas of mathematics learning based on the above aims.

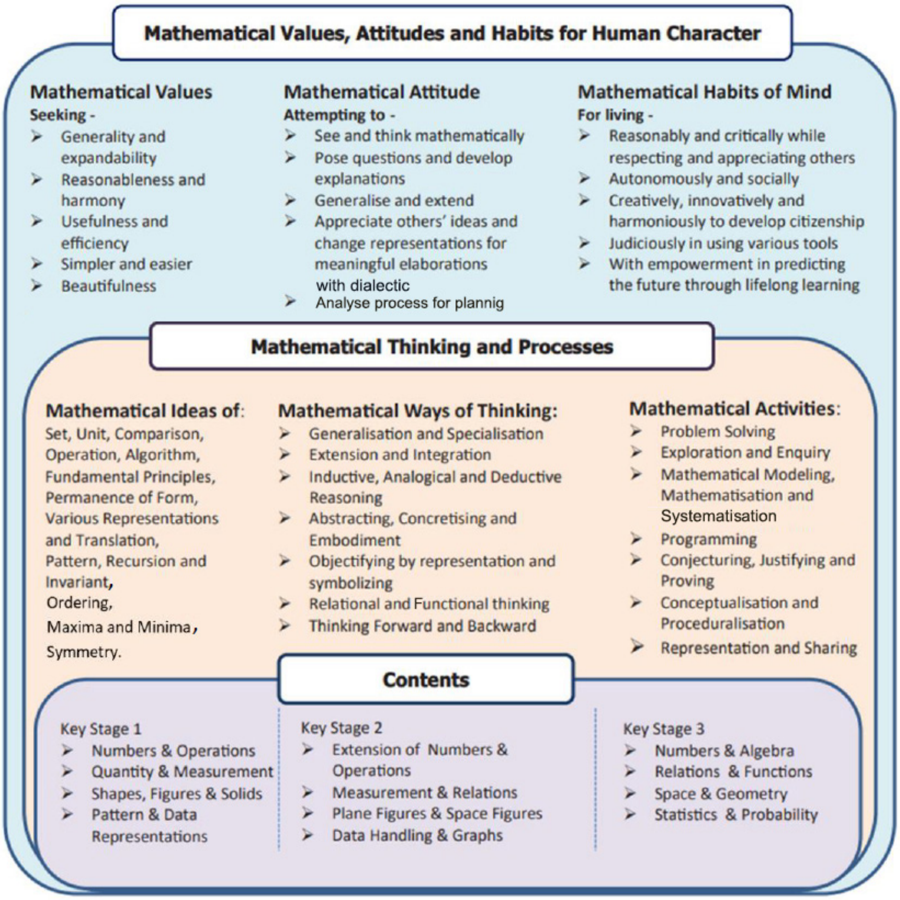

Figure 3. Revised CCRLS Framework in Mathematics*
*The terms in Figure 3 are explained at Appendix B

Mathematical Values, Attitudes and Habits for Human Character

To cultivate basic human characters, values, attitudes, and habits of mind must be developed through mathematics. It is a part of human characters appeared beyond mathematics. Values are the basis for setting objectives and making decisions for future directions. Attitudes are mindsets for attempting to pursue undertakings. Habits of mind are necessary for soft skills to live harmoniously in society. Mathematical values, mathematical attitudes and habits of mind are simultaneously developed and inculcated through learning the content knowledge. Essential examples of values, attitudes and habits of mind are given in Figure 3. Generalisable and expandable ideas are usually recognised as strong ideas on mathematical values. Explaining why proving is necessary for mathematics is a way of seeking reasonableness. Harmony and beautifulness are described in mathematical arts, the science of patterns and systems of mathematics. Usefulness and simplicity are used in the selection of mathematical ideas and procedures. Regarding mathematical attitudes, “seeing and thinking mathematically” means using the mathematics learned to see and think about objects. Posing questions and providing explanations such as the “why” and the “when” are the ordinary sequence for thinking mathematically. Changing representation to other ways, such as modelling, can overcome running out of ideas in problem-solving. The mindset of trying to understand others is the basis for explaining one’s ideas that the rest understand with appreciation. Producing a concept with a definition operationally is a manner of mathematics. In mathematical habits of mind, mathematical attitudes and values are necessary for reasoning critically and reasonably, enabling citizens to live meaningfully. Appreciating and respecting other ideas is also essential. Mathematics is developed independently for those who creatively, innovatively and harmoniously appreciate life. Seeking an easier and more effective manner of selecting appropriate tools is necessary. Mathematics is a subject that offers to challenge and experience competitiveness, appreciation with others, and develop the mindset for lifelong learning, personal development and social mobility.

Mathematical Thinking and Processes

Mathematical ideas, mathematical ways of thinking and mathematical activities are essential for developing creative human capital. Those are functioning in our life beyond mathematics. Mathematical ideas are process skills involving mathematical concepts. Mathematical thinking is a mathematical way of reasoning in general, which does not depend on specific concepts. Mathematical activities include problem-solving, exploration and inquiry. Mathematical processes, which include these components, are necessary skills to use in our lives, such as innovation in this society (e.g., the Internet of Things (IoT)). In education, competency, which refers to mathematical processes, is the basis for STEM and STEAM education and social science and economic education. In the era of generative AI, computational thinking and mathematical thinking have been enhanced. Due to the emergence of computers using mathematical language, mathematical thinking in our lives becomes an essential basis for computational thinking even though we do not do programming. Various mathematical thinking is embedded in computational thinking in our lives.

Mathematical ideas in general serve as the basis of content knowledge to promote and develop mathematical thinking. There are two types of mathematical ideas: Firstly, one is mathematical ideas in general. The second one is mathematical ideas for specific content. It is necessary to note that mathematical content cannot be used without mathematical ideas because content and ideas usually produce contexts. For example, some people believe that multiplication is an accumulation/a repeated addition. However, the idea of multiplication is that ‘any number can be seen as a unit for counting’. Without this idea, we cannot apply accumulation in a situation.

Some key ideas of mathematics in general are used as a special process. The fundamental ideas of set and unit lead to a more hierarchical and simple structural relationship. The ability to compare, operate, and perform algorithms of related functions enables efficient ways of learning mathematics and solving problems in learners’ lives with mathematics. In the case of a set, it is a mathematical idea related to conditions and elements. It is related to activity in grouping and distinguishing with other groups by conditions. For example, 3 red flowers and 4 white flowers become 7 flowers, if the condition of the set does not consider the colours. “A and non-A” is a simple manner to distinguish sets with logical reasoning. We use intervals such as $x > 0$, $x < 0$, and $x = 0$ for categorising. This situation can be seen in the hyperbolic graph, where . In the case of a unit, it is a mathematical idea that is related to a process to produce and apply the unit with operations. In some cases, trying to find the common denominator is the way to see the unit of two given quantities. A tentative unit such as arbitrary units can be set and applied locally whereas standard units are used globally. The combination of different quantities produces new measurement quantities such as distance to time produced speed. A square unit such as square centimetres is a unit of area. In programming, we usually find the repetitive parts as patterns and alternate them with the computer.

Mathematical thinking is well discussed by George Pólya. Inductive, analogical and deductive reasoning are major logical reasoning at school. However, deductive reasoning is enhanced with formal logic, whereas inductive and analogical reasoning are not well recognised. Pólya enlightens the importance of that reasoning in mathematics. In the process of mathematisation by Hans Freudenthal, objectifying the method is necessary. David Toll mentioned it using the term thinkable concept in the process of conceptual development. Pólya mentioned thinking forward and backward with ancient Greek terms synthesis and analysis. Mathematical activities are ways to represent a mathematical process. The problem-solving process was analysed by Pólya. He influenced problem solving with various strategies. Technology enhances the activities of conjecturing and visualising inquiries. Conceptualisation is done based on procedures such as 3+3+3+3=12, which becomes the basis for 4x3. The proceduralisation of multiplication is done through developing the multiplication table, the idea of distribution and mental operation.

Content

For cultivating well-qualified citizens, content knowledge of mathematics is essential. In the mathematics syllabus, the content is usually divided into the domains such as algebra and geometry. Domains are used for compartmentalisation. Strands are usually for mathematical connectivity. Human characters and creative human capital should be developed through mathematical processes. Values, attitudes and habits of mind are driving forces for engagement in mathematical processes. From this perspective, the content knowledge of mathematics cannot be realised without involving human character formations with mathematical process skills. Thus, instead of dividing content into domains as differences in mathematics, CCRLS preferred the words ‘strands’ to represent several connectivities. From the perspective of mathematical content, CCRLS is divided into three stages to indicate grade levels. Every stage has four content strands and the mathematical Process-Humanity strand.

Four Content strands and Mathematical Process-Humanity strand

Additionally, the Mathematical Process-Humanity strand makes clear the process skills such as communications and representations which should be taught at every stage. Between the stages, the names of the content strands are indirectly connected even though there is connectivity. However, beyond the content strands, connectivity is expected in the strands instead of domains. Mathematical Process-Humanity strand in every stage supports the connectivity of every standard in each stage. The names of strands for every key stage are shown in Figure 4.

Key Stages	Content Strands	Process Strands
Key Stage 1	Numbers and Operations Quantity and Measurement Shapes, Figures and Solids Pattern and Data Representations	Mathematical Process-Humanity
Key Stage 2	Extension of Numbers and Operations Measurement and Relations Plane Figures and Space Figures Data Handling and Graphs	Mathematical Process-Humanity
Key Stage 3	Numbers and Algebra Relations and Functions Space and Geometry Statistics and Probability	Mathematical Process-Humanity

Figure 4. Four Content Strands and One Process-Humanity Strand

In every stage, four content strands including several standards are mutually related. Between the stages, all strands in different key stages are mutually related. The same content strand names are used to indicate development and reorganisation beyond each stage. For example, “Numbers and Operations” in Key Stage 1, “Extension of Numbers and Operation” in Key Stage 2, and “Numbers and Algebra” in Key Stage 3 are well connected. These names of the content strands show the extension and integration of contents. For example, even and odd numbers can be taught at any stage with different definitions. At Key Stage 1, even numbers can be introduced as “counting by two” which does not include zero. In Key Stage 2, it can be re-defined by a number divisible by two. Finally, in Key Stage 3, it can be re-defined as a multiple of two in integers which includes zero. Although we use the same name as even numbers, they are conceptually different. The definition in Key Stage 1 is based on counting, Key Stage 2 is based on division and Key Stage 3 is based on algebraic notation. Expressing such theoretical differences requires names of strands for content to be distinguished. In the case of measurement, there is no strand name of measurement in Key Stage 3. Key Stage 1 relates to the quantity and setting of the units. In Key Stage 2, it extends to non-additive quantity beyond dimension. In Key Stage 3, the idea of unit and measurement is embedded in every strand. For example, the square root in the Numbers and Algebra strand is an irrational number which means unmeasurable, the Pythagorean Theorem in Space and Geometry strand is used for measuring, proportional function in the Relations and Functions strand is used for counting the number of nails by weight, and in Statistics and Probability strand, new measurement units are expressed such as quartile for boxplot.

To learn mathematics, the three components in Figure 3 should be embedded in every key stage as standards for the content of teaching. The “Mathematical values, attitudes, habits for the human character” component and the “Mathematical thinking and processes” component cannot exist without the “Content” component. The first two components can be taught only through teaching with the content. For teaching those three components at the same time, the Mathematical Process – Humanity strand is also included in every stage.

Learning Contexts to Link the Three Components

The interconnection of the three components is shown in Figure 5. The three components will not be ideally operationalised without appropriate contexts. In mathematics classrooms, teachers usually provide the necessary contexts of learning with real-world problem solving and mathematical task sequences.

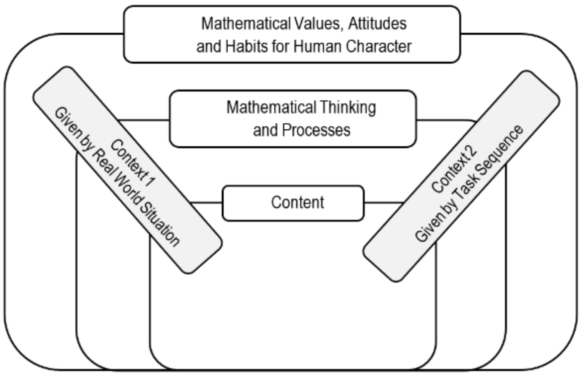

Figure 5. Interconnection of Components in CCRLS Framework in Mathematics

The ultimate aim of the CCRLS framework is to develop mathematical values, attitudes and human characters which are the essence of a harmonious society. This component is closely related to the affective domain of human character traits which correspond to soft skills that can be developed through appreciation. Concerning this, the acquisition of mathematics contents as hard skills and reflection on the thinking processes are needed to inculcate the capability of appreciation. The reflection is necessary for learners to recognise their cognitive skills derived from the contents. Even though contents appeared to be learned independently through acquisition, the mathematical thinking and process, as well as the appreciation of mathematical values, attitudes and habits for the human character, are possible to be developed through reflective experiences which include appreciation. Thus, ways of learning the three components can be characterised as in Figure 6.

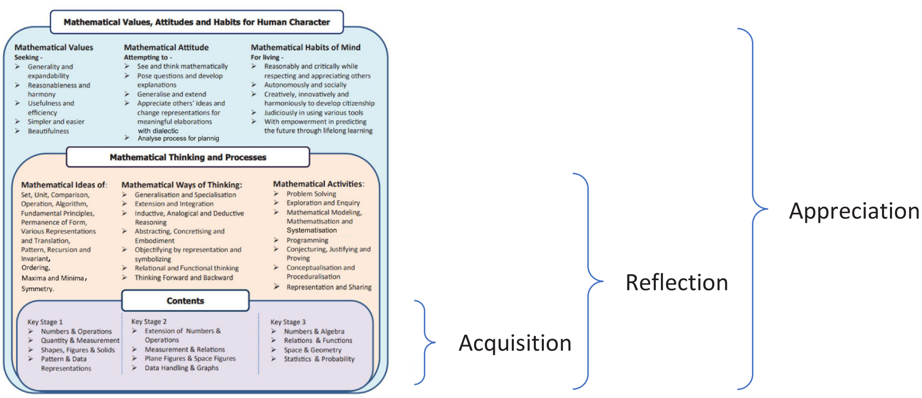

Figure 6. Acquired the content learning through reflection and appreciation

From the perspective of ‘Mathema’ in Ancient Greece (Inprasitha, Isoda & Araya, to appear) the ways of thinking and values such as dialectic discussion with critique can be learned through any subjects such as ethics, geometry and so on. The CCRLS framework in Figure 3 makes clear such ways of thinking and values from the perspective of current mathematics subjects. These days, algebraic representations as a unique universal language are necessary to be acquired through appropriate exercise as a second language as well.

The appropriateness of exercise is also clarified by the context given by the task sequence in Figure 5 for what elements students can learn as shown in Figure 1, through the process of reflection and appreciation in Figure 6. All three components are learning content which is necessary in the Era of Generative AI because Generative AI is used by natural language as well as algebraic notations and so on.

The sample contexts for learning mathematics in classrooms

Figure 5 illustrates two primary contexts which teachers usually provide to teach the three components at once. To clarify these two contexts in the classroom context, designing the following sample activities will be necessary to realise the three components in each classroom:

- Explore a problem with curiosity in a situation and attempt to formulate mathematical problems.
- Apply the mathematics learned, listen to others' ideas and appreciate the usefulness, power and beauty of mathematics.
- Enjoy classroom communications on mathematical ideas in solving problems with patience and develop persistence.
- Feel the excitement of "Eureka" with enthusiasm for the solutions and explanation of unknown problems.
- Think about ways of explanation using understandable representations such as language, symbols, diagrams and notation of mathematics.
- Discuss the differences in seeing situations before and after learning mathematics.
- Explain, understand others and conclude mathematical ideas.
- Explore ideas through inductive and deductive reasoning when solving problems to foster mathematical curiosity.
- Explore ideas with examples and counter examples.
- Imagine others, getting others perspectives, and then ask "if your saying is true, what will happen".
- Feel confident in using mathematics to analyse and solve contextual problems both in school and in real-life situations.
- Promote knowledge, skills and attitudes necessary to pursue further learning in mathematics.
- Enhance communication skills with the language of mathematics.
- Promote abstract, logical, critical and metacognitive thinking to assess their own and others' work.
- Foster critical reasoning for appreciating other's perspectives.
- Promote critical appreciation of the use of information and communications technology in mathematics.
- Appreciate the universality of mathematics and its multicultural and historical perspectives.

Those contexts are chosen to illustrate the interwoven links of the two components with contents. It looks like methods of teaching however all three components are the subject of teaching to develop students who learn mathematics by and for themselves. The Mathematical Process-Humanity strand in each stage illustrates the three components as contents of learning.

Appendix B

Terms of the Revised CCRLS Framework in Mathematics

Higher-order thinking is the curriculum terminology, but it is not specified in mathematics. Here, it is explained generally as acceptable terms from the perspective of mathematics in education. The following terms are samples that appear in SEA-BES: CCRLS (Figure 1) on Mathematical Thinking and Processes and Values and Attitudes. These terms are explained to make clear descriptions of the objectives of teaching. If you use these terminologies for writing the teaching objectives, you will be able to consider how you teach them in the process. Mathematical Thinking and Processes can be explained through (i) Mathematical Ideas, (ii) Mathematical Ways of Thinking, and (iii) Mathematical Activities. Mathematical values and attitudes for Human Character Formation are explained through (iv)Value and (v)Attitude. Mathematical Ideas and Ways of Thinking can be developed through reflection on the processes, while Values and Attitudes are developed through appreciation.

Mathematical Thinking and Processes

(i) Mathematical Ideas

Although every mathematics content embeds some necessary ideas, essential mathematical ideas are used in various situations/contexts. Mathematical ideas are not exclusive but function as complementary. The following are samples of crucial mathematical ideas.

Set

A set is a collection of elements based on certain conditions. When the condition of the set changes, the result of reasonings related to the set may change too. By the definition of figures, we can compare the inclusion relationship between figures if every figure is defined by the conditions. If it is the number of elements, the sets are compared by one-to-one correspondence. The idea of a set is reflected through activities that require us to think about the membership (by elements or conditions) of a set. In addition, activities involving subsets, cardinality and power are extended ideas of a set. The number of elements is called a cardinal or cardinal number (or set number). An ordinal number does not imply the number of elements. Other ideas include operations of sets such as union, intersection, complement, the ordered pair/combination of elements such as Cartesian products and dimension mapping. A number system is a set with structures of equality, order (greater, less than), and operations, which are developed and extended throughout the curriculum from natural numbers to complex numbers: the set of complex numbers does not have the structure of order.

Unit

Units are necessary for counting, measurement, number lines, operations, and transformation. It is represented as “denomination’ for discrete quantity, such as 1 “apple’ for situations involving counting, or continuous quantity, such as 1 gram for situations involving measurement. Mathematically, a unit indicates a number by mapping it with the quantity in a situation.

In a situation, it can be fixed based on the context of comparison, which can be a direct or indirect comparison. In this context, a remainder or a difference from a comparison can be used to fix a new arbitrary unit for measurement that is a fraction of the original unit. This process of determining a new unit is the application of the Euclidean algorithm for finding the greatest common divisor.

For the base-10 place value number system, every column is defined by units such as ones, tens, hundreds, etc. However, in other place value number systems such as the binary system, every column is defined by units such as ones, twos, fours and so on. Therefore, in a place value number system, the unit is not always multiple to the power of ten.

In multiplication, any number can be seen as a unit for counting. Other number systems are made up of different units. For the calendar system, the lunar calendar is based on 30 (29.5) days, while the solar calendar is based on 365 (365.25) days. On the other hand, the imperial and U.S. customary measurements include units in the base-12 and base-16 systems. In the ancient Chinese and Japanese systems, there were units in the base-4, including the base-16. On the other hand, the units used in different currency systems depend on various cultures and countries. However, many countries had lost the unit of 1/100 on their currency systems, which originated from “per centos”, which means percent. Even though the based-10 place value system represents the value of money, many currency systems use the units for 2, 5, and 25 in their denominations instead.

Unit for a new quantity can be derived from the ratio of different amounts. For example, the unit for speed $(\mathrm{km/h})$ is the ratio of distance (km) to time (h), which cannot be added directly. A car moves at $30\mathrm{km/h}$ and then at $20\mathrm{km/h}$ does not mean the car moves at 50km/h.

The identity element for multiplication is one, but the additive identity is zero. Identity for multiplication is the base for multiplicative and proportional reasoning. The inverse element for multiplication is defined by using one.

Comparison

In any mathematical investigation, particularly in the mathematics classroom, problem-solving approaches and comparisons of various ideas, representations and solutions are key activities for discussion and appreciation. This comparison is the nature of the mathematical activity to find better ideas.

When comparing amounts, concrete objects can be compared directly or indirectly without measurement units. As mentioned in the Unit, a **direct comparison** can be used to fix a new unit of measurement. In contrast, **indirect comparison** can promote logic for transitivity, including syllogism.

Comparison of multiple denominated numbers with different unit quantities on the same magnitude, such as $5.2\ \mathrm{m}$ and $5\ \mathrm{m}\ 12\ \mathrm{cm}$, can be done if they are represented by a single denominated number by the unified unit quantity, such as $520\ \mathrm{cm}$ and $512\ \mathrm{cm}$. Furthermore, comparing expressions with the same answers on the same operation, such as $2+4$, $3+3$ and $4+2$, can be used to find rules and patterns. For example, $2+4 = 4+2$ shows a commutative rule for addition, whereas $2+4 = 3+3$ can show a pattern when 1 is added to 2 and subtracted from 4; the sum is still the same.

Comparison of fractions is an activity to find the unit fraction. For comparison of fractions, such as $\frac{1}{2}$ and $\frac{1}{3}$, we have to find the unit fraction $\frac{1}{6}$ which can measure $\frac{1}{2}$ and $\frac{1}{3}$. $\frac{1}{6}$ is the common denominator for $\frac{1}{2}$ and $\frac{1}{3}$. The algorithm to find the unit fraction as the common denominator is called ‘reduction of fraction’.

On numbers, the relationship between two numbers can be equal, greater or less. The number set up to a real number is a total/linear order set; thus, two numbers can be compared to a real number set. However, a complex number as an extension of a real number cannot be compared directly because it has two dimensions.

On the number line as a real number, the size of the number (distance) is defined by the difference, $1=2-1=3-2=4-3=\ldots$ Here, the difference is the subtraction value as a binary operation, which can be seen as the equivalence class. Regarding an equivalence class, the value of operations can be compared. On a plane such as a complex plane, even though the number is not simply ordered, the size of the number (distance) is defined by the Pythagorean theorem. By using this definition, $|1| = |(\frac{\sqrt{2}}{2})(1 + i)| = |i| = |(\frac{\sqrt{2}}{2})(1 - i)| = \ldots$ the theorem produces the distance on the plane, and the distance can be compared.

As explained in the Unit of measurement, the magnitude is given by defining the unit of magnitude of measurement. One of the ways to produce the unit magnitude is a direct comparison, which provides the difference, and the Euclidean Algorithm produces the unit of measurement as the greatest common divisor originating from the difference.

Another way to produce the unit is by using the ratio and multiplication. Such a newly produced magnitude lost linearity. In Physics, ‘dB’ is the size of volume produced by the common logarithm of sound pressure. ‘dB’ fits well with a human's impression of the size of sounds on its linearity. It is known as Weber-Fechner's law that human senses are proportional to the logarithm of the stimulus. In science, a logarithmic scale is used for a semi-log graph and a log-log graph for demonstrating extended linearity, even if it is an exponential phenomenon. Logarithms produce the scale to illustrate multiplicative phenomena as additive phenomena.

Operations

Addition, Subtraction, multiplication and division are four basic arithmetic operations. These binary operations involve any two numbers with symbols of operations $+$, $-$, $\times$ and $\div$. Polynomial expressions such as $2x+3$ are seen as a combination of binary operations. Mental arithmetic may be used in the column method with the base-10 place value system. An operation is not just a rule but can be demonstrated using various representations. For example, an operation can be represented by the manipulation of concrete objects as well as expressions. However, the actual manipulating process is different from the operation process. To show it the same way as the operation, we cut off some parts of the actual process and represented them as diagrams. If students can omit such parts, they can think of it as an operation.

From Key Stage 3 up to the field theory at the university level, arithmetic operations are only expressed as addition and multiplication. The negative vector is represented using the minus symbol on the axiom of vector space.

Although arithmetic operations are well known, limited operations exist for each specific set of numbers, such as modulus. Depending on the context, it is called a function or algorithm.

Algorithm

An algorithm is a set of sequential activities in a unique situation to produce a particular solution to a set of tasks. The column method is based on the base-10 place value system. When using this method, 200 + 300 is done by just 2 + 3 on the hundreds place value. This algorithm is adapted for mental calculation. Representation of the column method is not universal like an expression as algebraic representation. The algorithm for the column method is fixed as a formal form based on every culture. However, it can be created by manipulating the base ten blocks. On the other hand, an algebraic form of operation such as 2+3=5 is the universal form.

A formula also functions like an algorithm. It can be applied without understanding its meaning. However, it can only be created with understanding and recognising its underlying structure or meaning. If we know the structure or meaning, such as ratio and proportionality, memorising it is unnecessary.

Fundamental Principles

Fundamental principles are rules related to mathematical structures and forms in general. **Commutativity, Associativity and Distributivity** are three fundamental principles for arithmetic operations. Commutativity does not work on subtraction and division. On the discussion of Distributivity, if a, b, and c are positive numbers, then the expressions $a(b+c)$, $(b+c)a$, $a(b-c)$, and $(b-c)a$ are different. However, if ***a***, ***b***, and ***c*** are positive and negative numbers, the four expressions can be seen as the same. Before the appearance of Algebraic Axiomatic Systems such as Group, Ring and Field, these principles existed as the principles.

There are also other fundamental principles for arithmetic operations to simplify the operations at the elementary level, such as the following:

$1 + 9 = 10$	$2 \times 3 = 6$	$8.1 \div 9 = 0.9$	$8.1 \div 9 = 0.9$
$\downarrow{+1}\quad\downarrow{-1}$	$\downarrow{\times 10}\qquad\downarrow{\times 10}$	$\downarrow{\times 10}\qquad\uparrow{\div 10}$	$\downarrow{\times 10}\quad\downarrow{\div 10}$
$2 + 8 = 10$	$20 \times 3 = 60$	$81 \div 9 = 9$	$8.1 \div 90 = 0.09$

Most of them are related to proportionality used by the Ancients.

Principles can be identified through a comparison of equations. They are necessary for explaining algorithms and thinking about how to calculate by using models and other representations. On the extension of numbers and operations, principles are used to discuss the permanence of form (see the permanence of form).

In geometry, the extendable nature of a line changes its functions in the curriculum. For example, the shape is extended to the figure; the edge, which may include the inner part of a shape, is extended to the side, which may not include the inner part of a figure. Then, the side is extended to a line, which enables the discussion of the possibility of escribed circles. In addition, parallel lines are necessary to derive the area formula for triangles with various heights.

Permanence of Form

The Principle of the Permanence of the Equivalence of Form, Hankel's Principle, is known as Commutativity, Associativity, and Distributivity for algebra for the field theory. It is a fundamental principle.

On the other hand, the permanence of form appeared in the history of mathematics in the 16th century and functioned to shift from arithmetic algebra to symbolic algebra. Peacock's Permanence of Form is not only the limited three rules like Hankel's but also applies to any algebraic symbolic form.

In Education, the form is not limited to fundamental principles but includes patterns, and the permanence of form can be used in various situations. It is especially used for the extension of numbers and operations from elementary level to secondary level education like the following:

$$
\begin{array}{ccccccccccc}
(+3) & + & (+2) & = & +5 & \qquad & (+3) & - & (+2) & = & (+1) \\
 & & \downarrow -1 & & \downarrow -1 & & & & \downarrow -1 & & \downarrow +1 \\
(+3) & + & (+1) & = & +4 & & (+3) & - & (+1) & = & (+2) \\
 & & \downarrow -1 & & \downarrow -1 & & & & \downarrow -1 & & \downarrow +1 \\
(+3) & + & 0 & = & +3 & & (+3) & - & 0 & = & (+3) \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
(+3) & + & (-1) & = & ? & & (+3) & - & (-1) & = & ?
\end{array}
$$

The '?' are unknown and have not yet been learned, but other parts are known. However, people could imagine the '?' by analogical reasoning with the idea of the permanence of the patterns. Here, the permanence of patterns is used as a hypothesis, making it possible to apply it to unknown cases. And it provides the necessary explanation for unknown '$-(-1)$' as known '$+(+1)$'.

The permanence of form appears at the initiation of numbers in Key Stage 1 and can be seen at all other key stages. For example, it explains the necessity of introducing zero (0), which does not have any necessity to represent it to count the existing objects.

Various Representations and Translations

Every specified representation provides some meanings based on its essential nature of representation, which can be produced by specified symbols and operations. Different representations have different natures and use different symbols and operations. Every representation has the limitation of interpretation on its nature. Thus, thinking by using only one specified representation provides the limitation of reasoning and understanding. If one type of representation is translated into another type of representation, then the representation can be interpreted in other ways. If the idea of specified representation with a certain embedded nature is translated into various representations, a rich and comprehensive meaning and use will be produced. However, to make the translation meaningful, it is necessary to know the way of translations, which consist of correspondences between symbols and operations on different representations.

For example, proportional number lines only function for teachers and students who know well how to represent the proportionality on the tape diagrams and number lines, and so on. If they know what it is, they can use it to explain and produce an expression.

Comprehensive learning of mathematics using various representations and translations is necessary; however, students have to learn how to represent it in other representations and translate, at first, between expressions and situations.

According to Isoda (2016, 2018), representation in mathematics is specified by context/objective, symbols, and operations of symbols, even if they are figural representations. A proportional number line is the symbol. Operations are multiple under proportionality. The objective is to find the answer to multiplication and division. The proportional number line just looks like teachers' explanation tool if they do not teach it as a way of representation. However, if teachers teach it as a way of representation to students, they can use it as a tool for thinking by and for themselves. For teaching proportional number lines, teachers must teach how to scale the lines using the idea of multiple because it is the way of operation for them.

Pattern, Recursion and Invariant

Mathematics is the science of patterns. In other words, the science of finding invariants. A pattern means the existence of an invariant, something with no change in repetition. In the case of numbers, they are usually related to natural number sequences and something constant. For example, sequence patterns such as the arithmetic sequence have the same difference, and the geometric sequence has the same ratio. In a table, if the first difference is constant, it is an arithmetic sequence, but if the second sequence is constant, it is a quadratic sequence. The table also shows some functional relationships between x and y. The pattern is usually found on the table when given pairs of numbers are lined in an appropriate order. Lining ordered pairs under natural numbers appropriately is necessary to find the pattern.

Recursion on natural number sequence is usually represented by the recurrence formula. In mathematics, it is used for complete induction. In the programming, recursion means just repetition in a limited set on the natural number for the recurrence formula, and it focuses on the part of complete induction; if $p(k)$ is true, then $p(k+1)$ is true. Instead of algebraic form, such recursive process on the equations is usually represented by the repetition of figural diagrams (representation), which use the previous drawing at the next step in a consistent manner.

In the case of a figure, a pattern is also found on the tessellation of the congruence triangle. It is an appropriate repletion of the same triangle. If we tessellate it, we can find the invariant properties of angles relating to parallel lines. On the tessellated design as a whole, we can also find translation, rotation and symmetry. We can also find the enlarged figure, which shows the similarity of the figures. Mandelbrot set on infinite geometry is the recursion of figures with the natural number sequence. In the programming, finite cases can be identified on the screen.

An invariant is a stronger word in mathematics if we compare it with the usage pattern, which means repetition in informatics, such as programming because it is necessary to be proved in the mathematics system. Dynamic Geometry Software (DGS) provides ways to find something invariant which should be proved. It is normal, nothing strange, that two lines meet at one point if it is not parallel. However, it is exceptional and strange if three lines meet at one point because three lines usually produce a triangle. Thus, constructing the circumcenter, incenter and centroid on any triangle is amazing and should be proven.

Graphing Tools (GT) for functions are the tools that represent invariant properties in Algebra and Analysis. On $f(x)=ax2+bx+c$, if it is drawn by GT with parameters a, b, c, and fix b and c as any constant and changes a real number, the graph of f(x) produces the family of functions on the screen. Any graph of the $f(x)$ family intersects with the y-axis on $(0, c)$, and $y=bx+c$ is the tangent line for any $f(x)$. We should prove these findings as invariant in mathematics based on the chosen mathematical system. Various proofs are possible if we change the theory in mathematics.

Ordering

Ordering is a way of comparison by using ordered numbers such as Natural Numbers, which have total/linear order, and ordering activities are bases for finding patterns and invariants. Natural Numbers are produced by Peano's Axiom, which includes the first number (1), the next number $(+1)$ and mathematical induction. The number sequence, such as $2, 5, 8, 11, 14, \ldots, \mathrm{n}{+}3, \ldots$ is already ordered by the first number (2), the next number $(+3)$, and the mathematical induction $a_{k+1} = a_k + 3$. It might be difficult to find the pattern if it is ill-ordered, such as $11, 8, 2, 14,$ and 5. If the order is given as $2, 5, 8, 11, 14,$ we can find the difference of 3 as an invariant and guess the next number as 17.

Ordering is the activity of arranging objects from an ill-ordered situation to a well-ordered situation. When a table is used to find the property of a function, the top row of the table is usually given by integers. Otherwise, we could not find any pattern. Thus, to teach the necessity of order, we should begin from an ill-ordered situation to find the appropriate order to show patterns and invariants.

Symmetry

Symmetry is used for reasoning on patterns. It is known in geometric reasoning by point symmetry and line symmetry. Mirror images in 3D usually provide symmetry between left and right but do not change the top and bottom because the mirror plane is fixed on our standing plane to our viewpoint.

Symmetry is also used for algebraic reasoning. For example, in the symmetrical equation $(x + y)^2 = 1,$ and it means $x^2 + 2xy + y^2 - 1 = 0,$ then it is $x^2 + 2xy + (y + 1)(y - 1) = 0$. Because of the symmetry, $y^2 + 2yx + (x + 1)(x - 1) = 0$ is also true. In an algebraic explanation for truth, we consider that the given equation means $x + y = \pm 1$ and substitute $y = -x \pm 1$ to the first result. However, it is unnecessary to do these operations because the given equation is symmetrical.

Maxima and Minima

When we have various possible solutions to real-world problems, special cases such as maxima and minima are usually focused. It is valuable depending on the issues of the real world. In computer programming, the algorithm to produce minimum steps which produce the answer in the shortest time has a significance for preferable. Some companies make their decisions to maximise profit with minimum efforts, such as cost cuts, even though it will reduce employment. For making decisions mathematically, we usually produce/apply functions (mathematical models) to represent the order and find maximum or minimum solutions. For example, to decide where we should open a new restaurant, the population density, defined by the function $(\mathrm{population})/(km^2)$ is used as an indicator to select the area. Costs for renting the place, employment and raw food materials are also considered. However, the humanity of the workforce is a necessary variable to success.

(ii) Mathematical Ways of Thinking

Mathematical ways of thinking support the students' thinking process by and for themselves when they appreciate their values. The following are well-known samples in Mathematics.

Generalisation and Specialisation

Generalisation is to consider the general under the given conditions of situations. Any task in mathematics textbooks is usually explained by using some examples, known as special cases; however, it is usually preferred to discuss the general ideas. Given that conditions are usually unclear in textbooks, the conditions become clear if students are asked to consider other cases, such as by saying 'for example'. In the upper grades, variable, domain, range, parameter, discreet, continuous, zero, finite and infinite become the terms of a set to consider the conditions for general.

Specialisation is to consider the thinkable example of situations and necessary to find the hidden conditions. Considering general with thinkable example is called generalisable-special example. In mathematics, the general theory is usually stronger than the local theory. It is one of the objectives for generalisation and specialisation.

In mathematical inquiry, the process usually goes from special to general cases to establish a stronger theory. Thus, the task sequence of every unit in textbooks usually progressed from special cases to general for generating simple procedures, exceptionally the tasks for exercise and training procedures, which usually progress from general to special because students already learned the general.

For students engaging in generalisation and specialisation by and for themselves, students have to produce examples. Thus, developing students who say ‘for example’ by and for themselves is a minimum requirement for teaching mathematics.

Extension and Integration

The extension means extending the structure beyond the known set. The product of multiplication is usually increased if both factors are natural numbers. However, if we extend it into fractions and decimals, there are cases where the products decrease.

On the extension of structure beyond set, learned knowledge produces the misconception explained by the over-generalisation of learned knowledge. In the mathematics curriculum, we cannot initiate numbers from fractions and decimals instead of natural numbers. Thus, producing misconceptions is inevitable in the mathematics curriculum and its learning. Even though it is a source of difficulty for students to learn mathematics, it is the most necessary opportunity to think mathematically and to justify the permanent ideas to be extended. After experiencing the extension with misconceptions, it is a moment of integration if students learn what ideas can be extended.

Extension and Integration is a mathematical process for mathematisation to reorganise mathematics. It implies that the school mathematics curriculum is a kind of net which connects the local theory of mathematics as a knot even though their strings/paths include various inconsistencies with contradictions. It is a long-term principle for task sequence to establish relearning on the spiral curriculum.

To develop a way of thinking about extension and integration, teachers need to make efforts to clarify the repetition of both the same patterns for extensions and the reflections after every extension in the classroom. For example, there are repetitions for the extensions of numbers from Key Stage 1 to 3. On every extension of numbers, there are discussions of the existence of numbers with quantity, the comparison with equality, greater and lesser, and constructing operations of numbers with the permanence of form. By reflecting on every repetition, students can learn what should be done on the extension of numbers.

The permanence of Form also extends known ideas for overcoming inconsistency, such as misconceptions as an over-generalisation.

Inductive, Analogical and Deductive Reasoning

The three reasoning are general ways of reasoning, as for the components of logical reasoning in any subject in our life. However, mathematics is the best subject in schools to teach them.

Inductive reasoning is the reasoning that generalises a limited number of cases to the whole set of situations. It is usually discussed in natural number sequential situations. Mathematical/complete induction is deductive reasoning as the form of proof.

Three considerable cases or more will be the minimum to be considered inductive reasoning. To promote inductive reasoning, teachers usually provide a table for finding the patterns; however, it is not the way to develop inductive reasoning. To promote inductive reasoning, students must consider various possible parameters in a situation and choose two or more parameters by fixing other parameters. Then, consider the relationship in the situation to know the cause and effect, and subsequently, set the ways to check the cases by well-ordering the cause and get the effect, followed by recording the data in a table. Teachers usually provide the table first to find the patterns that promote inductive reasoning, but it is not the way to develop inductive reasoning. It is the concern of students to develop inductive reasoning by learning the ordering of natural numbers because it is the cause of effect and beautifulness of patterns that originated from the order of natural numbers. A table is a tool for finding patterns but students cannot produce their inductive reasoning by and for themselves if it is given by teachers. Students have to learn how to order to find the pattern inductively.

Analogical reasoning is the reasoning to apply known ideas to the unknown set or situations when recognising similarities with the known set or situations. It is the most popular reasoning in our life. Most of the reasoning to find ways of solution for unknown-problem solving by using what we already know is analogical reasoning. Depending on the case, it is called abduction. Analogical reasoning is to recognise similarities between the unknown problem and the known problems.
Even though the rule of translations between different representations is not well established, analogical reasoning may function as a metaphor for understanding. Many teachers explain operations by using diagrams. It appears meaningful to provide a hint for solving. Still, most students cannot use the hint by and for themselves because they do not recognise the similarity in their analogy. To develop analogical reasoning, the most necessary way is to develop the habit of using what students learned before, by and for themselves. Providing assisting tasks before posing the unknown problem is also used as a strategy to find similarities.

Deductive reasoning is the reasoning to establish systems with components of already approved notions and given by using 'if… then' and logic for propositions such as the transitivity rule. 'If not' also functions for proof by contradiction as well as counter-examples. In cases where the rules of translations are well established, the translations of various representations still function under their limitations too. Various methods for proving such as a complete induction are also done by deductive reasoning.

Inductive and analogical reasoning are necessary to find ways of explaining and proving. Analytical reasoning, thinking backwards from the conclusion to the given, is also used, but it does not allow writing as a part of the formal proof done by deductive reasoning. Arithmetic and algebraic operations can be seen as automatised deductive reasoning. Most students do it just by recognising the structure of expression intuitively without explaining why. To clarify the reason, teachers need to ask why.

Knowing the objectives of reasoning is necessary to develop the three reasonings. Inductive reasoning is applied to find general hypotheses. Analogical reasoning is applied to see the unknown as known for problem solving. Deductive reasoning is applied to explain or prove the local system in general.

Abstracting, Concretising and Embodiment

Abstracting and **concretising** are changing perspectives relatively by changing representations such as expressions. Abstracting is usually done to make a structure clearer. Concretising is usually done to make ideas meaningful by concrete objects. For numerical expressions, manipulatives and diagrams function as concrete. For algebraic expressions, numerical expressions function as concrete objects. For both examples, abstract and concrete representations do not correspond one-to-one in translation because concrete representations usually have some limitations as models. However, concrete representations function as metaphors for abstract ideas.

Embodiment functions in both abstracting and concretising. When abstract ideas can be concretised, it implies those abstract ideas are embedded with some specified concrete ideas. When concrete ideas can be abstracted, it implies that some ideas in concrete are embedded into abstract ideas, but other ideas in concrete do not represent the abstract ideas. Both embodiments function for understanding ideas as metaphors, but their translations are limited only to corresponding contents. Thus, the embodiment of abstract ideas changes how to see concrete objects before and after the embodiment.

Objectifying by Representation and Symbolising

A mathematical representation can be characterised by its symbols and operations with specified purpose and context. In the process of mathematisation, lower-level operational matters are usually objectified as for new symbolising and its operations, likely arithmetic algebra such as simplify $a \times x = ax$.

Until Key Stage 2, (whole) numbers do not mean positive and negative numbers. The number in red on the financial matter is large if the number is 'large'. The number in black on the financial matter is large if the number is 'large'. Here, the meaning of 'large' is defined as the opposite of the number rays; thus, it cannot easily be compared to the numbers in red and black.

At Key Stage 3, as for integration, we have to alternate new symbols and operations. We represent the red number by the negative symbol ‘-’ and the black number by the positive symbol ‘+’ and integrate the one direction for comparison into a one-dimensional number line. Here, in Key Stage 3, ‘larger’ for comparison (an operational matter on number rays) on the lower level becomes the object of a higher level to produce the comparison (an operational matter on a number line) for new number symbols with positive and negative as a directed number. Later, by the algebraic sum, positive ‘+’, addition ‘+’, and negative ‘-‘ and subtraction ‘-‘ symbols are integrated as ‘plus’ and ‘minus’.

It is the process of mathematisation that objectifies the operational matter to establish new symbols and operations. This process of abstracting from concrete can be seen as the process of mathematisation.

Relational and Functional Thinking

Relational and functional thinking are ways of thinking that can be represented by relation and function if we need to describe them by mathematical notation. It was used as significant terminology to explain mathematical thinking in the Klein Movement 100 years ago. Relation in pure mathematics is a fundamental axiom. Relating to functional thinking, it is known as the ordered pairs between two sets. Mapping is a relationship, and ordering is also a relationship if both support a structure. An equivalence class is given as a relationship in which the axiom is defined by reflective $(x = x)$, symmetry $(x = y \rightarrow y = x)$ and transitive $(x = y,\ y = z \rightarrow x = z)$ properties, or $(x >= y)$ is given as a relation in which the axiom is defined by reflective $(x >= x)$, antisymmetric $(x >= y,\ y >= x \rightarrow x = y)$, and transitive $(\ x >= y,\ y >= z \rightarrow x >= z\ )$ properties.

A function is a binary relation between two sets that associates every element of the first set to exactly one element of the second set. In education, the second set usually consists of numbers. Relational thinking can be represented by various representations, such as graphs. It is an activity for students who do not know such representations, and teachers have to teach such representations on the necessity of students to engage in their activity. Students are welcome to create their necessary informal representations, such as graphs and diagrams, by themselves.

Historically, Hamley described the characteristics of functional thinking (1934). He defined functional thinking as having four components: Class, Order, Variable and Correspondence. Class is a set that can include equivalence in operations. For example, 5 can be seen as the value of $1+4$, $2+3$, $3+2$, $4+1$. Order is discussed in a set and between sets. Variables are domain and range, which are different from the original set and the destination set. Correspondence is discussed between domain and range. When the teacher provides the table, a variable and between variables are already ordered. To find the functional relationship with a pattern in a situation with various variables, we can change the selected variable in order and other variables fixed as constant and try to find the influence of selected variables. To find influential variables and to remove no influential variable, it has to continue to find functional relationships.

Functional thinking is useful to predict and control a situation. At the elementary level, proportionality and operations are used for functional thinking on this objective. Tables and graphs are helpful for knowing the changing properties of each function. Teachers usually provide tables and graph papers at the beginning of the experiment to find patterns and properties of the function. However, this is not the way to develop functional thinking because teachers take over the opportunities from students to consider and fix the sets, orders, variables, correspondences and so on by themselves.

The rate of change is used to check if a function has linearity. In this case, if it is constant, it is a linear function. If not, otherwise. The limit of the rate of change for finding the tangent is the definition of differentiation. Correlation in statistics is a relation but does not necessarily function the relation as a causal relation.

Thinking Forward and Backwards

Thinking forward and backwards are the terminologies of Polya. In Pappus of Alexandria (4^{th} Century A. D.), **thinking forward corresponds to synthesis** and **thinking backward corresponds to analysis**. Synthesis is the deductive reasoning (proving) from the given and known. In contrast, analysis is the reasoning from the conclusion to find the possible ways of reasoning from the given and known.

In Ancient Greece, analysis was a method of heuristics, which was the way to find adjoining lines on construction problems, and valance for area and volume problems. It hypothetically uses a conclusion to find the solutions. Ancient Egyptians set a tentative number for a solution and got a tentative answer; they compared it with the necessary answer and then adjusted the tentative number to produce an appropriate solution. In Descartes's era, they used an unknown x for an algebraic problem instead of a tentative number. Leibniz used an unknown limit x of the function for calculus. This hypothetic-heuristic reasoning begins from the conclusion, such as if the construction is achieved if the valance is kept, if the unknown x is given and if the limit of x exists. Since the analysis began with a hypothesis, without proving the given, people used to believe that analysis produced tautology, which is not allowed to be written in the system. In Basian probability, we assume unknown probability as $\mathrm{p(x)}$ and begin the reasoning. It is an analysis of Statistics and Probability.

On the other hand, the modern mathematics system itself begins from the axiom as a presupposition. Thus, if analytic reasoning becomes a part of presupposition, it is allowed in a written form. For this reason, the unknown x can be written in Algebra, and the limit of x can be written in Calculus. However, until such reformations of mathematics, analysis, and a method of heuristics, it is not allowed to be written as a part of a theory. This is why some mathematics textbooks look very difficult to understand; they have to be written in the form of deductive reasoning for constructing the system from an axiom that does not include heuristics and ways of finding. As a consequence, there are old-fashioned textbooks that are just oriented to exercise the procedures as rules without explaining why.

Current school textbooks are oriented to writing the problem-solving process with various solutions and misconceptions using what is already learned, including heuristics such as thinking backwards and so on. In Key Stages 1 and 2 standards, addition and subtraction are inverse operations, and multiplication and division are inverse operations to verify answers on operations. Such ideas are reformulated at the algebra in Key Stage 3.

The case where 'If your saying (conclusion) is true, it produces contradiction which we already knew' is known as a dialectic in communication. It was formalised as the proof by contradiction in mathematics. It is also a way of analysis for thinking backwards. In mathematical communication, thinking backwards is a part. Without the preparation of a lesson plan, which includes thinking backwards, teachers cannot realise the classroom communication, including misconceptions, because the counter-example is the component for the proof by contradiction. Through the communication of objectives and ways of reasoning, such as thinking backwards by using what students already learned, we can develop students' mathematical thinking.

(iii) Mathematical Activities

Mathematical Activities usually explain the teaching and learning process and embed mathematical ideas and thinking. As for the style of teaching approach, they are usually enhanced. However, the style of teaching itself is not made clear in them.

Problem Solving

Pure mathematicians inquire about problems that have never been solved yet, and they develop new theorems to solve their problems. It produces a part of the system. Such authentic activity is the model of problem solving in education because it usually embeds rich ideas, ways of thinking and values in mathematics. As mathematicians usually pose problems for themselves, the activity includes problem posing and reflection, which is necessary for establishing new theories.

In education, there are two major approaches for embedding them in learning.

The first is setting the opportunity to implement problem solving such as the unit or project. Here, solving the problem itself is an objective for students. It focuses on heuristics: it is usually observed unexpectedly, and planning it is inevitably not easy. Due to this difficulty, the problem-solving tasks are usually provided in two types. The first type focuses on mathematical modelling from the real world. The second type focuses on open-ended tasks for students because it provides the opportunity for various solutions.

The second one, the problem-solving approach, tries teaching content through problem solving in classes. The content here includes mathematical ideas and so on. This case is only possible if teachers prepare a task sequence that enables students to challenge the unknown task by using what students have already learned (Zone of Proximal Development, ZPD). In the problem-solving approach, the tasks given by teachers are planned so that students can learn the content, mathematical ideas and ways of thinking. For teachers, solving the tasks themselves is not the objective of their classes, but students reveal the objectives of teaching by teachers by recognising problems as problematic of an unknown and finding solutions. From students' perspective, by solving problems which appeared from the task as problematic, they can learn by and for themselves. Even if the teacher gives a task, the teacher has to focus on the problem which originated from the given task. If students feel the problematic, they can begin to think for themselves. If not, they must wait for teachers' explanations on solving the given task because they do not recognise it as being solved by and for themselves.

For the problem-solving approach, it is necessary to plan the class in preparation for future learning and use the learned knowledge. Textbooks such as Japanese textbooks are equipped with task sequences for this purpose. In such textbooks, heuristics is not accidental but purposeful because every task in the textbook will be solved using the already learned representation. It is called a 'guided discovery' under ZPD because it expects well-learned students on the learning trajectory and never expects genius students to produce unknown ideas. Thus, in Japan, the most necessary problem-solving approach to using well-configured textbooks is to establish the custom of using what is already learned and acquiring the necessary representations to represent ideas in future learning.

For observers, the method of teaching using the problem-solving approach in a class cannot be distinguished from the open-ended approach. The open approach is not necessary to prepare the task sequence because it is characterised by an independent open-ended task. If teachers set the open-ended task independently, it is the first approach. If teachers set the task sequence of open-ended tasks for learning mathematics, it is the second problem-solving approach. Both approaches have been known as Japanese innovation in textbooks since 1934 for the elementary level and since 1943 for the secondary level. Japanese textbooks until the middle school level currently equipped the task sequence for problem-solving approaches.

Even though problem solving in education resembles the activities of authentic mathematicians, it is not the same because the problems of mathematicians are usually unsolvable beyond decades. At the same time, tasks in the classroom can be explained by teachers who pose the problems. When teachers refer to problem solving in education, it includes various objectives such as developing mathematical ideas, thinking, values and attitudes. These terminologies are used in education to develop students who learn mathematics by and for themselves, while mathematicians only use some of them. In teacher education, if teachers only learned the mathematics content, they may lack the opportunity to learn the necessary terminology. If teachers do not know it, the higher order thinking becomes a black box which cannot be explained.

Exploration and Inquiry

Exploration has been enhanced in finding hypotheses by using technology such as Dynamic Geometry Software and Graphing Software. The software provides an environment for students to explore easily. Exploration of the environment produces a hypothesis.

The inquiry includes exploration as a part but orients to the justification and proving through reflections.

Students' questioning enhances both exploration and inquiry. Thus, the process and finding will depend on students' questioning sequence, not likely on problem solving by the task, which teachers can design before the class.

Mathematical Modelling, Mathematisation and Systematisation

Mathematical modelling is a necessary way to solve real-world problems. A mathematical model is hypothetically set by using mathematics to represent the situation of the problem. Mathematical answers based on the model are confirmed by interpretation. Modelling enhanced various possibilities for applying multiple representations in mathematics.

In education, modelling has been enhanced for problem solving after the new math movement, which recognised school mathematics with set and structure and enhanced numerical solutions by using computers, which became current mathematical science tools. In this era of Artificial Intelligence (AI) and Big Data, computational modelling is done through programming with algorithms, and it is also a part of mathematics.

Before mathematical modelling, mathematics functioned as a metaphor for exact science, theory and language for nature. In Ancient Greece, music and astronomy were exactly theorised under geometric representations. Today, mathematics has various theories with algebraic representations and numerical solutions, which need to consider the limited round number for the possible number of digits. Thus, like computer simulation, modelling means a hypothetical approach to the real world using universal mathematical language instead of exact science.

Mathematisation has two usages: The first is for science and engineering in establishing mathematical models and producing new mathematics theories based on the model. In physics, mathematical problem solving of nature has produced various theorisations in mathematics for solving problems in general. The second usage is used in education as the mathematics curriculum sequence principle, which enhances the reorganisation of mathematical experiences. Freudenthal (1973) explained that the means for organising at a lower level becomes the subject matter for reorganising, which is done by the new means of reorganising. It includes the process of extension and integration based on the prior learned knowledge. The second usage of mathematisation includes the establishment of local theory after the integration.

Systematisation means the establishment of a local theory. In pure mathematics systems,
if the conclusion of the proposition is proven as correct, all propositions, definitions and axioms used previously can be seen as necessary conditions for the conclusion. However, in school mathematics, there are so many hidden conditions which support the conclusion and the propositions are usually demonstrated locally. From the perspective of pure mathematics systems, reorganisation of local theory can be said in the process of systematisation.

Programming

Programming activity is the activity to represent the procedure for computer by programming language. Programming language is a kind of mathematical language for using computers. In programming, a situation is analysed and divided into parts for finding the convenient solution by using a computer. Then, excerpting the parts to be interpreted and its process is represented by a programming language to make it operational with a computer. When we compared programming language with other mathematical language that produce mathematical system, it has physical limitations such as the limitation of memory and limitation of time in the computer system. However, if we ignore such limitations, it is the mathematical language. Thus, programming activity can be seen as a kind of mathematical activity.

Conjecturing, Justifying and Proving

Conjecturing, justifying and proving have been used in the context of proof and refutation. In education, students conjecture hypotheses with reasoning on exemplars. Conjecture is conceived through generalisation and justifying with appropriate conditions. Proving includes not only the formal proof in a local system but also the various ways of explanations. A counterexample is a way of refutation. A counterexample is meaningful because it sets off the reasoning from which the conclusion is true.

In the dialectic discussion on the proof and refutation, a counterexample is produced by saying, "'If your saying is true 'or 'if your conclusion is true,' 'what will happen?'" This manner of discussion enhances hypothetical reasoning by getting others' perspectives, which promotes mutual understanding.

Conceptualisation and Proceduralisation

Conceptual knowledge is the knowledge to explain the meaning and is used for conscious reasoning, while procedural knowledge is the skillful knowledge used for unconscious-automatised reasoning. A unit of mathematics textbook usually begins with the initiation of new conceptual knowledge by using learned procedural-conceptual knowledge called conceptualisation. After the initiations, new conceptual knowledge formulates the new procedural knowledge for convenience, and the exercises produce proficiency in proceduralisation.

In mathematics, the proposition format ‘if…, then…’ is the basic format to represent knowledge; however, in school mathematics, it is impossible to make clear the proposition from the beginning because the ‘if’ part can be clarified later. For example, ‘number’ changes the meaning several times in the school curriculum. In multiplication, products become large if it is […] number. Until numbers are extended to decimal, […] part cannot be learned. On this problem, it is normal that students encounter difficulty in their learning and are challenged to produce appropriate knowledge if they are provided a chance to over-generalise knowledge for knowing […] part.

For this reason, to produce exact knowledge in mathematics, conceptualisation and proceduralisation are a journey that continues recursively in mathematics learning to change the view of mathematics. It is the opportunity to learn necessary mathematical ideas and ways of thinking and to develop value and attitude. The process of mathematisation can also be seen from the perspectives of conceptualisation and proceduralisation in the context of numbers and algebra.

From conceptual and procedural knowledge perspectives, conceptual change/progress is the nature of the mathematics curriculum. Even though you can suppose a concept like Plato’s idea, you cannot configure the curriculum to acquire the concept directly in school because it continuously progresses in the mathematics curriculum. Even if we target it with pure mathematics, using the pure mathematics textbook for school students might be challenging. For example, many primary school teachers write lesson plans to develop the number concept even though they cannot explain what the number concept is. Such a discussion produces illusions about what they are teaching. The possible way for the school curriculum is to describe clearly how the conceptual and procedural knowledge progress in each grade with tentative meanings and procedures.

Representations and Sharing

A mathematical representation comprises symbols, operations, and objectives (context) (Isoda, 2018). Solving algebra equations can be seen as specific ordered elements of every equation that has the same answer. The order of equations shows the context of the process of solving. The property of equality can explain the operation of equations between one equation to another equation. Each equation is a symbolic sentence.

If the operations of representation are missing, it is not a mathematical representation even though it has some artistic images, such as diagrams. When students draw diagrams, it is sharable if the rule of the drawing (operation) is shared. In a classroom, students produce their images in diagrams. It is helpful to encourage their explanation by themselves, but every answer is independent until knowing the hidden ideas, as in the comparison. Other students cannot re-present it until the ways of drawing (operation) are shared. Thus, comparing various representations is part of the process of recognising ideas for producing symbols and operations, and translations as in mathematical representations.

In mathematics, different representations use different symbols and operations. If there is a rule for correspondence between symbols and operations on different representations, it can be translated and produce rich meanings. It is the way to produce a mathematics system.

In mathematics, a representation system can be defined universally, which is the product of convention by mathematicians. For teachers, it looks far for students’ activity in the classroom. However, it is the opportunity for students to reinvent the representation and its system by considering the why and how. For example, producing a metre as the measurement quantity includes such activities: There are historical episodes on why and how ‘m’ was defined by Condorcet and others in the middle of the French Revolution. In Engineering, Informatics and Science, applied mathematicians usually try to produce new measurements based on the necessity of research to conceptualise the idea mathematically and operationally. Setting the measurable quantity is a part of mathematical modelling for real-world problem solving because if it is measurable, we can apply known mathematics.

From the perspective of Radical Constructivism (Glasersfeld, 1995), another function of representation is ‘re-present’ for understanding others and sharing. To understand what others are saying, we must re-present other's ideas in our minds. Mathematics is a possible subject for constructing others' ideas in each of our notebooks with reasoning from everyone. If not, students say that it is difficult to understand. In this meaning, re-present in one's mind what others are saying is the most necessary activity to share ideas in the mathematics classroom. The mathematical community becomes as enjoyable as long as everyone tries to re-present what others say. Thinking hypothetically, like 'if your saying is true', is also necessary to get others' perspectives.

Mathematical Values and Attitudes

(i) Mathematical Values Seeking for:

Values indicate the direction that we seek. Thus, they set the direction of thinking. In mathematical values, generalisable and expandable ideas are usually recognised as strong ideas. Proving is necessary in mathematics to seek reasonableness. Harmony and beautifulness are described not only as relating to mathematical arts but also in the science of patterns and the basic structure of the system of mathematics. Usefulness and simplicity are used in the selection of mathematical ideas and procedures. Here, each term does not mean an independent category but is related to a feeling, such as 'Simple is beautiful in Mathematics.' Values can be learned through reflection and appreciation. The moment to appreciate the values of ideas and the ways of thinking can be set when the various ideas and ways of thinking of the past and the present are compared. When students are making comparisons, various ideas and ways of thinking should be shown on the board. Teachers verbalise these values when students say 'Aha' and so on. To make clear the verbalisation, the uncomfortable situation, such as feeling problematic, is also meaningful because it provides the object for comparison on the later situation.

Generality and Expandability

School mathematics seeks to establish general theory as well as university mathematics. For example, in the case of multiplication, the following learning process proceeds to seek for general operation with extension. At the initiation stage of multiplication, if students acquire the multiplication table from Row 1 to Row 9, they are released from accumulation (repeated addition) and begin to learn multiplication as the binary operation. However, without the extension of numbers for multiplication, such as in Row 10, Row 11, and so on, students still have to do multiplication as a repeated addition. Should they have to memorise all of them? It is impossible! If they acquire column multiplication, they can multiply any digits with their acquired knowledge of the multiplication table. The column multiplication can extend to decimals if they can manage the position of the decimal point on the base-ten place value notation. In these extensions and generalisation of multiplication, the accumulation form functioned to learn multiplication as a repeated addition at the initiation stage. However, in the next stage, the column form under the place value system functioned to alternate the accumulation to the column multiplication as a binary operation. These are the processes for seeking generality and expandability in the case of multiplication.

In learning the value of seeking generality and expandability, teachers must begin from the un-general and before-expansion cases, even though teachers know the general and the final extended forms. As long as teachers try to teach their general knowledge, students will never get the opportunity to learn these values. A good teacher can design the process of generalisation and expansion and seeks to teach the values through reflection and appreciation. If students appreciate, they can learn the values and seek them by and for themselves.

Reasonableness and Harmony

Mathematics is reasonable. Good teachers usually ask why for developing students who will explain the reason by and for themselves by using what students have already learned. Here, a logical-reasonable explanation for using learned knowledge is not always deductive but more analogical and inductive and is related to numbers at the primary level. Significance and objective are also considered as the reason why. In teaching reasonableness, students may have the intuition to feel strange or good first and explain it by using the learned knowledge or something sharable. An illogical and unreasonable situation is necessary to create the opportunity to learn the reasons itself. This means good teachers know what is unreasonable for students by comparing what they already learned and plan the process to recognise the unreasonableness and reach appropriate reasons by using what they have already learned. In the problem-solving approach, the unknown problem itself is set by such a teacher to provide students with the feeling of unknown or unreasonableness. However, in the given appropriate task sequencing designed by good teachers, students can apply what they have learned to the unknown problem. Thus, on a well-designed task sequence, teachers can ask students why.

Mathematics is a harmonious/harmonised subject: how do you explain it? In Ancient Greece, harmony is a name for the subject of mathematics, as represented by the Pythagorean music scale developed by the ratio 1 to 2, 2 to 3 and 3 to 4. Buildings were also constructed using the special ratio in Ancient Greece. Some ratios were used to represent beautifulness (Eros) in the Era.

In the case of music, we have to consider the overtone for the development of scale and code. The height of scale (sound) is developed by a multiple; however, dividing real strings (code) itself for the development of the musical instruments themselves is division as the multiple of reciprocal. It was the origin of music as a subject of education in Europe. In current mathematics, harmony has limited use, such as the harmonic series for the extension of overtone. On another story, the current music scale itself is defined as equal temperaments, which are the products of multiple $2^{\frac{1}{12}}$ to produce a harmonised code in orchestration is not the ratio on the Pythagorean scale because it produces a unique growl in a special case.

Here, as for mathematics education, harmony as a seeking value is metaphorically used, such as the harmony produced by the current music orchestration, even if it is not the usage of current mathematics. Firstly, when mathematical notions are proved on a system, it is recognised that they already existed from the beginning in a system: It is called pre-supposed/established harmony, likely the providence such as the god already planned before. Up to the final moment to be proved in a system, the notions do not have a clear position; however, at the proved moment, it is a part of the system based on mathematical structure. Even though they are not proved in one system (theory), they can be used hypothetically as true. Each of them is proved on different systems (theories), not one system, and used at the same time as long as it works. However, it looks strange until the harmonised position in the system is reset, like the history of negative and imaginary numbers.

In music, different scales/temperament systems produce different sounds. However, the music appears the same. For example, Current Buch players use instruments with equal temperaments even if they use different cords. We are listening to and playing different Bach music; however, we still recognise it as Bach. Metaphorically, it implies that we recognise mathematical notions on the different systems and use them harmoniously through proven under different systems/theories, even though they should be proved in a system later. Indeed, the school mathematics curriculum is the possible sequence of local theories, which includes several contradictions. However, we can accept them harmoniously. In this metaphorical usage, a teacher has eds to prepare the class based on various mathematical systems/theories that students have already learned.

Secondly, with more metaphorical usage in a mathematics classroom, each student looks at a player to produce mathematical notions harmoniously with collaboration. Current orchestration is possible through each player and conductor’s contributions and collaborations. Each player is a necessary component to make a resonance of code. Metaphorically, it is a form of verbal and non-verbal communication in the mathematics classroom where each student’s idea produces harmonised reasoning as one whole and appreciates the likely resonance of code. Although there may be students who do not talk verbally, they may contribute by non-verbal manners such as attitude and eye line and so on. A well-harmonised lesson can be seen through such attitudes and so on.

Usefulness and Efficiency

Mathematics usually alternates the meanings that are necessary for reasoning to the procedure that automates and compresses the reasoning. Students can learn the efficiency of the procedure if this alternation is recognised and done by students. If teachers only teach the final procedure from the beginning without any meaning, students lose the opportunity to learn efficiently. Even though teachers take over the opportunity from students, teachers can teach the usefulness of the application problem after they teach the procedure. In the previous example of multiplication, accumulation is necessary at the beginning. However, if we memorise the multiplication table at once and use it for column multiplication, it is unnecessary to consider the accumulation. What good things you learned about column multiplication can be explained if they learned the accumulation before column multiplication. What are the good things you learned about positive and negative numbers? These questions are first answered by usefulness and efficiency, even though there are other values. To answer these questions, students might be able to compare before and after.

Simpler and Easier

These values are similar to the values of usefulness and efficiency; however, simpler values are usually used for selecting procedures and setting the mathematical form. In the development of the procedure, we chose simpler forms and so on. For example, numbers are read from the largest place value. Thus, children usually try to add from the largest place initially. However, in the column addition, if there is regrouping, they have to rewrite the number several times in the process. It becomes simple if they decide to add from the lowest place value. To explain the procedure, we use a diagram such as a proportional number line to make it easier to produce the expression and interpret the meaning. It would be easier to understand if students could draw a diagram to explain the meaning.

Beautifulness

In mathematics teaching, beautifulness is used on several occasions with various values such as simple reasoning and thus beautiful. If we find a structure/pattern/invariant, it is beautiful. If we make a line with equal signs in the operation of the equation easier to see, then it is beautiful. For students to feel the beautifulness, teachers need to set students to feel the ugly/not beautiful situation and change it to feel beautiful. For example, it is just a heap of coins before the arrangement of the same coins. However, if the coins are arranged in vertical bars as in ordering natural numbers, we can alternate the numbers to the height. If the coins are regrouped by the height of 10 coins, we can easily count by 10. Comparisons before and after are necessary to discuss how beautiful the order is.

Beautifulness is also discussed in specific mathematical ideas. In number and operation, the answer of operation should be a number, then $1 \div 3 = \frac{1}{3}$, thus $\frac{1}{3}$ is a number even if it is $0.\dot{3}$. $\frac{1}{3}$ looks more beautiful than $0.\dot{3}$. In algebra, the general form of the quadratic function is $y = ax^2 + bx + c$ and the standard form of the quadratic function is $y = a(x - \alpha)^2 + \beta$. If we do not know how these two forms are beautiful, we cannot operate a quadratic function. In a figure, symmetry is usually found in the tessellation of a figure. Symmetry is the word to represent the beautifulness of geometry and algebra. On the sequence, the recursion on the numbered sequence is beautiful because it is a representation of an invariant pattern.

(ii) Mathematical Attitude Attempting to:

Here, attitude means one's mindset. Similar to mathematical value, it will be learned through reflection and appreciation in a social context. From a social perspective, it can be developed as a part of human character in mathematics classrooms through competitive and sympathetic experiences for excellent ideas. Excellence is usually related to representing mathematical ideas and ways of thinking and qualified by mathematical value through comparison. Like mathematics and seeking values, mathematics is a competitive subject that tries to produce innovative ideas faster than others, but it should generally be appreciable and useful ideas under specified conditions. Such an idea demonstrates one's excellence in the community. Secondly, mathematics is a sympathetic subject that allows us to appreciate others through re-presenting others' ideas. If others re-present one's ideas and share, and the others would like to use them, they all become the owners of excellent ideas. If there are no reactions from others for one's sharing action, one's excellence may not survive in their community. Thus, to show excellence, one has to make an effort to explain one's idea as simple and reasonable as possible so that it will be re-presented by others.

See and Think Mathematically

Every content in mathematics provides the ways to see. If students learned multiplication as any number can be seen as a unit for counting, they try to array the objects for counting by the unit for each. If each box has the same number of objects and the number of the boxes is known, we can apply multiplication. Such a thing is the way to see what should be learned when they learn multiplication. If students learn the significance of some representations such as proportional number lines and if they would like to use it, they may use it at the next task. If students appreciate the way of thinking on some representations such as generalisation by using permanence of form in a sequential pattern and if they would like to use it, they may use it at the next opportunity. Thus, to develop an attitude of attempting to see and think mathematically, students are necessary to have the opportunity to discuss the ways to see and think in every learning content.

Pose Questions and Develop Explanations

In the beginning, if students cannot pose questions by and for themselves, teachers have to pose the questions. However, through certain repetitions of similar questioning by teachers, teachers must develop students who pose similar questions by and for themselves. In the beginning, teachers have to say 'for example' by themselves if students never say it. However, they shift it to a question 'Tell me other examples.' If students can produce their examples, and teachers recognise students can say 'for example' in their explanation, then teachers challenge the question 'What do you want to do, next?' If students can show their example for this question, it means they internalised the way of problem posing by using the word 'for example' because it asks to produce different situations or representations for a given original task. Such questions are posed when students feel problematic or uncomfortable for others in context.

Mathematical explanations are done using necessary representations. The necessity of explanations originated from problematic, however, to produce possible explanations for problematics, it is necessary to utilise or find necessary representations.

Generalise and Extend

Without the experience of generalising and extending, reflecting and appreciating, students could not learn the value of generalisation and extension. The attitude of attempting to generalise and extend will be possible as long as such an experienced student would like to do. We must use what has already been learned for generalising and extending. However, students are not sure which learned knowledge should be generalised for what objective. For the unknown situation, they have to use something learned. Misconception is known as over-generalisation. To develop an attitude of attempting to generalize and extend, teachers and other students have to accept it and consider the learned which was overgeneralized. To grow the attitude to generalize and extend, we need to accept any possibility of generalization because it is evidence to be generalized and extended by oneself.

Appreciate others' Ideas and Change Representations for Meaningful Elaborations with Dialectic

Even though misconceptions, from the perspective of the dialectic approach, a necessary idea is to implement dialectic discussions such as 'If your saying is true what will happen?' For example, to know what A is, we should know what non-A is. If two ideas, A and B, appear, even if B is an overgeneralised idea, through the discussions of A and B, the original idea A evolves to idea A' with a comparison of B. For meaningful elaboration on the other side, A has to get the perspective of B, and B has to get the perspective of A. If each side understands the perspectives of the other, it is possible to discuss with each other the question 'If your saying is true, what will happen?' If a counterexample is necessary, we may need to develop different examples of the extended situation and hear what others say. In these discussions, to change the representation to be understandable for another side will be necessary to implement meaningful elaborations for each other. The generalisable idea will be chosen as the situation for dialectic discussion.

Analyse Process and Plan

For planning a problem solving, we need to find ways to reach a conclusion or solution. When the conclusion is given in the problem that seek proving, the reasoning from the conclusion which try to revert to the given is an *analysis* (Thinking backwards). On the other hand, in problems in our lives such as carpenters who use civil engineering, the final deadline and the conclusion such as the product are usually given at the beginning.

In this case, *analysis* means breaking the process to possible parts, planning the process in each part and integrating them within the schedule is planning. In STEAM education, it is a necessary part of *designing*. In mathematics, even though an unknown problem, we try to solve it analogically or inductively. These are also analyses. In the case of analogical analysis, we try to compare it with known problems and adapt the known ways. In the case of inductive analysis, we select one parameter in various parameters and change the only selected parameter, step by step. It is a kind of simulation when we already established the computational model in the computer system by mathematical modeling. In experimental science and engineering, the inductive analysis is necessary for planning experiments.

APPENDIX C

Strand: Mathematical Process – Humanity [K3MH]

In the era of Generative AI, the critical argument in mathematics is enhanced through communication with others beyond Key Stage 2. These proposed challenging activities will promote metacognitive thinking at different levels of arguments to make sense of mathematics. Translating real-life activities into mathematical models and solving problems using appropriate strategies are emphasised in functional situations. The process of doing mathematical activities involves patience, which develops perseverance in learners and helps them take responsibility for their learning. At this stage, the habitual practice of self-learning will eventually build confidence. Thus, the opportunity for challenges to extend mathematics and the ability to plan sequences of future learning is also enhanced.

Standards:

Enjoying problem solving through various questioning and conjecturing for extension of operations into algebra, space and geometry, relationship and functions, as well as statistics and probability [K3MH1]
Enjoying measuring space using calculations with various formulae [K3MH2]
Producing proof in geometry and algebra [K3MH3]
Utilising tables, graphs and expressions in situations [K3MH4]
Using diagrams for exploring possible and various cases logically [K3MH5]
Exploring graphs of functions by rotation, by symmetry and by translation of proportional function [K3MH6]
Understanding ways for extension of numbers [K3MH7]
Designing a sustainable life with mathematics [K3MH8]
Utilising ICT tools as well as other technological tools [K3MH9]
Promoting creative and global citizenship for sustainable development of society in mathematics [K3MH10]
Represent recursive process by using manipulatives, drawings and tables for finding patterns [K3MH11]

[K3MH1]
Enjoying problem solving through various questioning and conjecturing for extension of operations into algebra, space and geometry, relationship and functions, as well as statistics and probability[94]

i. Pose questions to extend numbers and operations into positive and negative numbers, algebraic operations, and further extension into polynomial operations, and numbers with square roots
ii. Pose questions to solve linear equations, simultaneous equations and simple quadratic equations
iii. Pose assumptions in geometry as objects of argument and proof
iv. Pose questions to transform three-dimensional objects into two-dimensional shapes and vice versa
v. Pose questions in relations and functions for knowing properties of different types of function
vi. Pose questions to explore data handling and to know the structure of distributions
vii. Pose questions that apply PPDAC in relation to statistical problem solving
viii. Pose questions in relation to “equally likely” events
ix. Pose assumptions to discuss the hypothesis based on the sample and population
x. Pose conjectures such as if x increases and y decreases, then it is an inverse proportion

[K3MH2]
Enjoying measuring space using calculations with various formulae[95]

i. Extend the number line to positive and negative numbers and compare the size of numbers with the idea of absolute value
ii. Derive the square root using unit squared paper through the idea of the area
iii. Explain the expansion of polynomials using area diagrams
iv. Use the projection of space figures to plane figures using the Pythagorean theorem
v. Apply similarity and simple trigonometry for measurement
vi. Use common factors to explain factorisation of the area of a rectangle based on the area of a square

[K3MH3]
Producing proof in geometry and algebra

i. Have an assumption through exploration and produce propositions
ii. Justify the proposition using examples and counter-examples to achieve understanding

[94] *Connected to the three strands, namely Numbers and Algebra, Relations and Functions, and Space and Geometry.*
[95] *Connected to the three strands, namely Numbers and Algebra, Relations and Functions, and Space and Geometry.*

iii. Rewrite propositions from sentences to mathematical expressions by using letters and diagrams
iv. Search the ways of proving by thinking backwards from the conclusion and thinking forward from the given
v. Show the proof and critique for the shareable and reasonableness
vi. Deduce other propositions in the process of proving and after proving using what if and what if not
vii. Adapt ways of proving to other similar propositions of proof
viii. Explain the written proof in geometry and algebra by the known
ix. Revise others' explanations meaningfully

[K3MH4]
Utilising tables, graphs and expressions in situations[96]

i. Explore the properties of functions by using tables, graphs and expressions and establish the fluency of connections among them for interpreting functions in the context
ii. Analyse the distribution of raw data by using tables, graphs and expressions in daily life

[K3MH5]
Using diagrams for exploring possible and various cases logically[97]

i. Use number line with inequality to identify range and set
ii. Use a circle to identify the relationship between the circumference and the central angle (acute, obtuse and right)
iii. Use a rectangle and rotate a point on the side rectangle to draw the graph of the area
iv. Use a tree diagram for thinking about all possible cases sequentially

[K3MH6]
Exploring graph of functions by rotation, by symmetry and by translation of proportional function[98]

i. Use the slope of a graph for the proportional function to rotate the graph or to determine the point of intersection
ii. Explore to know the nature of two simultaneous equations by using translation
iii. Use the *y*-axis, *x*-axis and as the line of symmetry to explore the proportional function
iv. Explain the linear function graph by translating the proportional function.

[K3MH7]
Understanding ways for extension of numbers[99]

i. Extend the numbers based on the necessity of solving equations such as $x + 5 = 3$ and , and show examples for demonstrating the existence, such as on the number lines, and understand it as a set
ii. Compare the size of numbers and identify how to explain the order of numbers and their equivalence
iii. Extend operations to keep the form[100] beyond the limitations of meaning[101]

[K3MH8]
Designing models for sustainability using mathematics[102]

i. Discuss and utilise probabilities in life, such as weather forecasting, for planning
ii. Design cost-saving lifestyle models through comparison of data such as cost of electricity, water consumption, and survey
iii. Plan emergency evacuation such as heavy rain and landslide where the calculations on the amount

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96. *Connected to the strand on Relations and Functions.*
97. *Connected to the two strands of Relations and Functions and Space and Geometry.*
98. *Connected to the two strands of Numbers and Algebra and Relations and Functions.*
99. *Connected to the strand on Numbers and Algebra.*
100. *There are three meanings of the form: (1) Permanence of form means "keep the pattern of operation" such as (-3)x(+2)=-6, (-3)x(+1)=-3, (-3)x0=0, and (-3)x(-1)=+3, and (-3)x(-2)=+6. Here, the product of the pattern increases by 3; (2) The form means "Principle of the permanence of equivalence form" which means to keep the law of commutativity, associativity and distributivity; and (3) The form means the axiom of field in Algebra. Normally, in education, we only treat (1) and(2).*
101. *For the extension of numbers to positive and negative numbers, beyond the limitations of meaning such as subtracting smaller numbers from larger numbers. For the extension of numbers to irrational numbers, beyond the limitation of meaning such as rational numbers is quotient (value of division).*
102. *Connected to the three strands, Numbers and Algebra, Relations and Functions and Space and Geometry.*

[K3MH9]
Utilising ICT tools as well as other technological tools

i. Use dynamic geometry software for assumption, specialisation and generalisation
ii. Use a graphing tool for comparison of the graph and knowing the properties of a function
iii. Use data to analyse statistics with software
iv. Use internet data for the discussion of sustainable development
v. Use calculators for operations in the necessary context
vi. Use a projector for sharing ideas such as project surveys, reporting and presentation
vii. Use the idea of function to control a mechanism
viii. Use ICT tools for conjecturing and justifying to produce the object of proving.

[K3MH10]
Promoting creative and global citizenship for the sustainable development of society using mathematics

i. Utilise notebooks, journal books and appropriate ICT tools to wisely record and produce good ideas for sharing with others
ii. Prepare and present ideas using posters, projectors, pamphlets and social media to promote good practices in society
iii. Promote the beautifulness, reasonableness and simplicity of mathematics through contextual situations in the society
iv. Listen to other's ideas and ask questions for better designs, craftsmanship and innovations
v. Utilise information, properties, models and visible representations as the basis for making intelligent decisions
vi. Utilise practical arts, home economics, financial mathematics and outdoor studies to investigate local issues for improving the welfare of life

[K3MH11]
Represent recursive process by using manipulatives, drawings and tables for finding patterns

i. Analysing situations by multiple action processes and alternating them with manipulatives, drawings and tables
ii. Find the structure of repetitions which is used in repeated actions previously
iii. Confirm recursive structures by using manipulatives, drawings and tables
iv. Use the established recursive structures to produce new situations

APPENDIX D

Key Stage 1

Key Stage 1 (KS1) serves as the foundation of knowledge covering the basic facts and skills developed through simple hands-on activities, manipulation of concrete objects, pictorial and symbolic representations. This stage focuses on arousing interest, enjoyment and curiosity in the subject through exploration of patterns, characterisation, identification and describing shapes, performing the four fundamental operations, identifying its algorithm, and understanding basic mathematical concepts and skills experienced in daily life. Calculations of quantities will also be established to carefully and wilfully understand the attribution of objects used to make direct and indirect comparisons.

Strand: Numbers and Operations [K1NO]

The number is introduced with situations, concrete objects, pictorial, symbolic representations and extended based on knowledge and skills learned. Ways of counting and distributions are extended to addition, subtraction, multiplication and division. The base ten number system is the key to extending the numbers and operations for standard algorithms in vertical form. Also, various procedures of calculations and algorithms are focused on. Models and diagrams are used for extensions instead of concrete materials themselves. Number sense will be developed by establishing fluency in calculations connected to situations and models. Fractions and decimals are introduced with manipulatives.

Topics:

Introducing Numbers up to 120 [K1NO1]
Introducing Addition and Subtraction [K1NO2]
Utilising Addition and Subtraction [K1NO3]
Extending Numbers with Based Ten System up to 1 000 000 Gradually [K1NO4]
Producing Vertical Forms for Addition and Subtraction and Acquiring Fluency of Standard Algorithms [K1NO5]
Introducing Multiplication and Produce Multiplication Algorithm [K1NO6]
Introducing Division and Extending it to Remainder [K1NO7]
Introducing Fractions and Extending to Addition and Subtraction of Similar Fractions [K1NO8]
Introducing Decimals and Extending to Addition and Subtraction [K1NO9]

Introducing Numbers up to 120 [K1NO1]

[K1NO1-1]
Enjoying counting orally and manipulatively with number names, without symbolic numerals

i. Develop fluency in the order of number names and use them based on situations[1]
ii. Set the initial object for counting, the direction of counting and recognise the last object with one-to-one correspondence
iii. Distinguish the original situation with concrete objects and the representation of counting chips, blocks or marbles

[K1NO1-2]
Understanding and using the cardinality and ordinality of numbers with objects and numerals[2] through grouping activities, corresponding and ordering, and developing number sense[3] and appreciating its beautifulness

i. Group objects for counting with conditions such as cups, flowers, and rabbits in situations and introduce numerals
ii. Obtain fluency in counting concrete objects and understand counting on, and recognise the necessity of zero

1. *The denomination such as 3 cups, 2 cups, and one cup is described in strands on the Quantity and Measurement.*
2. *Inclusive in reading and writing of numerals.*
3. *Units for counting that describe the Quantity and Measurement.*

iii. Compare different sets by one-to-one correspondence and recognize larger, smaller or equal with appreciation in drawing paths between objects.
iv. Compose and decompose numbers to strengthen the number sense[4] and cardinality
v. Understand the difference between ordinal and cardinal numbers and use them appropriately in situations and challenge the mixed sequence.
vi. Acquire number sense[5] and appreciate the beautifulness of ordering numerals with and without concrete objects

[K1NO1-3]
Introducing the base-ten system with groupings of 10 and extending numbers up to 120[6]

i. Extend numbers to more than 10 with base-ten manipulative representing numbers in ones and tens, and appreciate the base-ten numeration system
ii. Extend the order sequence of numbers to more than 10 in relation to the size of ones and tens and compare numbers using numerals in every place value
iii. Introduce number lines to represent the order of numbers and for comparison starting from zero and counting by ones, twos, fives, and tens
iv. Enjoy various ways of the distribution of objects with counting such as playing cards, and explain it and enhance the number sense
v. Draw a diagram for representing the size of the number with base-ten blocks such as a flat (square) for a unit of hundred and a rectangular bar for a unit of ten

Introducing Addition and Subtraction [K1NO2]

[K1NO2-1]
Understanding situations for addition up to 10 and obtaining fluency in using addition in situations

i. Introduce situations (together, combine, and increase) for addition and explain it orally with manipulative to define addition for operation
ii. Develop fluency in addition expressions using a composition of numbers for easier calculation with number sense for the composition of numbers
iii. Apply addition with fluency in learners' lives

[K1NO2-2]
Extending addition to more than 10 and obtaining fluency in using addition in situations

i. Extend addition situations and think about how to answer using the idea of making 10 with decomposition and composition of numbers
ii. Explain the idea of addition with place value using base-ten blocks
iii. Develop fluency in addition expressions to more than 10 for easier calculation
iv. Apply the addition fluency in learners' lives.

[K1NO2-3]
Understanding situations for subtraction up to 10 and obtaining fluency in using subtraction in situations

i. Introduce subtraction situations (remaining, complement, and difference) and explain orally with manipulative to define subtraction for operation[7]
ii. Develop fluency in subtraction expressions using the decomposition of numbers for easier calculation
iii. Apply subtraction fluency in learners' lives

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4. *Relationship of composing and decomposing numbers becomes the preparation for addition and subtraction for inverse operation.*
5. *Number pattern is discussed under Pattern and Data Representations.*
6. *For discussing the difference of hundred twenty is not twelve ten in English.*
7. *Distinguish minuend and subtrahend.*

[K1NO2-4]
Extending subtraction to more than 10 and obtaining fluency in` using subtraction in situations

i. Extend subtraction with situations and think about how to answer using the idea of 10 with addition and subtraction of numbers (composition and decomposition of numbers)
ii. Explain the idea of subtraction in place value using base ten blocks
iii. Develop fluency in subtraction expressions to more than 10 for easier calculation
iv. Apply subtraction fluency in learners' daily life

Utilising Addition and Subtraction [K1NO3]

[K1NO3-1]
Utilising addition and subtraction in various situations and understanding their relationships

i. Understand the difference between addition and subtraction situations with tape diagrams
ii. Explain subtraction as an inverse of addition situations with tape diagrams
iii. Understand addition with three numbers, subtraction with three numbers and combination of addition and subtraction situations
iv. Apply addition and subtraction in various situations such as in ordering numbers

Extending Numbers with Base-Ten System Up to 1 000 000 Gradually [K1NO4]

[K1NO4-1]
Extending numbers using base-ten system up to 1 000

i. Experience counting 1 000 by using various units and appreciate the necessity of the base-ten system
ii. Extend the order of numbers to more than 1 000 in relation to the size of ones, tens and hundreds
iii. Use a partial number line to compare the size of numbers through a translation of the size of every digit with the appropriate scale
iv. Represent appropriate diagram to show the size of numbers without counting such as three of hundreds mean 30 of tens and visualise the relative size of numbers
v. Represent larger or smaller numbers by the symbol of inequality

[K1NO4-2]
Extending numbers using a base-ten system up to 10 000

i. Visualise the 10 000 by using thousand, hundred, ten and one as units
ii. Extend the order sequence of numbers to more than 10 000 in relation to the size of ones, tens, hundreds and thousands
iii. Use a number line with an appropriate scale to show the size of numbers and the relative size of numbers while focusing on the scale

[K1NO4-3]
Extending numbers using base-ten system up to 1 000 000[8]

i. Extend numbers up to 1 000 000 and learn the representation of the place value for grouping every 3-digit numeral system up to a million
ii. Write large numbers using the grouping of a 3-digit numeral system[9] such as thousand as a unit and compare numbers in relation to it
iii. Develop number sense such as larger and smaller based on comparison of place values through visualisation of the relative size of numbers

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8. *One million is too big for counting and is introduced only for learning the three-digit system.*
9. *3-digit numeral system such as 123 times thousand equals the same way of reading plus thousand. In the case of Chinese, the four-digit numeral system is used.*

Producing Vertical Form Addition and Subtraction[10] and Acquiring Fluency in Standard Algorithms [K1NO5]

[K1NO5-1]
Thinking about the easier ways for addition and subtraction and producing vertical form algorithms

i. Think about easier ways of addition or subtraction situations and use models with base-ten blocks meaningfully for representing the base-ten system
ii. Produce and elaborate efficient ways and identify the standard algorithms[11] in relation to the base-ten system with appreciation
iii. Explain the algorithms of borrowing and carrying with regrouping of base-ten models
iv. Acquire fluency in addition and subtraction algorithms

[K1NO5-2]
Acquiring fluency in standard algorithms for addition and subtraction and extending up to 4-digit numbers

i. Extend the vertical form addition and subtraction through the extension of numbers and appreciate the explanation using the base-ten block model
ii. Develop fluency in every extension up to 3-digit numbers and simple case for 4-digit numbers

[K1NO5-3]
Developing number sense[12] for estimation[13] and using a calculator judiciously for addition and subtraction

i. Develop number sense for mental arithmetic with estimation for addition or subtraction of numbers
ii. Identify necessary situations to use calculators judiciously in real life.
iii. Appreciate the use of calculators in the case of large numbers for finding the total and the difference

Introducing Multiplication and Produce Multiplication Algorithm [K1NO6]

[K1NO6-1]
Introducing multiplication and mastering multiplication table

i. Understand the meaning of multiplication[14] situations with models using the idea of addition and distinguishing from the common addition to find the total number
ii. Produce a multiplication table in the case of counting by 2 and 5 with array diagrams, pictures or block models and extend it until 9 and 1 with an appreciation of patterns[15]
iii. Develop a sense for multiplication through mental calculation with fluency
iv. Use multiplication in daily life, differentiating multiplication in various situations with the understanding that any number can be a unit for counting in multiplication

[K1NO6-2]
Producing multiplication in vertical form and obtaining fluency

i. Think about easier ways of multiplication in the case of numbers greater than 10 using array diagrams and block models
ii. Develop multiplication in vertical form using multiplication table, array, model, and base-ten system with appreciation
iii. Extend the multiplication algorithm to 3-digit times 2-digit numbers
iv. Obtain fluency in the standard algorithm for multiplication
v. Use estimation with the multiplication of tens or hundreds in life
vi. Compare the multiplication expressions which is larger, smaller or equivalent
vii. Appreciate the use of calculators sensibly in life in the case of large numbers

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10. *Understanding the relationship between addition and subtraction is discussed under Pattern and Data Representations.*
11. *Various algorithms are possible and there is no one specific form because depending on the country, the vertical form itself is not the same. Here, the standard algorithm means the selected appropriate form.*
12. *Money system is discussed under Measurement and Relations.*
13. *Rounding numbers are treated in key stage 2 under Measurement and Relations.*
14. *Meaning of area is described in Measurement and Relations.*
15. *Multiplication row of 1 is not a repeated addition.*

Introducing Division and Extending It to Remainder [K1NO7]

[K1NO7-1]
Introducing division with two different situations and finding the answers by multiplication

i. Understand division with quotative and partitive division for distribution situations
ii. Think about how to find the answer to division situations by distribution using diagrams, repeated subtractions and multiplication
iii. Obtain fluency to identify answers of division through the inverse operation of multiplication
iv. Appreciate the use of multiplication table for acquiring mental division

[K1NO7-2]
Extending division into the case of remainders and using division for distribution in daily situations

i. Extend division situations with remainders and understand division as a repeated subtraction with remainders
ii. Obtain fluency in the division and apply it in daily situations
iii. Understand simple cases of the division algorithm

Introducing Fractions and Extending to Addition and Subtraction of Similar Fractions [K1NO8]

[K1NO8-1]
Introducing simple fractions such as halves, quarters and so on using paper folding and drawing diagrams

i. Introduce simple fractions using paper folding and drawing diagrams in the context of part-whole relationship
ii. Use "*a half of*" and *"a quarter of*" in a daily context such as half a slice of bread
iii. Count a quarter for representing one quarter, two quarters, three quarters, and four quarters
iv. Compare and explain simple fractions in the case where the whole is the same

[K1NO8-2]
Extending fractions using tape diagram and number line to one, and think about how to add or subtract similar fractions for producing a simple algorithm

i. Extend fractions to more than one unit quantity for representing the remaining parts (unit fraction) such as measuring the length of tape, recognising the remaining parts as a unit measure of length, and understanding proper and improper fractions
ii. Appreciate fractions with quantities in two ways; firstly, the whole is a unit of quantity and secondly, based on the number of unit fraction
iii. Compare fractions in the case where the whole is the same and explain it with a tape diagram or number line, and develop fraction number sense such as with quantities and so on
iv. Think about how to add or subtract similar fractions with a tape diagram or number line and produce a simple algorithm with fluency

Introducing Decimals and Extending to Addition and Subtraction [K1NO9]

[K1NO9-1]
Introducing decimals to tenths, and extending addition and subtraction into decimals

i. Introduce simple decimals to tenths by remaining part such as using a tape diagram with appreciation
ii. Compare the size of decimal numbers on a number line with the idea of place value
iii. Extend addition and subtraction of decimals utilising the place value system in the vertical form up to tenths
iv. Think about appropriate place value for applying addition and subtraction in life

Strand: Quantity and Measurement [K1QM]

Attributes of objects are used to make direct and indirect comparisons. The non-standard and standard units are also used for comparison. Counting activities denominate units of quantities such as cups for volumes, arm-length and hand-spans for length. Standard units such as metre, centimetre, kilogram and litre are introduced. Time and durations which are not the base ten system are introduced. Money is not a complete model for a base-ten system. The concept of conservation of quantities will be established through the calculation of quantities. The sense of quantity is developed through the appropriate selection of measurement tools.

Topics:

Comparing Size, Directly and Indirectly, Using Appropriate Attributes and Non-Standard Units [K1QM1]
Introducing Quantity of Length and Expand It to Distance [K1QM2]
Introducing Quantity of Mass for its Measurement and Operation [K1QM3]
Introducing Quantity of Liquid Capacity for its Measurement and Operation [K1QM4]
Introducing Time and Duration and its Operation [K1QM5]
Introducing Money as Quantity [K1QM6]

Comparing Size, Directly and Indirectly, Using Appropriate Attributes and Non-Standard Units [K1QM1]

[K1QM1-1]
Comparing and describing quantity using appropriate expression

i. Compare two objects directly by attributes instead of stating in length and amount of water such as longer or shorter and less or more
ii. Compare two objects indirectly using non-standard units to appreciate the unification of units
iii. Use appropriate denomination[16] of quantity (such as the number of cups) for counting and appreciating the usage of units for quantity in a suitable context

Introducing Quantity of Length and Expanding to Distance [K1QM2]

[K1QM2-1]
Introducing centimetre for length and extending to millimetres and metre

i. Compare the length of different objects and introduce centimetres with a calibrated tape[17] of one centimetre
ii. Demonstrate equivalent length with addition and subtraction such as part-part whole
iii. Extend centimetre to millimetre to represent remaining parts with ideas of equally dividing and the idea of making tens
iv. Extend centimetre to metre to measure using a metre stick
v. Estimate the length of objects and select appropriate tools or measuring units for measurement with fluency
vi. Convert mixed and common units of length for comparison[18]
vii. Convert mixed and common units of length when adding or subtracting in acquiring the sense for quantity

[K1QM2-2]
Introducing distance for the extension of length

i. Introduce kilometres to measure distance travelled using various tools and appreciate the experiences of measuring skills
ii. Distinguish the distance travelled and the distance between two places on the map
iii. Compare mixed units of length with an appropriate scale on a number line

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16. *Denomination is necessary for learning the group of counting. It also describes pattern and data representations and number and operation both of Key Stage 1.*
17. *The plane tape can be used for direct and indirect comparison by marking. If a non-standard unit scales the tape, we can use it for measurement. If the tape is scaled by one centimetre, we can define the length of the centimetre.*
18. *Which one is longer, 2 m 3 cm or 203 mm?*

Introducing Quantity of Mass and Its Measurement and Operation [K1QM3]

[K1QM3-1]
Introducing gram for mass and extending to kilogram and tons

i. Compare the mass of different objects directly using balance and introduce gram
ii. Demonstrate equivalent mass with addition and subtraction such as part-part whole
iii. Extend gram to kilogram, measure with a weighing scale
iv. Extend kilogram to metric ton through the relative measure (such as 25 children, each weighing 40 kilograms)
v. Estimate the mass of objects and select appropriate tools or measuring units for measurement with fluency
vi. Convert mixed and common units of mass for comparison
vii. Convert mixed and common units of mass for addition and subtraction in acquiring the sense of quantity

Introducing Quantity of Liquid[19] Capacity and Its Measurement and Operation [K1QM4]

[K1QM4-1]
Introducing litre for capacity of liquid and extending to millilitre

i. Compare the amount of water in different containers and introduce litre with measuring cups of 1 litre
ii. Demonstrate equivalent capacity with addition and subtraction such as part-part whole
iii. Extend litre by decilitre/100-millilitre cup for representing remaining parts with ideas of equally dividing and making 10, and extend until millilitre
iv. Estimate the capacity of containers and select the appropriate measuring unit
v. Convert mixed and common units of capacity for comparison
vi. Convert mixed and common units of capacity for addition and subtraction in acquiring the sense of quantity

Introducing Time and Duration, and Its Operation [K1QM5]

[K1QM5-1]
Introducing analogue time and extending to duration

i. Tell and write analogue time of the day corresponding with different activities in daily life such as morning, noon, afternoon, day and night.
ii. Show time by using a clock face with an hour hand and a minute hand
iii. Understand the relative movement of clock hands

[K1QM5-2]
Extending clock time to a duration of one day[20]

i. Introduce duration in hours and minutes based on the beginning time and end time of activities
ii. Express time and duration on a timeline, and understand duration as the difference between two distinguished times
iii. Addition and subtraction of duration and time
iv. Extend time and duration to seconds
v. Convert mixed and common units of duration for comparison
vi. Estimate the duration of time and select an appropriate measuring unit for measurement with fluency and appreciate the significance of time and duration in life
vii. Appreciate the difference in time depending on the area (time zone) and the seasons

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[19]. *The density which explains the relationship between mass and liquid capacity is usually learned in science at a later stage. In the case of CCRLS Science, it starts in Key Stage 2 such as 1 cubic centimetre of water is equivalent to 1 gram.*
[20]. *Calendar is possible in the keys stage 1 under Pattern and Data Representation.*

Introducing Money as Quantity [K1QM6]

[K1QM6-1]
Introducing money as quantity and use as the model of the base-ten system[21]

i. Introduce units of money using notes and coins and determine the correct amount of money
ii. Use counting by fives and so on for the base-10 system
iii. Appreciate the fluency in the calculation of money with all the four operations
iv. Appreciate number sense for the conversion and transaction of money in daily life

Strand: Shapes, Figures and Solids [K1SF]

Basic skills of exploring, identifying, characterising and describing shapes, figures and solids are learned based on their features. Activities such as paper folding enable the exploration of various features of shapes. Identification of similarities and differences in shapes and solids enables classification to be done for defining figures. Using appropriate materials and tools, relationships in drawing, building and comparing the 2D shapes and 3D objects are considered. Through these activities, the skills for using the knowledge of figures and solids will be developed. The compass is introduced to draw circles and mark scales of the same length.

Topics:

Exploring shapes of objects [K1SF1]
Characterising shapes for figures and solids [K1SF2]
Explaining positions and directions [K1SF3]

Exploring Shapes of Objects [K1SF1]

[K1SF1-1]
Exploring shapes of objects to find their attributes

i. Roll, fold, stack, arrange, trace, cut, draw, and trace objects (blocks such as boxes, cans and so on) to know their attributes
ii. Use attributes of blocks for drawing the picture by tracing shapes on the paper and explain how to draw it with the shapes
iii. Create patterns of shapes (trees, rockets and so on) by using the attributes and recognise the characteristics of shapes[22]
iv. Appreciate functions of shapes of objects in learners' life
v. Appreciate the names of shapes in daily life by using one's mother tongue

Characterising the Shapes for Figures and Solids [K1SF2]

[K1SF2-1]
Describing figures with characters of shapes

i. Use characteristics of shapes for understanding figures (quadrilateral, square, rectangle and triangle, right angle, same length)
ii. Introduce line and right angle with relation to activities such as paper folding and use it for describing figures with simple properties, such as a triangle has 3 lines
iii. Classify triangles by specific components, such as side, vertex and angle (right-angled triangle, equilateral, isosceles) and then know the properties of each classification
iv. Reorganising rectangular shapes and squared shapes as figures by using the right angle and length of sides

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21. *Coins and notes are dependent on the country. Some countries use currency units of twenty and twenty-five in coins or notes. These forms are not appropriate for the model of the base-10 system.*
22. *Pattern of shapes is discussed in Key Stage 1 under Pattern and Data Representations.*

[K1SF2-2]
Describing solids with characteristics of shapes

i. Use characteristics of shapes to understand solids such as boxes can be developed by six rectangular parts with simple properties
ii. Develop boxes with the properties
iii. Appreciate solids around daily life by considering the functions of solids

[K1SF2-3]
Drawing a circle and recognising the sphere based on the circle

i. Think about how to draw a circle and find the centre and radius
ii. Draw a circle with an instrument such as a compass
iii. Enjoy drawing pictures using the function of circles such as Spirograph
iv. Find the largest circle of the sphere with a diameter and identify the sphere by its centre and radius
v. Appreciate the use of circles and spheres in daily life such as manholes, and the difference between a soccer ball and a rugby ball

Explaining Positions and Directions [K1SF3]

[K1SF3-1]
Exploring how to explain a position and direction

i. Identify simple positions and directions of an object accurately using various ways such as in my perspective, in your perspective in the classroom, and the left, right, front, back, west, east, north, south and with measurement
ii. Draw the map around the classroom with consideration of locations
iii. Design a game to appreciate the changing of positions and directions in a classroom

Strand: Pattern and Data Representations [K1PD]

Various types of patterns such as the number sequence and repetition of shapes are considered. The size of pictures can be represented by the number sequence. Tessellation of shapes and paper folding can be represented by the repetition of shapes. Exploration of patterns and features is also considered to represent the data structure using pictographs and bar graphs. Patterns and features produce the meaning of data and represent mathematical information. Patterns are represented by diagrams and mathematical sentences which are also used for communication in identifying and classifying situations to produce meaningful interpretations. In the era of Generative AI, natural language reasoning, which includes drawings, will be enhanced because it runs through using natural language.

Topics:

Using Patterns under the Number Sequence [K1PD1]
Producing Harmony of Shapes using Patterns [K1PD2]
Collecting Data and Represent the Structure [K1PD3]

Using Patterns under the Number Sequence[23] [K1PD1]

[K1PD1-1]
Arranging objects for beautiful patterns under the number sequence

i. Know the beautifulness of patterns in cases of arranging objects based on number sequence
ii. Arrange objects according to number sequence to find simple patterns
iii. Arrange expressions such as addition and subtraction to find simple patterns

23. *Number sequence is discussed in Key Stage 1 under Numbers and Operations.*

iv. Express the representation of patterns using placeholders (empty box)
v. Enjoy the arrangement of objects based on number sequence in daily life
vi. Find patterns on number tables such as in calendars[24]

Producing harmony of shapes using patterns[25] [K1PD2]

[K1PD2-1]
Arranging tiles of different or similar shapes to create harmony

i. Know the beautifulness of patterns in cases of arranging the objects based on shapes, colours and sizes
ii. Arrange objects according to shapes, colours and sizes to show patterns
iii. Arrange boxes according to shapes, colours and sizes to create a structure
iv. Arrange circles and spheres for designing
v. Enjoy the creation based on different shapes, colours and sizes in daily life

Collecting data and representing the structure [K1PD3]

[K1PD3-1]
Collecting data through categorisation to get information

i. Explore the purpose of why data is being collected
ii. Grouped data by creating similar attributes on the denomination[26] of categories and counting them (check mark and count)
iii. Think about what information is obtained from the tables with categories and how to use it

[K1PD3-2]
Organising the data collected and representation using pictograms for easy visualisation

i. Produce the table and pictograms from collected data under each category
ii. Interpretation of tables and pictograms as a simple conclusion about the data being presented.
iii. Appreciate pictograms through collecting data and adding data in daily activities in learners' life

[K1PD3-3]
Representing a data structure by using a bar graph to predict the future of communities

i. Understand how to draw bar graphs from a table using data categories and sort the graph to show its structure
ii. Appreciate ways of presenting data such as using tables, pictograms and bar graphs with sorting for predicting their future communities
iii. Appreciate the use of data for making a decision

24. *Time and duration are discussed in key stage 1 under Quantity and Measurement.*
25. *Harmony of shapes will be discussed in Key Stage 1 under Shapes, Figures and Solids.*
26. *Denomination will be learned in Key Stage 1 under Quantity and Measurement.*

Strand: Mathematical Process – Humanity [K1MH]

Enjoyable mathematical activities are designed to bridge the standards in different strands. Exploration of various number sequences, skip counting, addition and subtraction operations help to develop a number sense that is essential to support explanations of contextual scenarios and mathematical ideas. Mathematical ways of posing questions in daily life are also necessary to learn at this stage. The ability to select simple, general and reasonable ideas enables effective future learning. The application of number sense provides a facility for preparing sustainable life. The use of ICT tools and other technological tools provides convenience in daily life. At the initial stage, concrete model manipulation is enjoyable, however, drawing a diagram is most necessary for explaining complicated situations by using simple representation. This manner is necessary to develop computational and mathematical thinking.

Standards:

Enjoying problem solving through various questioning for four operations in situations [K1MH1]
Enjoying measuring through setting and using the units in various situations [K1MH2]
Using blocks as models and their diagrams for performing operations in base ten [K1MH3]
Enjoying tiling with various shapes and colours [K1MH4]
Explaining ideas using various and appropriate representations [K1MH5]
Selecting simple, general and reasonable ideas which can apply to future learning [K1MH6]
Preparing a sustainable life with a number sense [K1MH7]
Utilising ICT tools such as calculators as well as other tools such as notebooks and other instruments such as clocks [K1MH8]
Promoting creative and global citizenship for sustainable development of neighbourhood using mathematics [K1MH9]
Represent recursive process by using manipulatives and drawings for finding patterns [K1MH10]

[K1MH1]
Enjoying problem solving through various questioning for four operations in situations[27]

i. In addition, pose questions for altogether and increase situations
ii. In subtraction, pose questions for remaining and differences situations
iii. In multiplication, pose questions for the number of groups situations
iv. In division, pose questions for partition and quotation situations
v. Enjoy questioning by using a combination of operations in various situations
vi. In operations, pose questions to find easier ways of calculation
vii. Use posing questions for four operations on measurements in daily life

[K1MH2]
Enjoying measuring through setting and using the units in various situations[28]

i. Compare directly and indirectly
ii. Set tentative units from differences for measuring
iii. Give appropriate names (denominations) for counting units
iv. Use measurement for communication in daily life
v. Use tables and diagrams for showing the data of measures

[K1MH3]
Using blocks as models and their diagrams for performing operations in base ten[29]

i. Show increasing and decreasing patterns using blocks
ii. Show based ten system using blocks; the unit cube is 1, the bar stick is 10 and the flat block represents 100
iii. Explain the addition and subtraction algorithm in vertical form using a base-ten block model
iv. Explain multiplication table with grouped blocks
v. Explain division using equal distribution of blocks and repeated subtraction of blocks
vi. Use the number of blocks for measurement in daily life

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27. *It is related to Numbers and Operations and Quantity and Measurement both in Key Stage 1.*
28. *It is related to Quantity and Measurement and Pattern and Data Representations both in Key Stage 1.*
29. *It is related to Pattern and Data Representations and Numbers and Operations, both in Key Stage 1.*

[K1MH4]
Enjoying tiling with various shapes and colours[30]

i. Appreciate producing beautiful designs with various shapes and finding the pattern to explain it
ii. Reflect, rotate and translate to produce patterns
iii. Cut and paste various shapes and colours to form the box and ball such as developing the globe from a map

[K1MH5]
Explaining ideas using various and appropriate representations[31]

i. Explain four operations using pictures, diagrams, blocks and expressions for developing ideas
ii. Explain measurement using measuring tools, tape diagrams, containers and paper folding for sharing ideas
iii. Make a decision on how to explain the figures and the solids by using manipulative objects or diagrams or only verbal explanation
iv. Explain patterns using diagrams, numbers, tables and expressions with a blank box
v. Ask questions using terms such as why, how, what, if and if not, and reply using examples and 'for example' in discussions
vi. Change the representation and translate it appropriately into daily life

[K1MH6]
Selecting simple, general and reasonable ideas which can apply to future learning[32]

i. Discuss the argument for the easier ways for addition and subtraction algorithms in vertical form
ii. Extend the algorithm to large numbers for convenience and fluency
iii. Use the pattern of increase in the multiplication table for convenience
iv. Use multiplication tables for finding the answers to division

[K1MH7]
Applying number sense[33] acquired in Key Stage 1 for preparing sustainable life[34]

i. Use mathematics for the minimum and sequential use of resources in situations
ii. Estimate for efficient use of resources in situations
iii. Maximize the use of resources through an appropriate arrangement in space
iv. Understand equally likely of resources in situations

[K1MH8]
Utilising ICT tools such as calculators as well as other tools such as notebooks and other instruments such as clocks[35]

i. Use calculators for addition of multiple numbers in situations
ii. Use mental calculations for estimations
iii. Use a balance scale to produce equality and inequality
iv. Use cups, tapes, stopwatches, and weighing scales for measuring distances and weights
v. Use calculators to explain the calculation process by solving backwards and understanding the relationship between addition and subtraction, multiplication and division
vi. Enjoy using notebooks to exchange learning with each other such as mathematics journal writing
vii. Enjoy presentations with board writing
viii. Use various tools for conjecturing and justifying

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30. *It is related to Shapes, Figures and Solids and Pattern and Data Representations, both in Key Stage 1.*
31. *It is related to all strands in Key Stage 1.*
32. *It is related to Numbers and Operations and Pattern and Data Representations, both in Key Stage 1.*
33. *It is related to Numbers and Operations, Pattern and Data Representations and Quantity and Measurement, all included in Key Stage 1.*
34. *Sustainable development goals were crafted at the 70th Session of the United Nations General Assembly and indicated as universal values in education.*
35. *STEM education is enhanced. Mathematics is the major and base subject for STEM Education in Key Stage 1 hence, technological contents are included in Mathematics.*

[K1MH9]
Promoting creative and global citizenship for sustainable development of neighbourhood using mathematics

i. Utilise notebooks and journal books to record and find good ideas and share them with others
ii. Prepare and present ideas using posters to promote good practices in the neighbourhood
iii. Listen to other's ideas and ask questions for better creation
iv. Utilise information, properties and models as a basis for reasoning
v. Utilise practical arts and outdoor studies to investigate local issues for improving the welfare of life

[K1MH10]
Represent recursive process by using manipulatives and drawings for finding patterns

i. Analysing situations by multiple action processes and alternating them with manipulatives and drawings
ii. Find the structure of repetitions which is used in repeated actions previously

APPENDIX E

Key Stage 2

Key Stage 2 (KS2) is learned based on the Key Stage 1. This stage shows the extension of numbers and operations, measurement and relations, plane figures and space figures, data handling and graphs. This stage shows the extension of the four operations to daily use of numbers such as decimals and fractions, allows the use of mathematical terminologies investigations and establishing the ground for analysing, evaluating and creating in learners' lives. Appreciating the beauty of the structure of mathematics will enable them to enjoy and sustain their learning, providing the basis for Key Stage 3.

Strand: Extension of Numbers and Operations [K2NO]

Numbers are extended to multi-digits, fractions and decimals. Multiplication and division algorithms are completed with fluency. Fractions become numbers through the redefinition as a quotient instead of a part-whole relationship. Multiplication and division of decimals and fractions are also explored to develop procedures for the calculation. Various representations are used to elaborate and produce meaning for the calculation. Number sense, such as approximating numbers and the relative size of numbers and values, is enhanced for practical reasoning in the appropriate context of life.

Topics:

Extending Numbers with Base Ten Up to Billion and also to Thousandths with a Three-Digit Numeral System Gradually [K2NO1]
Making Decision of Operations on Situations with Several Steps and Integrate them in One Expression and Think About the Order of Calculations and Producing the Rule (PEMDAS) [K2NO2]
Producing the Standard Algorithm for Vertical Form Division with Whole Numbers [K2NO3]
Extending the Vertical Form Addition and Subtraction with Decimals to Hundredths [K2NO4]
Extending the Vertical Form Multiplication and Division with Decimals and Finding the Appropriate Place Value such as Product, Quotient and Remainder [K2NO5]
Using Multiples and Divisors for Convenience [K2NO6]
Introducing Improper and Mixed Fractions and Extending to Addition and Subtraction of Fractions to Dissimilar Fractions [K2NO7]
Extending Fractions as Numbers and Integrate [K2NO8]
Extending Multiplication and Division to Fractions [K2NO9]

Extending Numbers with Base Ten Up to a Billion and also to Thousandths with the Three-Digit Numeral System Gradually [K2NO1]

[K2NO1-1]
Extending numbers using the base-ten system up to a billion[36] with the three-digit numeral system[37]

i. Adopt the three-digit numeral system, extend numbers up to a billion with the idea of relative size of numbers
ii. Compare numbers such as larger, and smaller with the base-ten[38] system of place values through visualisation of the relative size of numbers using cube, plane (flat), bar (long) and unit

[K2NO1-2]
Extending decimal numbers to hundredths, and thousandths[39]

i. Use the idea of quantity and fractions, and extend decimal numbers from tenths to hundredths

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36. *Billion is too large for counting and it is introduced in the three-digit system under the relative size of number.*
37. *In the British system, it is referred to as short scale.*
38. *Metric system names of units are discussed under measurement and relations.*
90. *Under the three-digit system, if we teach until thousandths, we can extend by three-digits.*

i. Compare decimal numbers such as larger, and smaller with the base-ten system of place value
ii. Adopt the ways of extension up to thousandths and so on, and compare the relative sizes

Making Decisions of Operations on Situations with Several Steps and Integrating them in One Expression, thinking about the Order of Calculations and Producing the Rule (PEMDAS) [K2NO2]

[K2NO2-1]
Finding easier ways of calculations using the idea of various rules of calculations[40] such as the associative, commutative and distributive rules

i. Find the easier ways of addition and subtraction and use them, if necessary, such as the answer is the same if add the same number to the subtrahend and minuend
ii. Find the easier ways of multiplication and division and use them in convenient ways such as 10 times multiplicand produce the product 10 times
iii. Use associative, commutative and distributive rules of addition and multiplication for easier ways of calculation, however, the commutative property does not work in subtraction and division
iv. Appreciate the use of simplifying rules of calculations

[K2NO2-2]
Thinking about the order of calculations in situations and producing rules and order of operations

i. Integrate several steps of calculation into one mathematical sentence
ii. Produce the rule of PEMDAS and apply it to the multi-step situation
iii. Think about the easier order of calculation and acquiring fluency in PEMDAS and rules with appreciation

Producing the Standard Algorithm Using Vertical Form Division with Whole Numbers [K2NO3]

[K2NO3-1]
Knowing the properties of division and using it for an easier way of calculation

i. Find the easier ways of division and use them, if necessary, such as the answer is the same if multiplying the same number by the dividend and divisor
ii. For confirmation of the answer of division, use the relationship among divisor, quotient and remainder and appreciate the relationship

[K2NO3-2]
Knowing the algorithm of division in vertical form and acquiring fluency

i. Know the division algorithm with tentative quotient and confirm the algorithm by the relationship among divisor, quotient and remainder
ii. Interpret the meaning of quotient and remainder in situations
iii. Acquire fluency for division algorithm in the case of up to 3-digit whole number divided by 2-digit
iv. Think about the situations with or without remainder in relation to situations for quotative and partitive division

Extending the Vertical Form Addition and Subtraction with Decimals to Hundredths [K2NO4]

[K2NO4-1]
Extending the vertical form addition and subtraction in decimals to hundredths

i. Extend the vertical form addition and subtraction to hundredths[41] place and explain it with models
ii. Appreciate the use of addition and subtraction of decimals in their life

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40. *In Measurement and Relations, the use of constant sum, difference, product and quotient are described. (e.g.25 -21 = 4, 26–22 = 4, 27– 23 = 4)*
41. *Discussion of decimals to hundredths is related to the use of money. It is a minimum requirement. If teaching to hundredths, further extension of place value can be understood.*

Extending the Vertical Form Multiplication and Division with Decimals and Finding the Appropriate Place Value Such as Product, Quotient and Remainder [K2NO5]

[K2NO5-1]
Extending the multiplication from whole numbers to decimal numbers

i. Extend the meaning of multiplication with the idea of measurement by the number of unit length for multiplication of decimal numbers and use diagrams such as number lines to explain them with appreciation in situations
ii. Extend the vertical forms multiplication of decimals up to 3 digits by 2 digits with consideration of the decimal places step-by-step
iii. Obtain fluency using multiplication of decimals with sensible use of calculators in learners' life
iv. Develop number sense in the multiplication of decimals[42] such as comparing sizes of products before multiplying

[K2NO5-2]
Extending the division from whole numbers to decimal numbers

i. Understand how to represent division situations using diagrams such as number lines, and extend the diagram of decimal numbers for explaining division by decimal numbers
ii. Extend the division algorithm in the vertical form of decimal numbers and interpret the meaning of decimal places of quotient and remainder with situations
iii. Acquire fluency in the division algorithm of decimals up to 3 digits by 2 digits with consideration of decimal places step-by-step
iv. Obtain fluency using division of decimals with sensible use of calculators in learners' life
v. Develop number sense in the division of decimals such as comparing sizes of quotients before multiplying
vi. Distinguish the situations with decimal numbers of multiplication and division

Using Multiples and Divisors for Convenience [K2NO6]

[K2NO6-1]
Using multiples and divisors for convenience with appreciation to enrich number sense

i. Understand sets of numbers by using multiples and divisors
ii. Find common multiples and appreciate their use in situations, and enrich number sense with figural representations such as an arrangement of rectangles to produce a square
iii. Find a common divisor and appreciate its use in situations, and enrich number sense with figural representations such as dividing a rectangle into pieces of square
iv. Understand numbers as a composite of multiplication of numbers as factors[43]
v. Appreciate ideas of prime, even and odd numbers in situations using multiples and divisors
vi. Acquire the sense of numbers to see the multiples and divisors for convenience

Introducing Improper and Mixed Fractions and Extending to Addition and Subtraction of Fractions to Dissimilar Fractions [K2NO7]

[K2NO7-1]
Extending fractions to improper, mixed and equivalent fractions

i. Extend fractions to improper and mixed fractions using a number line of more than one by measuring with a unit fraction[44]
ii. Find ways to determine equivalent fractions with number lines and with the idea of multiple numerators and denominators
iii. Compare fractions using number lines and the idea of multiple

42. *Applying the idea of multiplication into ratio, percent and proportion is discussed in Measurement and Relations.*
43. *This idea is related to the strand on Measurement and Relations in Key Stage 2, for the area of a rectangle.*
44. *Extension of a fraction to more than one is done by using the fraction with quantity for a situation such as* $\frac{4}{3}m$.

[K2NO7-2]
Extending addition and subtraction of similar fractions to improper and mixed fractions, and dissimilar fractions

i. Extend addition and subtraction of similar fractions to proper and mixed fractions with explanations using models and diagrams
ii. Extend addition and subtraction into dissimilar fractions with explanations using diagrams and common divisors
iii. Acquire fluency in addition and subtraction of fractions with appreciation of ideas to produce the same denominators

Extending Fractions as Numbers and Integrate[45] [K2NO8]

[K2NO8-1]
Seeing fractions[46] as decimals and seeing decimals as fractions

i. See fractions as decimals using division and define quotient with divisible which includes repetition of the remainder
ii. See decimals as fractions such as hundredths are per hundred
iii. Compare decimals and fractions and order them on a number line

Extending Multiplication and Division to Fractions [K2NO9]

[K2NO9-1]
Extending multiplication to fractions

i. Extend multiplication to fractions with situations using diagrams such as number lines step by step, and find the simple algorithm for the multiplication of fractions
ii. Acquire fluency in the multiplication of fractions
iii. Develop number sense[47] of multiplication of fractions such as comparing sizes of products before multiplying

[K2NO9-2]
Extending division to fractions

i. Extend division to fractions with situations using diagrams such as number lines step-by-step
ii. Acquire fluency in the division of fractions
iii. Develop number sense of division of fractions such as comparing sizes of quotients before dividing

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45. *Selecting the appropriate denomination of quantities and units for a fraction in the context* $(\frac{2}{3})m$ *is two of* $\frac{1}{3}m$ *and the whole is 1m, however,* $\frac{3}{3}m$ *is* $3x(\frac{1}{3})m$*; the structure is the same as tens are ten of units and discussed under Measurement and Relations.*
46. *Fraction as a ratio is introduced in Measurement and Relations.*
47. *Applying the idea of the multiplication of fractions into ratio, proportion, percentage and base is discussed in Measurement and Relations.*

Strand: Measurement and Relations [K2MR]

Additive quantities, such as angles, areas, and volume, and relational quantities, such as population density and speed, are introduced. Additive quantity can be introduced by establishing the standard unit, the same as the Quantity and Measurement of Key Stage 1. Relations of quantities in situations are discussed with patterns such as sum constant, difference constant, product constant and quotient constant using tables and represented by mathematical sentences and letters. Proportion and ratio are introduced with representations of diagrams, graphs, and tables for multiplication, and connected with decimals and fractions. Percent is introduced with diagrams in relation to ratio and proportion. Relational quantity is produced by different quantities with the understanding of ratio. The area of a circle is discussed through a proportional relationship between the radius and the circumference. Ideas of ratio and proportion are fluently applied to real-world problem solving.

Topics:

Introducing Angle and Measuring it [K2MR1]
Exploring and Utilising Constant Relation [K2MR2]
Extending Measurement of Area in Relation to Perimeter [K2MR3]
Extending Measurement of Volume in Relation to Surface [K2MR4]
Approximating Quantities [K2MR5]
Extending Proportional Reasoning to Ratio and Proportion [K2MR6]
Producing New Quantities Using Measurement Per Unit [K2MR7]
Investigating the Area of a Circle [K2MR8]
Exchanging Local Currency with Currency in the ASEAN Community [K2MR9]
Extending the Relation of Time and Use of Calendar in Life [K2MR10]
Converting Quantities in Various System of Units [K2MR11]
Showing Relationships Using a Venn Diagram [K2MR12]

Introducing Angle and Measuring It[48] [K2MR1]

[K2MR1-1]
Introducing angle by rotation, enabling measure and acquire fluency using the protractor

i. Compare the extent of rotation and introduce degree as a unit for measuring an angle
ii. Recognise right angle is 90 degrees, and adjacent angle of two right angles is 180 degrees, and 4 right angles are 360 degrees
iii. Acquire fluency in measuring angles using the protractor
iv. Draw equivalent angles with addition and subtraction using multiples of 90 degrees
v. Appreciate measurement of angles in geometrical shapes and situations in life[49]

Exploring and Utilising Constant Relation [K2MR2]

[K2MR2-1]
Exploring equal constant relation with the utilisation of letters to represent placeholders[50]

i. Explore two possible unknown numbers such that their sum (or difference/product/quotient) is constant,[51] for example, $\square+\Delta=12$ ($\square$ and Δ are placeholders).
ii. Use letters instead of placeholder[52] (empty box) to derive an equivalent relation
iii. Understand the laws for operations (e.g. associative, commutative and distributive etc.) to explain the simpler way of calculation
iv. Appreciate the use of diagrams such as number lines and areas to represent relations when finding solutions

48. *Right angle is learned at Key Stage 1 in Shapes, Figures and Solids for explaining the properties of figures.*
49. *Conservation of angles will be re-learnt in a triangle under Key Stage 2 Plane Figures and Space Figures.*
50. *The idea for the use of Numbers and Operations (Key Stage 2) in finding easier ways of calculations with the idea of rules of calculations.*
51. *Conservation of angles will be re-learnt in a triangle under Key Stage 2 Plane Figures and Space Figures.*
52. *The idea for the use of Numbers and Operations (Key Stage 2) in finding easier ways of calculations with the idea of rules of calculations.*

Extending Measurement of Area in Relation to Perimeter [K2MR3]

[K2MR3-1]
Introducing area and producing a formula for the area of a rectangle

i. Compare the extent of an area and introduce its unit, and distinguish it from the perimeter
ii. Introduce one square centimetre (as a unit for area and its operation using addition and subtraction
iii. Investigate areas of rectangles and squares and produce the formula of the area[53]
iv. Extend square centimetre to square metre and square kilometre for the measure of large areas
v. Convert units and use appropriate units of areas with fluency
vi. Draw the equivalent size of a rectangular area based on a given area with the factors of a whole number[54]
vii. Appreciate the use of areas in daily life such as comparing land sizes.

[K2MR3-2]
Extending the area of a rectangle to other figures to derive formulae

i. Explore and derive a formula for the area of a parallelogram by changing its shape to a rectangle without changing its area
ii. Explore and derive a formula for the area of a triangle by bisecting a rectangle into two triangles without changing its area
iii. Appreciate the idea of changing or dividing shapes of a rectangle, parallelogram, or/and triangle for deriving the area of other figures
iv. Use formulae to calculate areas in daily life

Extending Measurement of Volume in Relation to Surface [K2MR4]

[K2MR4-1]
Introducing volume from the area and deriving the formula for cuboid

i. Compare the extent of volume and introduce its unit, and distinguish it from the surface
ii. Introduce one cubic centimetre as the unit for volume and its addition and subtraction
iii. Investigate the volume of a cuboid and cube and produce the formulae
iv. Extend cubic centimetre to cubic metre to measure large volume
v. Convert units and use appropriate units of volume with fluency
vi. Appreciate the use of volume in life such as comparison of the capacity of containers

[K2MR4-2]
Extending the volume of a cuboid to other solid figures to derive formulae

i. Extend the formula for the volume of a cuboid as base area x height for exploring solid figures such as prism and cylinder
ii. Extend the formula for the volume of a prism and a cylinder to explore and derive the volume formula of a pyramid and cone
iii. Use the formulae to calculate volume in daily life

Approximating with Quantities [K2MR5]

[K2MR5-1]
Approximating numbers with quantities depending on the necessity of contexts

i. Understand the ways of rounding such as round up and round down
ii. Use rounding as an approximation for deciding on the quantity with related context
iii. Critique approximation beyond the context with a sense of quantity such as based on the relative size of units

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53. *Multiplications were studied in Key Stage 1 Number and Operations.*
54. *The idea of composite numbers such as 2 times 10 equals 5 times 4 is related to factors in extending the numbers and operations at the same Key Stage 2.*

Extending Proportional Reasoning to Ratio[55] and Proportion [K2MR6]

[K2MR6-1]
Extending proportional reasoning to ratio and percent for comparison

i. Understand ratio as a relationship between two same quantities or between two different quantities (the latter idea is rate)[56]
ii. Express the value of a ratio by quotient such as the rate of two different quantities[57]
iii. Understand percent as the value of a ratio with the same quantities[58] and the necessity of rounding
iv. Understand proportional reasoning for ratio as part-whole and part-whole relationships
v. Apply the rule of three[59] in using ratio

[K2MR6-2]
Extending proportional reasoning to proportion

i. Extend proportional reasoning to multiplication tables as equal ratios and understand proportions
ii. Understand proportion by multiple and a constant quotient, not changing the value of the ratio[60]
iii. Demonstrate simple inverse proportion by constant product[61]
iv. Express proportion in a mathematical sentence by letters and graph[62]
v. Use properties of proportionality to predict and explain phenomena in daily life

Producing New Quantities Using Measurement Per Unit [K2MR7]

[K2MR7-1]
Producing new quantities using measurement per unit

i. Introduce average as units for distribution and comparison of different sets of values
ii. Introduce population density with the idea of average and appreciate it for comparison
iii. Introduce speed with the idea of average and appreciate it for comparison
iv. Appreciate using diagrams such as number lines and tables to decide the operations on the situations of measurement per unit quantity
v. Comparing the context of different quantities with the idea of average as rate[63]
vi. Apply the idea of measurement per unit quantity in different contexts[64]

Investigating the Area of a Circle [K2MR8]

[K2MR8-1]
Areas of a circle are discussed through the relationship between the radius and circumference

i. Investigate the relationship between the diameter of a circle and its circumference using the idea of proportion
ii. Investigate the area of a circle by transforming it into a triangle or parallelogram and find the formula of the circle

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55. *Band graph and pie chart for representing ratio are discussed under Key Stage 2 under the strand Data Handling and Graphs.*
56. *Ratio of different quantities is a rate. The ratio of the same quantities has a narrow meaning of ratio.*
57. *The value of a fraction as a ratio is not necessarily a part of a whole in situations. Fraction as a ratio is usually used in the context of multiplication situations, where the denominator is the base or a unit for comparison.*
58. *Percent is used in Data Handling and Graphs.*
59. *Rule of three is the method on the table to find one unknown term from the three known terms using proportional reasoning such as;*

a	c
?	d

60. *Enlargement is discussed in Key Stage 2 under Plane Figures and Space Figures. The graph is treated at Key Stage 2 under the strand Data Handling and Graphs.*
61. *Proportion and Inverse proportion are necessary in Key Stage 3 in science.*
62. *This will be discussed in detail in the same Key Stage 2 under Data Handling and Graphs.*
63. *On Number and Operations Key Stage 2, the rate is the value of division as quotient.*
64. *Using measurement per unit quantity with fluency to make logical judgments in daily life, refer to Key Stage 2 Data Handling and Graphs.*

i. Estimate the area of inscribed and circumscribed shapes based on a known formula of area[65]
ii. Enjoy to estimate the area of irregular shapes with fluency in life

Exchanging Local Currency with Currency in the ASEAN Community [K2MR9]

[K2MR9-1]
Exchanging local currency in the ASEAN community with the idea of rate

i. Extend the use of ratio for currency exchange (rate of exchange)
ii. Apply the four operations for money in appropriate notation in life
iii. Appreciate the value of money

Extending the Relation of Time and Use of Calendar in Life [K2MR10]

[K2MR10-1]
Extending the relation of time and use of calendar in life

i. Convert time in the 12-hour system with the abbreviation a.m. and p.m. to the 24-hour system and vice versa
ii. Investigate the numbers in a calendar to relate days, weeks, months and years using the idea of number patterns
iii. Appreciate the significance of various calendars in life

Converting Quantities in Various System of Units [K2MR11]

[K2MR11-1]
Converting measurement quantities based on international and non-international systems with the idea of base-10

i. Convert measurement system of metre and kilogram with prefixes deci-, centi-, and milli-, and with deca, hecto-, and kilo-
ii. Convert measurement system of litre with cubic centimetre
iii. Convert measurement system of area using are (a) and hectare (ha) with square meter
iv. Convert measurement of local quantities with standard quantities
v. Understand the unit system with power, such as metre, square metre and cubic metre

Showing Relationship Using a Venn Diagram [K2MR12]

[K2MR12-1]
Using a Venn diagram to show relationships between numbers and figures for making a clear logical deduction

i. Sort objects by their defining characteristics
ii. Show relationships of squares, rectangles, rhombus, parallelograms, trapeziums and quadrilaterals by using a Venn diagram
iii. Show the relationship between numbers
iv. Critique ambiguous reasoning by using a Venn diagram to make clear a definition

65. *Relationships on polygons and circles are discussed in Key Stage 2 under Plane Figures and Space Figures.*

Strand: Plane Figures and Space Figures [K2PS]

Through tessellation, figures can be extended through plane figures. Parallelograms and perpendicular lines are tools to explain the properties of triangles and quadrilaterals as plane figures. They are also needed for identifying and recognising symmetry and congruency. Plane figures are used to produce solids in space and vice versa. Opening faces of solids would produce plane figures referred to as nets. Activities related to building solids from plane figures are emphasised and encouraged to facilitate finding the area of a circle through numerous sectors of the circle to construct a rectangle. Circles are used for explaining the nets of cylinders.

Topics:

Exploring Figures with their Components in the Plane [K2PS1]
Exploring Space Figures with their Components in Relation to the Plane [K2PS2]
Exploring Figures with Congruence, Symmetry and Enlargement [K2PS3]

Exploring Figures with their Components in the Plane [K2PS1]

[K2PS1-1]
Exploring figures with their components in the plane and using their properties

i. Examine parallel lines and perpendicular lines by drawing with instruments
ii. Examine quadrilaterals using parallel and perpendicular lines, and identify parallelogram, rhombus, and trapezium by discussion
iii. Find properties of figures through tessellations such as a triangle where the sum of the angles is 180 degrees, a straight angle
iv. Extend figures to polygons and expand them to circles by knowing and using their properties

Exploring Space Figures with their Components in Relation to the Plane [K2PS2]

[K2PS2-1]
Exploring rectangular prisms and cubes with their components

i. Identify the relationship between faces, edges and vertices for drawing sketch
ii. Explore nets of a rectangular prism and find the corresponding position between components
iii. Explore the perpendicularity and parallelism between the faces of a rectangular prism
iv. Explain positions in rectangular prisms with the idea of 3-dimensions

[K2PS2-2]
Extending rectangular prism to other solids such as prisms and cylinders

i. Extend the number of relationships between faces, edges and vertices for drawing of sketch
ii. Explore nets of prisms and cylinders, and find the corresponding position between components
iii. Distinguish prism and cylinder by the relationship of their faces

Exploring Figures with Congruence, Symmetry and Enlargement [K2PS3]

[K2PS3-1]
Exploring the properties of congruence

i. Explore properties of figures which fit when overlapped and identify conditions of congruency with corresponding points and sides
ii. Draw congruent figures using minimum conditions and confirm by measuring angles and sides
iii. Appreciate the usefulness of congruent figures by tessellation

[K2PS3-2]
Exploring the properties of symmetry

i. Explore the properties of figures which reflect and identify conditions of symmetry with line and its correspondence
ii. Draw symmetrical figures using conditions in an appropriate location
iii. Appreciate the usefulness of symmetry in designs

[K2PS3-3]
Exploring the properties of enlargement[66]

i. Explore properties of figures in finding the centre of enlargement in simple cases such as a rectangle
ii. Draw an enlargement of a rectangle using the ratio (multiplication of the value of the ratio)[67]
iii. Appreciate the usefulness of enlargement in the interpretation of a map

66. *General cases will be discussed in Key Stage 3, Space and Geometry.*
67. *Ratio and rate are discussed in Key Stage 2 under Measurement and Relations.*

Strand: Data Handling and Graphs [K2DG]

The process of simple data handling is introduced through data representation such as using a table, bar graph, line graph, bar chart and pie chart. Graphs are utilised depending on the qualitative and quantitative data used such as bar graphs for distinguishing and counting in every category. The discussion of producing the line graph includes taking data at specific intervals, using suitable scales, and using slopes. A histogram is necessary for interpreting the data representation of social studies and science and is also used as a special type of bar graph. Average is introduced based on the idea of ratio for making the dispersion of bar chart even, and used for summarising and comparing data on a table. *Problem-Plan–Data- Analysis–Conclusion* (PPDAC) cycles are experienced through the process of data handling by using those data representation skills. Those are necessary for using mathematical thinking and computational thinking in our lives.

Topics

Arranging Tables for Data Representations [K2DG1]
Drawing and Reading Graphs for Analysing Data [K2DG2]
Using Graphs in the PPDAC Cycle [K2DG3]
Applying Data Handling for Sustainable Living [K2DG4]

Arranging Tables for Data Representations [K2DG1]

[K2DG1-1]
Collecting and arranging data

i. Explore how to collect multi-category data based on a situation
ii. Explore how to arrange and read multi-category data on appropriate tables.
iii. Appreciate the use of multi-category tables in situations

Drawing and Reading Graphs for Analysing Data [K2DG2]

[K2DG2-1]
Drawing and reading line graphs to know the visualised pattern as the basis for the tendency of change

i. Introduce line graphs based on appropriate situations such as rainfall, temperature and others
ii. Distinguish line graph from bar graph for observation such as increase, decrease, and no-change
iii. Introduce the graph of proportion using the idea of a line graph and read the gradient by a constant ratio[68]
iv. Appreciate the line graph in various situations

[K2DG2-2]
Drawing and reading band graphs and pie charts for representing ratios in a whole[69]

i. Explore how to scale the band or circle to represent the ratio or percent
ii. Use the band graph and pie chart for comparison of different groups
iii. Appreciate the band graph and pie chart in a situation

[K2DG2-3]
Reading histogram[70] for analysing frequency distribution

i. Draw a simple histogram[71] from the frequency table of situations
ii. Read various histograms for analysing data distribution
iii. Use averages[72] (mean) to compare different groups in the same situation with histograms

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68. *Proportions are learnt in the Key Stage 2 under Measurement and Relations.*
69. *Ratio is learnt under Key Stage 2 Measurement and Relations.*
70. *How to draw a histogram is discussed in Key Stage 3 under Statistics and Probability. Reading histograms is necessary in social studies and science.*
71. *Using ICT for drawing graphs will be mentioned in mathematical activities.*
72. *Mean is introduced as average in Key Stage 2 under Measurement and Relations.*

Using Graphs in the PPDAC[73] Cycle [K2DG3]

[K2DG3-1]
Identifying appropriate graphs for problem solving in data handling using the PPDAC cycle

i. Analyse a problem situation and discuss the expected outcomes before collecting data to clarify the purpose of a survey
ii. Plan the survey for the intended purpose
iii. Collect the data based on the purpose of the survey
iv. Use appropriate graphical representation which is most suitable for the purpose
v. Appreciate the use of graphs before making the conclusion

Applying Data Handling for Sustainable Living [K2DG4]

[K2DG4-1]
Applying data handling for sustainable development[74] and appreciating the power of data handling for predicting the future

i. Read data related to sustainable development on SDGs and adopting positive views for the betterment of society
ii. Understand the idea of probability as ratio and percentage in reading the data for situations related to sustainable development
iii. Experience implementing a project of reasonable size in data handling to achieve sustainable development and appreciate the power of data handling

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73. *PPDAC itself will be described in the mathematical activities later.*
74. *This standard is related to SDG as inter-subject content between social studies and science.*

Strand: Mathematical Process – Humanity [K2MH]

As a follow-up of Key Stage 1, activities are designed to enable an appreciation of knowledge and skills learned and the ways of learning, such as applying knowledge of number sense to solve daily problems. Mathematical processes such as communication and reasoning explain mathematical problems and modelling. The ability to connect and reason mathematical ideas would trigger excitement among learners. Discussions of misconceptions are usually enjoyable and challenging. Mathematics learning usually begins from situations at Key Stage 1. In Key Stage 2, the development of mathematics is possible through discussions for the extension of the forms. Appreciation of ideas and representations learned becomes part of the enjoyable activities. The application of learning becomes meaningful through the consistent use of representations such as diagrams.

Standards:

Enjoying problem-solving through various questioning and conjecturing for extension of operations into decimals and fractions with proportionality and new quantities such as area and volume [K2MH1]
Enjoying measuring through settings and using the area and volume in situations [K2MH2]
Using ratio and rate in situations [K2MH3]
Using number lines, tables, and area diagrams for representing operations and relationships in situations [K2MH4]
Establishing the idea of proportion to integrate various relations with the consistency of representations [K2MH5]
Enjoying tiling with various figures and blocks [K2MH6]
Producing valuable explanations based on established knowledge, shareable representations and examples [K2MH7]
Performing activities of grouping and enjoy representing with Venn diagram [K2MH8]
Experiencing the PPDAC (Problem-Plan-Data-Analysis-Conclusion) cycle and modelling cycle in simple projects in life [K2MH9]
Preparing a sustainable life with number sense and mathematical representations [K2MH10]
Utilising ICT tools as well as notebooks and other technological tools [K2MH11]
Promoting creative and global citizenship for sustainable development of community using mathematics [K2MH12]
Represent recursive process by using manipulatives, drawings and tables for finding patterns [K2MH13]

[K2MH1]
Enjoying problem-solving through various questioning and conjecturing for extension of operations into decimals and fractions with proportionality and new quantities such as area and volume[75]

i. Pose questions to develop a division algorithm in vertical form using multiplication and subtraction
ii. Pose questions to develop multiplication and division of decimal numbers using the idea of proportionality with tables and number lines
iii. Pose questions to develop multiplication and division of fractions using the idea of proportionality with tables, area diagrams and number lines
iv. Pose questions to extend multiplication and division algorithm in vertical form to decimal numbers and discuss decimal points
v. Pose questions to use decimals and fractions in situations
vi. Pose questions to use area and volume in life
vii. Pose questions to use ratio and rate in life
viii. Pose conjectures based on ideas learned, such as when multiplying, the answer becomes larger

[K2MH2]
Enjoying measuring through settings and using the area and volume in situations

i. Compare directly and indirectly areas and volumes
ii. Set tentative units from difference for measuring area and volume[76]
iii. Give the formula for the area and volume for counting units
iv. Use measurement for communication in daily life

75. *Proportions are learnt in the Key Stage 2 under Measurement and Relations.*
76. *Ratio is learnt under Key Stage 2 Measurement and Relations.*

[K2MH3]
Using ratio and rate in situations[77]

i. Understand division as partitive (between different quantities) and quotative (between the same quantity) in situations
ii. Develop the idea of ratio and rate utilising the idea of average and per unit with tables and number lines
iii. Communicate using the idea of population density and velocity in life

[K2MH4]
Using number lines, tables, and area diagrams for representing operations and relations in situations[78]

i. Represent proportionality on number lines with the idea of multiplication tables
ii. Use number lines, tables, and area diagrams to explain operations and relations of proportionality in situations

[K2MH5]
Establishing the idea of proportion to integrate various relations with the consistency of representations[79]

i. Use the idea of proportion as the relation of various quantities in life
ii. Identify through the idea of proportion using tables, letters, and graphs
iii. Adopt the idea of proportion to angles, arcs and areas of circles
iv. Adopt the idea of proportion to area and volume
v. Adopt the idea of proportion to enlargement
vi. Use ratio for data handling such as percent and understand the difficulties of extending it to proportion

[K2MH6]
Enjoying tiling with various figures and blocks[80]

i. Appreciate producing parallel lines with a tessellation of figures
ii. Explain the properties of figures in tessellations by reflections, rotations and translations
iii. Develop nets from solids and explain the properties of solids by each of the component figures
iv. Use the idea of tiling for calculating the area and volume

[K2MH7]
Producing valuable explanations based on established knowledge, shareable representations and examples

i. Establish the habit of explanation by referring to prior learning and ask questions using terms such as why, how, what, if and if not, and reply using examples and 'for example' in a discussion
ii. Assessing the appropriateness of explanations using representations such as generality, simplicity and clarity
iii. Use other's ideas to produce a better understanding
iv. Use inductive reasoning for extending formulae

[K2MH8]
Performing activities of grouping and enjoying representing with Venn diagram

i. Use the idea of the Venn diagram for social study
ii. Understand classifications based on characteristics and represent them by using Venn diagrams

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77. *Ratio and proportion bridge multiplication and division in a situation of two quantities with reference to the Extension of Numbers and Operations and Measurement and Relations.*
78. *This is a bridge to the Extension of Numbers and Operations and Measurement and Relations.*
79. *Bridge to the three strands, Measurement and Relations, Plane Figures and Space Figures and Data Handling and Graphs.*
80. *Connected to the two strands, Measurement and Relations, and Plane Figures and Space Figures.*

[K2MH9]
Experiencing the PPDAC (Problem-Plan-Data-Analysis-Conclusion) cycle and modelling cycle in simple projects in life

i. Understand the problem of context
ii. Plan appropriate strategies to solve the problem
iii. Gather data and analyse using appropriate methods and tools
iv. Draw a conclusion with justification based on data analysis

[K2MH10]
Preparing a sustainable life with number sense and mathematical representations[81]

i. Use minimum and sequential use of resources in situations
ii. Use data with number sense, such as order of quantity and percentage, for the discussion of matters related to sustainable development
iii. Estimate the efficient use of resources in situations
iv. Maximise the use of resources through an appropriate arrangement in a space such as a room
v. Understand “equally likely” of resources in situations

[K2MH11]
Utilising ICT tools as well as notebooks and other technological tools

i. Use internet data for the discussion of matters related to sustainable development
ii. Distinguish appropriate or inappropriate qualitative and quantitative data for using ICT
iii. Use calculators for organising data such as average
iv. Use calculators for operations in necessary context
v. Use projectors for sharing ideas as well as board writing
vi. Enjoy using notebooks to exchange learning experiences with each other such as in mathematics journal writing
vii. Use protractors, triangular compasses, straight edges, and clinometers for drawing and measuring
viii. Use the idea of proportionality to use mechanisms such as rotating once and moving twice (wheels, gears)
ix. Use various tools for conjecturing and justifying

[K2MH12]
Promoting creative and global citizenship for sustainable development of community using mathematics

i. Utilise notebooks, journal books and appropriate ICT tools to record and find good ideas and share them with others
ii. Prepare and present ideas using posters and projectors to promote good practices in the community
iii. Listen to others’ ideas and ask questions for producing better designs
iv. Utilise information, properties, models and visible representations as the basis for reasoning
v. Utilise practical arts, home economics and outdoor studies to investigate local issues for improving the welfare of life

[K2MH13]
Represent recursive process by using manipulatives, drawings and tables for finding patterns

i. Analysing situations by multiple action processes and alternating them with manipulatives, drawings and tables
ii. Find the structure of repetitions which is used in repeated actions previously
iii. Confirm recursive structures by using manipulatives, drawings and tables

81. *It is related to Numbers and Operations, Pattern and Data Representations and Quantity and Measurement all under Key Stage 1.*

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