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筑波大学
University of Tsukuba

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SEAMEO Basic Education Standards (SEA-BES): Common Core Regional Learning Standards (CCRLS) in Mathematics for Primary and Lower Secondary Levels *Second Edition*

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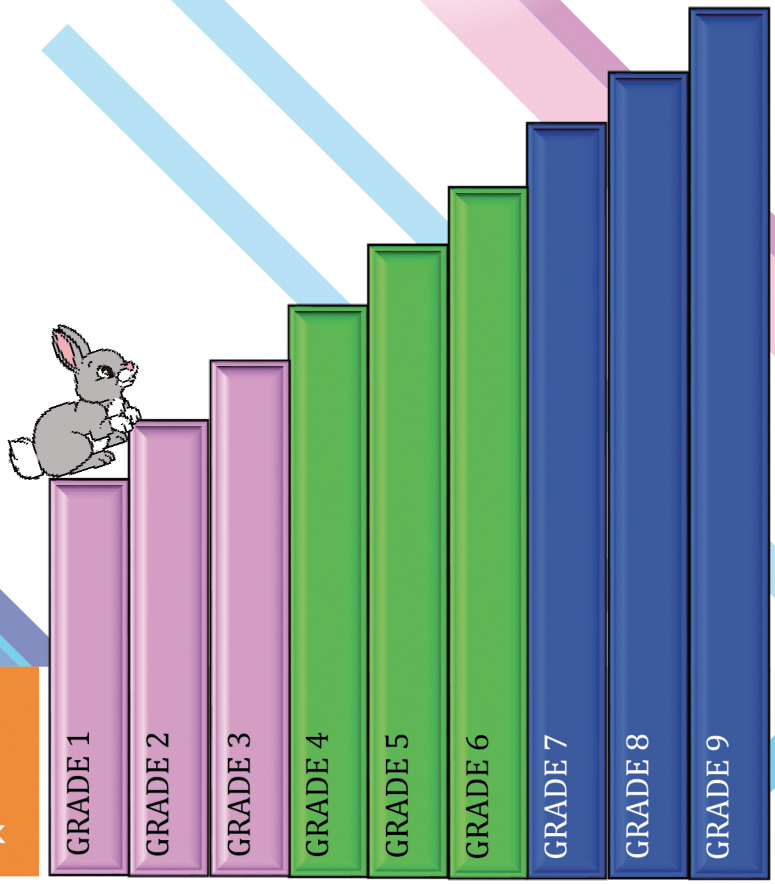

Editors
Isoda Masami
Teh Kim Hong
Gan Teck Hock

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FOREWORD

I congratulate SEAMEO RECSAM for presenting the second edition of the SEAMEO Basic Education Standards (SEA-BES): Common Core Regional Learning Standards (CCRLS) in Mathematics, building on the foundation established by the first edition published in 2017, under the endorsement in the Educational Ministers Meeting. This book encompasses the collaborative efforts of SEAMEO member countries, particularly SEAMEO RECSAM and SEAMEO Qitep in Mathematics (SEAQIM), and the University of Tsukuba (UT), Japan as an affiliated member of SEAMEO.

For the 21st-century curriculum framework, we need to consider soft skills, hard skills, cognitive skills, as well as values and attitudes. The core of the curriculum framework is the value and attitude. The original CCRLS in Mathematics has already included values, attitudes, and thinking skills. Such aspects are also enhanced and emphasised in the Learning Compass of the OECD Education 2030.

In the era of generative artificial intelligence (AI), the second edition clarifies the bases for computational thinking from the perspective of mathematics. Mathematical ideas, thinking and processes, values and attitudes are explained more clearly. We sincerely recommend those terminologies for describing the curriculum objectives and classroom practices for student learning. It is commendable that three guidebooks, namely Mathematic Challenges for Classroom Practices at the Lower Primary Level, Upper Primary Level and the Lower Secondary Level, have been published and revised to support the teachers/educators.

Acknowledgement is directed to UT-CRICED for providing continuous cooperation and commitment to developing the CCRLS in Mathematics. Furthermore, supporting resources in the form of Teachers' guidebooks for lower primary, upper primary, and lower secondary school mathematics have also been produced by SEAMEO RECSAM, SEAQIM and UT-CRICED in subsequent projects since the first edition. The SEAMEO STEM-Ed project, Unplugged Computational Thinking, provided the opportunity for cooperation and collaboration to concretise computational thinking as mathematical thinking. SEAMEO STEM-Ed project members are STEM-ED; CRICED, University of Tsukuba; CIAE, University of Chile; SEAQIM; SEAMOLEC; RECSAM and IRDTP, Khon Kaen University.

Last but not least, I expressed my most profound appreciation to all contributing members to this project. As a team, we shall strive toward strengthening the professionalism of teachers and educators for the well-being of our learners. The dedication and enthusiasm of everyone involved in the project are the cornerstones of our success.

DATUK DR HABIBAH ABDUL RAHIM
Director, SEAMEO Secretariat, Bangkok, Thailand

FOREWORD

In the fast-paced modern world, the significance of laying a sturdy foundation in mathematics cannot be overstated. It equips students with analytical tools, problem-solving skills, and critical thinking abilities essential for success across various fields. This very ethos lies at the core mission of the Southeast Asian Ministers of Education Organisation: Regional Centre for Education in Science and Mathematics (SEAMEO RECSAM).

It is a distinct honour for me to represent SEAMEO RECSAM in light of this transformative journey. Our vision is to be a leading centre for quality science and mathematics education, dedicated to promoting and enhancing Science and Mathematics Education in the member countries. This commitment drives our efforts to develop initiatives such as the Common Core Regional Learning Standards (CCRLS) in Mathematics which is endorsed by the educational ministers, now in its second edition. This book aims to equip teachers with knowledge and skills to guide their students to develop contemporary skills crucial for thriving in our dynamic world. This initiative brings us closer to inspiring Southeast Asian students to learn mathematics that will spark wonder and joy in their hearts instead of invoking fear.

The CCRLS in Mathematics emphasises student agency. It empowers students to participate actively in their educational journey and ignites a lifelong passion for mathematics. This remarkable achievement speaks volumes about the relentless dedication of curriculum professionals, educators, and local experts who poured their hearts into crafting this invaluable resource. The accessibility of the document and its user-friendly structure ensures that all stakeholders can seamlessly engage with it. Furthermore, producing three other accompanying teacher's guidebooks also serves as a treasure trove of resources, enriching classroom experiences and fostering meaningful learning interactions.

At SEAMEO RECSAM, we believe that education goes beyond the confines of textbooks and classrooms; it's about nurturing well-rounded individuals proficient in mathematics and embodying values of empathy, integrity, and social responsibility. As teachers embark on their journey with the Second Edition of SEA-BES: CCRLS in Mathematics, they are not just teaching their students to acquire mathematical skills; they are also cultivating a mindset that values collaboration, diversity, and global citizenship. The development of the Second Edition of SEA-BES: CCRLS in Mathematics aims to nurture teachers who can foster and create an educational environment where students not only excel academically but also emerge as compassionate leaders who can navigate complexities and contribute positively to the betterment of society.

Together, let us embark on this exhilarating journey toward a future where teachers can nurture within every Southeast Asian student a love for learning through mathematics, empowerment, and inspiration. Lastly, I would like to extend my heartfelt gratitude to all those whose remarkable efforts and steadfast dedication have brought this vision to fruition.

Dr. AZMAN BIN JUSOH
Centre Director, SEAMEO RECSAM, Penang, Malaysia

FOREWORD

I am pleased to present the second edition of the SEAMEO Basic Education Standards (SEA-BES): Common Core Regional Learning Standards (CCRLS) in Mathematics for Primary and Lower Secondary Levels. The new edition is necessary to keep abreast with the current curriculum development frameworks and align with the Learning Compass of the OECD Education 2030.

In retrospect, the production of the initial CCRLS document was a very challenging yet rewarding endeavour right from its initiation until the publication of the document in 2017. The constraints of limited utilisation within a confined community of readership had created another opportunity to develop a series of three guidebooks from 2018 until 2023, catered for teachers and educators to help them understand the framework and the learning standards in the CCRLS. The three guidebooks are Mathematics Challenges for Classroom Practices at the Lower Primary Level, Upper Primary Level and Lower Secondary Level. In the process of preparing the three guidebooks, some shortcomings in the writing of the learning standards were identified. For clarity, many corrections and improvements have been made since the first guidebook was prepared in 2018. Hence, the Revised CCRLS Framework in Mathematics was used in the three guidebooks. Likewise, terminologies for mathematical thinking and processes are explained and included as an appendix to enhance the understanding of the framework. Compiling and restructuring all changes as a second edition is necessary at this juncture. Besides this, concretising computational thinking as mathematical thinking (STEM-ED, 2024) provided the opportunity to streamline the scope of higher-order thinking, encompassing computational thinking as an aspect of mathematical thinking. In addition, SEAMEO Qitep in Mathematics (SEAQIM) supports the use of CCRLS in Mathematics through their MaRWA project and the Encyclopaedia project.

In these engagements, the support of CRICED University of Tsukuba in funding and professional expertise is highly appreciated. In addition, I would like to acknowledge the contribution of supporting institutions, namely, RECSAM and SEAQIM, in reviewing the document. Last but not least, I would like to thank my team of co-editors, Teh Kim Hong and Gan Teck Hock, for working committedly on this project since 2018. I hope this new edition will be shared widely in SEAMEO countries and beyond to upgrade the quality of mathematics education.

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PREFACE FOR SECOND EDITION

SEAMEO RECSAM, with its mandate to promote science and mathematics education in the Southeast Asian region, successfully published the SEAMEO Basic Education Standards (SEA-BES): CCRLS in Mathematics and Science (SEAMEO RECSAM, 2017) with the collaboration of SEAMEO centres in member countries and University of Tsukuba (UT), Japan, as the SEAMEO affiliate member. The standards document emphasises the competency of the OECD (2005), which is aligned with formulating a 21st-century curriculum that cultivates basic human characters, creative human capital, and well-qualified citizens through mathematics and science.

In the past decade, Artificial Intelligence (AI) and data science based on mathematical sciences become the core of the revolution promoting Digital Transformation (DX) and STEM production. Similarly, in the past few years, Generative AI promoted the necessity of competency for everyone to handle AI using natural language. The second edition of CCRLS in mathematics was revised from the 2017 first edition to be aligned with these newest perspectives.

The objective of CCRLS in Mathematics is to provide the analytical tools for higher-order thinking Skills (HOTS) from the perspective of mathematical thinking. It also provides descriptive terminologies in the mathematics curriculum that concretise the development of mathematical thinking in classrooms. OECD Learning Compass 2030 proposed the concept of Student Agency, which refers to students' capacity to take responsible action to effect positive change in their lives and the world. In other words, it means that students who can learn mathematics through thinking and communicating by and for themselves will positively influence their learning. The CCRLS in Mathematics Framework consists of three core components: (i) the Contents, (ii) Mathematical Thinking and Processes, and (iii) Mathematical Values, Attitudes and Habits for Human Character. All these components are also aimed at developing the Student Agency. This second edition elaborates the standards more clearly and adds some necessary terminologies of mathematics related to computational thinking in the Generative AI Era (APEC, InMside; Isoda & Araya et al. 2021).

Based on these backgrounds, to permeate the CCRLS in Mathematics (2017) for improvement of teaching, RECSAM collaborated further with UT-CRICED to embark on an additional project starting in 2018 to produce three volumes of the teachers' guidebook, written based on the regional standards in mathematics. The guidebooks are titled Mathematics Challenges for Classroom Practices at the i) Lower Primary Level, ii) Upper Primary Level and iii) Lower Secondary Level. Contents presented in the three books are task-based and aimed to help users understand and interpret the standards. Ideas in books are helpful, serving as resources for classroom teaching. Due to the demand for understanding the standards, CCRLS in Mathematics has continuously been edited since 2018 to stay current with the latest developments in Mathematics content and the newest reform issues. These guidebooks improved perspectives and explanations for the CCRLS in Mathematics to overcome ambiguity issues and stay current.

Additionally, in March 2023, at SEAMEO RECSAM, UT-CRICED with the University of Chile and SEAMEO STEM-ED proposed to begin the project on 'Unplugged Computational Thinking'. The project also extends its collaboration with SEAMEO RECSAM, SEAMEO Qitep in Mathematics, SEAMEO Regional Open Learning Center (SEAMOLEC) and Khon Kaen University. It has provided fruitful discussions on the relationship between Computational and Mathematical Thinking in the one-year project (SEAMEO STEM-ED; Kritchasai, Isoda & Araya, 2024). Similarly, SEAQIM has continuously supported the CCRLS project by using the document in their MaRWA and the Encyclopedia projects.

With these constant efforts in mathematical emphasis and perspective since 2018, the CCRLS document has continuously improved. With the complete publication of the guidebooks, the minor edited version of the CCRLS in mathematics is also completed. For the reasons discussed above and the editing done, it is necessary for CCRLS in Mathematics, the first edition of CCRLS, 2017, to be published as a separate document for the second edition.

The second edition covers a minor revision for Chapters 1 to 4 on the description of the mathematics part and two appendixes. Chapter 1, Framework for CCRLS in Mathematics, has been rewritten to improve its clarity, whereas Chapters 2, 3 and 4 illustrate the learning standards with minor revisions on Key Stages 1, 2 and 3, Respectively. Appendix A illustrates the terminologies explained for Mathematical Thinking and Processes and the Mathematical Values and Attitudes discussed in the framework. It is hoped that the explanation of terminologies will enable readers to comprehend the content of the second edition better. Appendix B retains the joint chapter on the development of SEA-BES CCRLS in Mathematics and Science in the first edition (2017) to explain the emergence of the project and the tremendous efforts by several contributors to produce the first edition.

Simultaneously, the three guidebooks for CCRLS in Mathematics, "Mathematics Challenges for Classroom Practices at the Lower Primary, Upper Primary, and Lower Secondary Levels" by Teh, Gan & Isoda (2024) are revised to support the teachers, trainers, educators and curriculum developers in interpreting CCRLS in Mathematics.

We look forward to your taking up the challenges of producing innovative classroom practices by utilising those books.

Masami Isoda
Editorial Team

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- Roberto Araya Colouring book - Araya, R. (2022). *Colorea Ideas, Pensamiento Computacional*.

- Roberto Araya, Maitree Inprasitha, Sofian Tajul Arus and Orlando Gonzalez for APEC InMside 2021.

- STEM-ED, SEAQIM, SEMOLEC, Khon Kaen.

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Editors
Masami Isoda
Editorial Team

Chapter 1

Framework for CCRLS in Mathematics

The first edition of SEA-BES: CCRLS in Mathematics and Science (2017) emphasised the direction of CCRLS for creating a harmonious ASEAN society in this competitive era through mutual understanding in the context of OECD Competency (2005), which had claimed success in establishing a well-functioning society through challenging their competitive world. The CCRLS in Mathematics (2017) was developed to promote the reform of the 21st-century curriculum, which cultivates basic human characters, creative human capital, and well-qualified citizens through mathematics. Here, we would refer to the 21st-century curriculum frameworks to explain the necessity of some pillars in the mathematics curriculum, which should be illustrated later.

Current Issues in Curriculum Reform

In the first quarter of the 21st century, the era of the 4th Industrial Revolution to establish a new society has been ongoing. The past decade has seen Artificial Intelligence (AI) and data science based on mathematical sciences become the core of the revolution which promotes Digital Transformation and STEM production. In the past few years, Generative AI has promoted the necessity of competency for everyone to handle AI using natural language. The second edition of CCRLS in mathematics was prepared to align with these newest perspectives.

In maintaining literacy of the framework for curriculum development, the second edition of the CCRLS in Mathematics kept the three major components as pillars for the 21st-century curriculum: First, for cultivating basic human characters through mathematical values, attitudes, and habits of mind and second, developing creative human capital equipped with necessary process skills and thirdly, acquiring mathematical knowledge and skills. These three components are mutually considered for cultivating well-qualified citizens.

The first and second components are usually explained as Higher Order Thinking Skills (HOTS) in ASEAN countries for the 21st-century curriculum. For example, in APEC (2017), Datuk Dr Habibah Abdul Rahim proposed the triangle model for the 21st-century curriculum as shown in Figure 1. (cf. Developing Computational Thinking on AI and Big Data Era for Digital Society - Recommendations from APEC InMside I Project, APEC, 2021, p2).

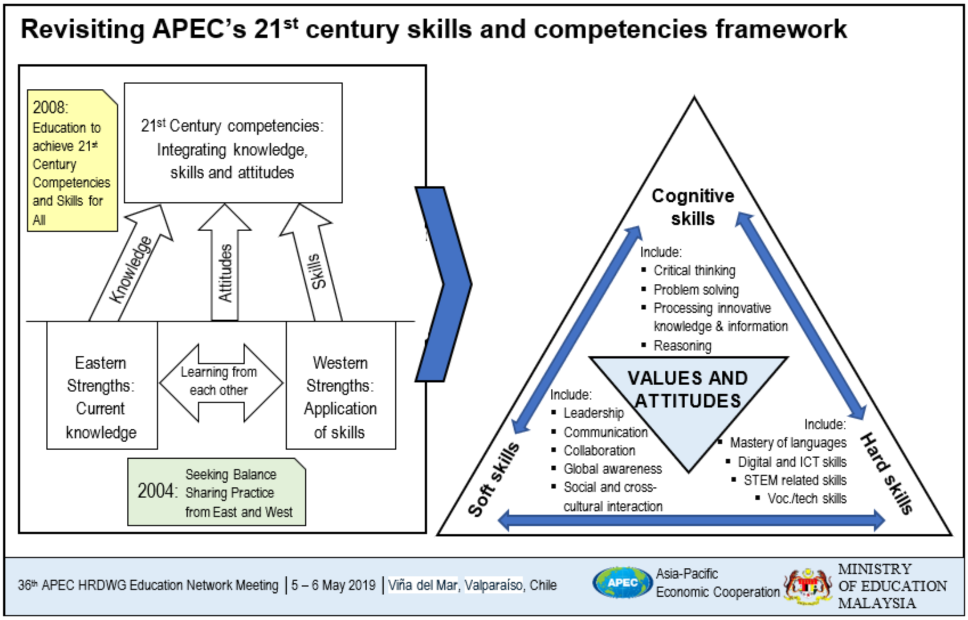

Figure 1. 21st-Century Curriculum Framework by Dr. Habibah (2019)

The objective of CCRLS in Mathematics is to provide the analytical tools for Higher-Order Thinking Skills (HOTS) from the perspective of mathematical thinking. It also provides descriptive terminologies in the mathematics curriculum, which concretise the development of mathematical thinking in every classroom.

In 2019, the OECD proposed the Student Agency, which had already learned to learn challenging ways to solve its various problems. In other words, students can learn mathematics by and for themselves. Figure 2 shows a proposed learning compass that directs learning based on knowledge, skills, values and attitudes as well as Figure 1. All components in the curriculum framework are also aimed at developing the Student Agency; however, they did not provide any analytical tools for mathematics or HOTS. For example, what does value have to do with mathematics? Although HOTS are a general framework, they never explain what it is in mathematics. The role of CCRLS in mathematics is to concretise values & attitudes, and knowledge & various skills in mathematics.

Figure 2. OECD Education 2030: Learning Compass

Regarding mathematical thinking, the APEC InMside Project (2019-2021) provided the Curriculum Framework for Computational and Statistical Thinking, and SEAMEO STEM-ED began the Unplugged Computational Thinking Project (2024) as a follow-up of the InMside project. To align with the latest perspective, this second edition elaborates on the standards more clearly and adds some necessary terminologies related to computational thinking in mathematics.

To clarify how these three components function in the mathematics framework, the nature of mathematics is considered initially, and the aims of CCRLS in mathematics are deduced from it. Lastly, the format of describing the CCRLS in mathematics will be shown before writing every standard.

Nature of Mathematics

Mathematics has been recognised as a necessary literacy for citizenship, not only for living economically but also for establishing a society with fruitful arguments and creations for better living.

In Ancient Greek, 'Mathema' meant the subject of learning, a learning theme which developed the ways of thinking by using their mother language, Greek. Currently, it means a natural language for using Generative AI. Even Ethics is a subject of Mathema in Ancients. Since that era, mathematics has been taught as the training of reasoning by using language for all academic subjects with visual and logical-symbolic representations.

This is especially significant after Descartes' proposal, where algebraic representations became sophisticated and spread as a universal language for any mathematical sciences, which is necessary for learning science and engineering. The narrow image of mathematics in the school curriculum is deeply influenced by his proposed universal mathematics, which tries to represent any subject in algebraic form that cannot be managed without learning algebraic operation as an artificial language. Even programming language is influenced by Algebraic representations.

In the current era, school mathematics has expanded its role to establish 21st-century skills through reviewing mathematics as the science of patterns for future prediction and designing with big data, producing innovation for technological advancement and business models. Mathematics is an essential subject for establishing common reasoning for the sustainable development of society through viable arguments in understanding each other and developing critical reasoning as a habit of mind. This means mathematics should be learned as the basis for all subjects.

For clarifying the framework in CCRLS on mathematics and by knowing the role of mathematics education, the humanistic and philosophical natures of mathematics are confirmed as follows.

Humanistic nature of mathematics

The humanistic nature of mathematics is explained by the attitudes of competitiveness and understanding of others with sympathetic minds on the episodes of challenging mathematicians such as Blaise Pascal, Rene Descartes, Isaac Newton and Gottfried Wilhelm Leibniz.

For example, suppose you read the letter from Blaise Pascal to Pierre de Fermat. In that case, you will recognise Pascal's **competitive attitude** toward Fermat's intelligence and his **sympathetic attitude** toward Fermat: Pascal demanded that

Fermat understand his excellence in finding Pascal's Triangles beyond Fermat's achievement.

On the other hand, if we read Pascal's Pensées, we recognise how Pascal denied Descartes's geometry, which used an algebraic form from the aspect of Ancient Greek geometry, which used natural language. Even Descartes himself tried to overcome the difficulties of ancient geometry through algebra. If you read the letter from Descartes to the Royal Highness Elisabeth, you will recognise how Descartes appreciated and felt happy that Elisabeth used his algebra ideas in geometry. Despite being a princess, Elisabeth has continuously been learning mathematics in her life.

The contrast between Pascal and Descartes as mathematicians implies that even mathematicians can have different identified mathematics. Pascal established his identity in mathematics based on geometry from Ancient Greece, whereas Descartes established his identity in mathematics based on Algebra originated from Ancient Arab.

There were discussions on who developed calculus between Britain and Continental mathematicians. In that context, Johann Bernoulli, a continental mathematician, posed a question in the journal about the Brachistochrone problem. The answer is the locus of the point on the circumference of the circle when it rotates on the line. No one replied, and Bernoulli extended the deadline for the answer and asked Newton to reply. Newton answered it within a day. Finally, six contributions of the appropriate answer, including Newton and other Continental mathematicians, were accepted. However, even in Calculus, Newton, well-known for using algebraic representation, preferred physical representation for his Calculus in his Philosophiæ Naturalis Principia Mathematica. However, Continental mathematicians progressed from algebraic form to functional form. Those differences implied the differences between their identities in mathematics.

All those episodes show that mathematics embraces the humanistic nature of proficiency for competitiveness and understanding others to share ideas. Those Mathematicians' ideas are deeply related to their identity in mathematics.

Philosophical nature of mathematics

The philosophical nature of mathematics can be explained from ontological and epistemological perspectives. From the ontological perspective, mathematics can be seen as a system for universal understanding and a common scientific language. Plato and Aristotle are usually compared on this perspective. In Platonism, Plato believes that the existence of the world of "idea" and mathematics existed in the world of "idea". In this context, mathematical creation is usually explained by the word "discover", which means taking out the cover from which it already exists. At the moment of discovery, the reasonableness, harmony and beautifulness of a mathematical system are usually felt.

Aristotle tried to explain how to reach an idea from the "material" to the "form". This explains that abstract mathematics can be understood with concrete materials using terms such as "modelling", "instruments", "embodiment", "metaphor", and "change representation". Mathematical inventions such as creating definitions are usually based on the embodiment or metaphor of notions. For example, the triangle as a figure is usually defined based on the triangular images of a shape. From an ontological perspective, anyone can understand and acquire mathematics. If acquired, it is a common scientific language used to express any subject. Once ideas are represented using the shared common language, the world can be autonomously perceived in the same view. From the ontological perspective, 'I understand what you are saying in mathematics' means to **re-**present others' ideas in one's mind. The ontological meaning of mathematics is a unique subject that can **re-**present others' ideas precisely in one's mind as long as we try to understand them.

As mentioned, a computer programming language can be seen as a mathematical language. Ontologically, the Computer programme has also a similar nature: It exists like an idea mathematically and logically, but it was invented as an artificial language that runs electronically with human actions on the physical causality in time and the action using a computer is understood like the extension of the human body.

From the epistemological perspective, mathematics can be developed through processes which are necessary to acquire mathematical values and ways of thinking. From this perspective, idealism and materialism are compared. In the context of Hegel, a member of German idealism, Imre Lakatos explained the development of mathematics through proof and refutation using counter-examples. In other words, beyond contradiction is the nature of the mathematical activity, and it provides the opportunity to think mathematically to overcome contradictions. In this context, mathematics is not a fixed system but an expandable system that can be restructured through a dialectic process to construct viable arguments. Plato also used dialectic to reach ideas with counter-examples in mathematics. The origin of dialectic in Philosophy is known as the origin of indirect proof in Mathematics. Dialectics have been the content of teaching in mathematics since Ancient Greece.

In education today, dialectics are a part of critical thinking for creation. George Pólya and Hans Freudenthal give parallel perspectives for mathematical developments. For the discovery of mathematics, Pólya explained mathematical problem-solving processes with mathematical ideas and ways of thinking in general. Freudenthal enhanced the activity to re-organise mathematics with the term mathematisation on the principle of reinvention. Genetic epistemologist Jean Piaget established his theory for operations based on various theories, including the discussion of Freudenthal and explained the mathematical development of operations by the term reflective abstraction. Reflection is also a necessary activity for mathematisation by Freudenthal. These are perspectives of constructivism.

On materialism, under the Vygotskyian perspective, intermediate tools such as language become the basis for reasoning in the mind. Under his theory, high-quality mathematical thinking can be developed depending on high-quality communication in mathematics classrooms. This is the reason why Dialectical-critical discussion has been enhanced in mathematics classes.

From the epistemological perspective, mathematics can be developed through the processes of (dialectic) communication, problem-solving and mathematisation, which include the re-organisation of mathematics. Those processes are necessary to acquire mathematical values and ways of thinking through reflection.

Aims of Mathematics in CCRLS

The aims of mathematics in CCRLS for developing basic human characters, creative human capital, and well-qualified citizens in ASEAN for a harmonious society are as follows:

- Develop mathematical values, attitudes and habits of mind for human character,
- Develop mathematical thinking and be able to engage in appropriate processes,
- Acquire proficiency in mathematics contents and apply mathematics in appropriate situations.

The Framework for CCRLS in Mathematics was developed based on the three components with discussions of mathematics' humanistic and philosophical nature. Due to the current necessity of the Era of Generative AI, the framework was revised, as shown in Figure 3. This framework also depicts the concrete ideas of mathematics learning based on the above aims.

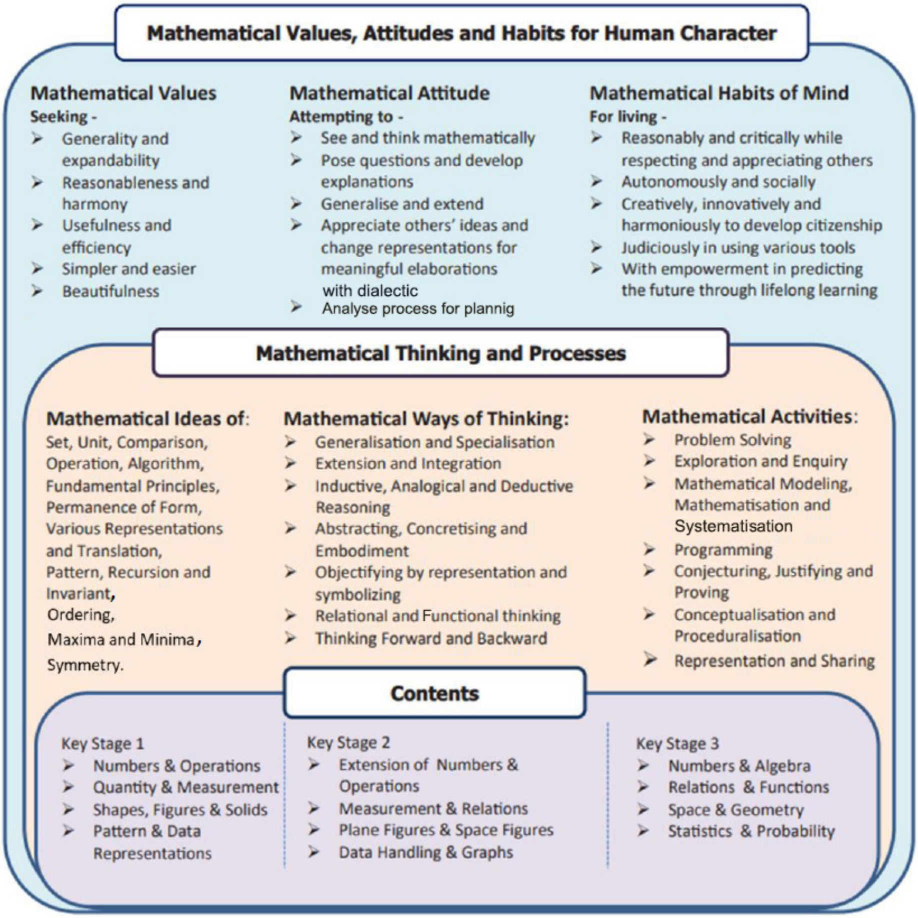

Figure 3. Revised CCRLS Framework in Mathematics*

*The terms in Figure 3 are explained at Appendix A

Mathematical Values, Attitudes and Habits for Human Character

To cultivate basic human characters, values, attitudes, and habits of mind must be developed through mathematics. It is a part of human characters appeared beyond mathematics. Values are the basis for setting objectives and making decisions for future directions. Attitudes are mindsets for attempting to pursue undertakings. Habits of mind are necessary for soft skills to live harmoniously in society. Mathematical values, mathematical attitudes and habits of mind are simultaneously developed and inculcated through learning the content knowledge. Essential examples of values, attitudes and habits of mind are given in Figure 3. Generalisable and expandable ideas are usually recognised as strong ideas on mathematical values. Explaining why proving is necessary for mathematics is a way of seeking reasonableness. Harmony and beautifulness are described in mathematical arts, the science of patterns and systems of mathematics. Usefulness and simplicity are used in the selection of mathematical ideas and procedures.

Regarding mathematical attitudes, "seeing and thinking mathematically" means using the mathematics learned to see and think about objects. Posing questions and providing explanations such as the "why" and the "when" are the ordinary sequence for thinking mathematically. Changing representation to other ways, such as modelling, can overcome running out of ideas in problem-solving. The mindset of trying to understand others is the basis for explaining one's ideas that the rest understand with appreciation. Producing a concept with a definition operationally is a manner of mathematics. In mathematical habits of mind, mathematical attitudes and values are necessary for reasoning critically and reasonably, enabling citizens to live meaningfully. Appreciating and respecting other ideas is also essential. Mathematics is developed independently for those who creatively, innovatively and harmoniously appreciate life. Seeking an easier and more effective manner of selecting appropriate tools is necessary. Mathematics is a subject that offers to challenge and experience competitiveness, appreciation with others, and develop the mindset for lifelong learning, personal development and social mobility.

Mathematical Thinking and Processes

Mathematical ideas, mathematical ways of thinking and mathematical activities are essential for developing creative human capital. Those are functioning in our life beyond mathematics. Mathematical ideas are process skills involving mathematical concepts. Mathematical thinking is a mathematical way of reasoning in general, which does not depend on specific concepts. Mathematical activities include problem-solving, exploration and inquiry. Mathematical processes, which include these components, are necessary skills to use in our lives, such as innovation in this society (e.g., the Internet of Things (IoT)). In education, competency, which refers to mathematical processes, is the basis for STEM and STEAM education and social science and economic education. In the era of generative AI, computational thinking and mathematical thinking have been enhanced. Due to the emergence of computers using mathematical language, mathematical thinking in our lives becomes an essential basis for computational thinking even though we do not do programming. Various mathematical thinking is embedded in computational thinking in our lives.

Mathematical ideas in general serve as the basis of content knowledge to promote and develop mathematical thinking. There are two types of mathematical ideas: Firstly, one is mathematical ideas in general. The second one is mathematical ideas for specific content. It is necessary to note that mathematical content cannot be used without mathematical ideas because content and ideas usually produce contexts. For example, some people believe that multiplication is an accumulation/a repeated addition. However, the idea of multiplication is that 'any number can be seen as a unit for counting'. Without this idea, we cannot apply accumulation in a situation.

Some key ideas of mathematics in general are used as a special process. The fundamental ideas of set and unit lead to a more hierarchical and simple structural relationship. The ability to compare, operate, and perform algorithms of related functions enables efficient ways of learning mathematics and solving problems in learners' lives with mathematics. In the case of a set, it is a mathematical idea related to conditions and elements. It is related to activity in grouping and distinguishing with other groups by conditions. For example, 3 red flowers and 4 white flowers become 7 flowers, if the condition of the set does not consider the colours. "A and non-A" is a simple manner to distinguish sets with logical reasoning. We use intervals such as $x > 0$, $x < 0$, and $x = 0$ for categorising. This situation can be seen in the hyperbolic graph, where $y = \frac{1}{x}$. In the case of a unit, it is a mathematical idea that is related to a process to produce and apply the unit with operations. In some cases, trying to find the common denominator is the way to see the unit of two given quantities. A tentative unit such as arbitrary units can be set and applied locally whereas standard units are used globally. The combination of different quantities produces new measurement quantities such as distance to time produced speed. A square unit such as square centimetres is a unit of area. In programming, we usually find the repetitive parts as patterns and alternate them with the computer.

Mathematical thinking is well discussed by George Pólya. Inductive, analogical and deductive reasoning are major logical reasoning at school. However, deductive reasoning is enhanced with formal logic, whereas inductive and analogical reasoning are not well recognised. Pólya enlightens the importance of that reasoning in mathematics. In the process of mathematisation by Hans Freudenthal, objectifying the method is necessary. David Toll mentioned it using the term thinkable concept in the process of conceptual development. Pólya mentioned thinking forward and backward with ancient Greek terms synthesis and analysis. Mathematical activities are ways to represent a mathematical process. The problem-solving process was analysed by Pólya. He influenced problem solving with various strategies. Technology enhances the activities of conjecturing and visualising inquiries. Conceptualisation is done based on procedures such as 3+3+3+3=12, which becomes the basis for 4x3. The proceduralisation of multiplication is done through developing the multiplication table, the idea of distribution and mental operation.

Content

For cultivating well-qualified citizens, content knowledge of mathematics is essential. In the mathematics syllabus, the content is usually divided into the domains such as algebra and geometry. Domains are used for compartmentalisation. Strands are usually for mathematical connectivity. Human characters and creative human capital should be developed through mathematical processes. Values, attitudes and habits of mind are driving forces for engagement in mathematical processes. From this perspective, the content knowledge of mathematics cannot be realised without involving human character formations with mathematical process skills. Thus, instead of dividing content into domains as differences in mathematics, CCRLS preferred the words 'strands' to represent several connectivities. From the perspective of mathematical content, CCRLS is divided into three stages to indicate grade levels. Every stage has four content strands and the mathematical Process-Humanity strand.

Four Content strands and Mathematical Process-Humanity strand

Additionally, the Mathematical Process-Humanity strand makes clear the process skills such as communications and representations which should be taught at every stage. Between the stages, the names of the content strands are indirectly connected even though there is connectivity. However, beyond the content strands, connectivity is expected in the strands instead of domains. Mathematical Process-Humanity strand in every stage supports the connectivity of every standard in each stage. The names of strands for every key stage are shown in Figure 4.

Key Stages	Content Strands	Process Strands
Key Stage 1	Numbers and Operations Quantity and Measurement Shapes, Figures and Solids Pattern and Data Representations	Mathematical Process-Humanity
Key Stage 2	Extension of Numbers and Operations Measurement and Relations Plane Figures and Space Figures Data Handling and Graphs	Mathematical Process-Humanity
Key Stage 3	Numbers and Algebra Relations and Functions Space and Geometry Statistics and Probability	Mathematical Process-Humanity

Figure 4. Four Content Strands and One Process-Humanity Strand

In every stage, four content strands including several standards are mutually related. Between the stages, all strands in different key stages are mutually related. The same content strand names are used to indicate development and reorganisation beyond each stage. For example, “Numbers and Operations” in Key Stage 1, “Extension of Numbers and Operation” in Key Stage 2, and “Numbers and Algebra” in Key Stage 3 are well connected. These names of the content strands show the extension and integration of contents. For example, even and odd numbers can be taught at any stage with different definitions. At Key Stage 1, even numbers can be introduced as “counting by two” which does not include zero. In Key Stage 2, it can be re-defined by a number divisible by two. Finally, in Key

Stage 3, it can be re-defined as a multiple of two in integers which includes zero. Although we use the same name as even numbers, they are conceptually different. The definition in Key Stage 1 is based on counting, Key Stage 2 is based on division and Key Stage 3 is based on algebraic notation. Expressing such theoretical differences requires names of strands for content to be distinguished. In the case of measurement, there is no strand name of measurement in Key Stage 3. Key Stage 1 relates to the quantity and setting of the units. In Key Stage 2, it extends to non-additive quantity beyond dimension. In Key Stage 3, the idea of unit and measurement is embedded in every strand. For example, the square root in the Numbers and Algebra strand is an irrational number which means unmeasurable, the Pythagorean Theorem in Space and Geometry strand is used for measuring, proportional function in the Relations and Functions strand is used for counting the number of nails by weight, and in Statistics and Probability strand, new measurement units are expressed such as quartile for boxplot.

To learn mathematics, the three components in Figure 3 should be embedded in every key stage as standards for the content of teaching. The "Mathematical values, attitudes, habits for the human character" component and the "Mathematical thinking and processes" component cannot exist without the "Content" component. The first two components can be taught only through teaching with the content. For teaching those three components at the same time, the Mathematical Process – Humanity strand is also included in every stage.

Learning Contexts to Link the Three Components

The interconnection of the three components is shown in Figure 5. The three components will not be ideally operationalised without appropriate contexts. In mathematics classrooms, teachers usually provide the necessary contexts of learning with real-world problem solving and mathematical task sequences.

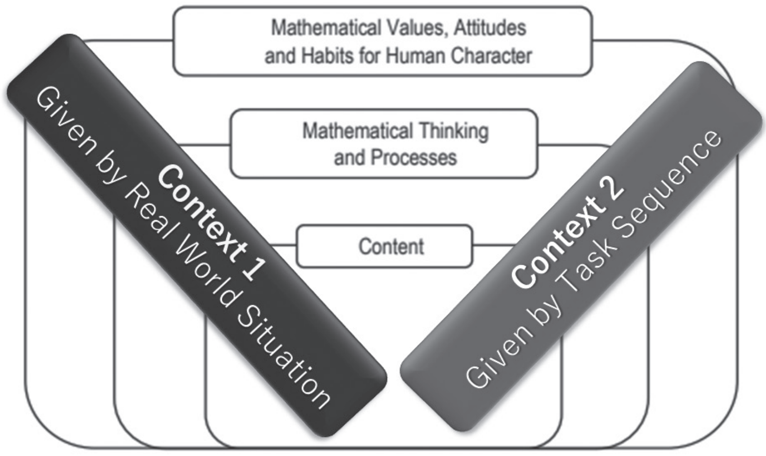

Figure 5. Interconnection of Components in CCRLS Framework in Mathematics

The ultimate aim of the CCRLS framework is to develop mathematical values, attitudes and human characters which are the essence of a harmonious society. This component is closely related to the affective domain of human character traits which correspond to soft skills that can be developed through appreciation. Concerning this, the acquisition of mathematics contents as hard skills and reflection on the thinking processes are needed to inculcate the capability of appreciation. The reflection is necessary for learners to recognise their cognitive skills derived from the contents. Even though contents appeared to be learned independently through acquisition, the mathematical thinking and process, as well as the appreciation of mathematical values, attitudes and habits for the human character, are possible to be developed through reflective experiences which include appreciation. Thus, ways of learning the three components can be characterised as in Figure 6.

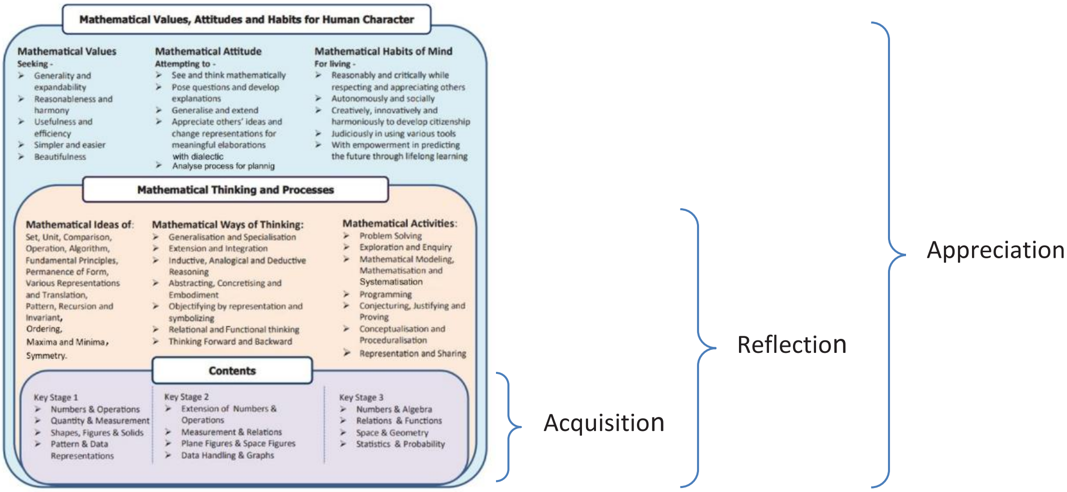

Figure 6. Acquired the content learning through reflection and appreciation

From the perspective of 'Mathema' in Ancient Greece (Inprasitha, Isoda & Araya, to appear) the ways of thinking and values such as dialectic discussion with critique can be learned through any subjects such as ethics, geometry and so on. The CCRLS framework in Figure 3 makes clear such ways of thinking and values from the perspective of current mathematics subjects. These days, algebraic representations as a unique universal language are necessary to be acquired through appropriate exercise as a second language as well. The appropriateness of exercise is also clarified by the context given by the task sequence in Figure 5 for what elements students can learn as shown in Figure 1, through the process of reflection and appreciation in Figure 6. All three components are learning content which is necessary in the Era of Generative AI because Generative AI is used by natural language as well as algebraic notations and so on.

The sample contexts for learning mathematics in classrooms

Figure 5 illustrates two primary contexts which teachers usually provide to teach the three components at once. To clarify these two contexts in the classroom context, designing the following sample activities will be necessary to realise the three components in each classroom:

- Explore a problem with curiosity in a situation and attempt to formulate mathematical problems.
- Apply the mathematics learned, listen to others' ideas and appreciate the usefulness, power and beauty of mathematics.
- Enjoy classroom communications on mathematical ideas in solving problems with patience and develop persistence.
- Feel the excitement of "Eureka" with enthusiasm for the solutions and explanation of unknown problems.
- Think about ways of explanation using understandable representations such as language, symbols, diagrams and notation of mathematics.
- Discuss the differences in seeing situations before and after learning mathematics.
- Explain, understand others and conclude mathematical ideas.
- Explore ideas through inductive and deductive reasoning when solving problems to foster mathematical curiosity.
- Explore ideas with examples and counter examples.
- Imagine others, getting others perspectives, and then ask "if your saying is true, what will happen".
- Feel confident in using mathematics to analyse and solve contextual problems both in school and in real-life situations.
- Promote knowledge, skills and attitudes necessary to pursue further learning in mathematics.
- Enhance communication skills with the language of mathematics.
- Promote abstract, logical, critical and metacognitive thinking to assess their own and others' work.
- Foster critical reasoning for appreciating other's perspectives.
- Promote critical appreciation of the use of information and communications technology in mathematics.
- Appreciate the universality of mathematics and its multicultural and historical perspectives.

Those contexts are chosen to illustrate the interwoven links of the two components with contents. It looks like methods of teaching however all three components are the subject of teaching to develop students who learn mathematics by and for themselves. The Mathematical Process-Humanity strand in each stage illustrates the three components as contents of learning.

Description of the Format of SEA-BES CCRLS in Mathematics

Curriculum standards in every country are written under the specific given format for all school subjects under the regulation and policy of the Ministry of Education. The direction of reform is reflected and embedded in the format.

The format of writing the CCRLS is produced under Figures 1, 2, and 3, and the content strands and the process-humanity strand with the defined context as follows:

- Each key stage has four content strands and one mathematical process humanity strand.
- The content strands describe the content standards in each strand.
- The mathematical process-humanity strands describe the elements covered under the mathematical thinking and process component; and the values, attitudes and habits for the human character component in Figure 3. Here, the word "process" refers to mathematical thinking and processes. The word "humanity" refers to the elements in the components of mathematical values, attitudes and habits of mind.

The mathematical process-humanity standards of each key stage imply the context for learning those two components. The descriptions of every standard under the process-humanity strands are commonly discussed across different content strands within the same key stage. The Mathematical process-humanity strand includes three processes (Mathematical Ideas, Thinking and Activities) and the three humanities (Mathematical Values, Attitudes and Habits of mind for citizens to live).

The above classroom contexts explain the processes for developing mathematics, such as argumentation for understanding others and establishing sophistication through critique, applying mathematics through modelling and representation, and extension and generalisation, which are essential for developing mathematics. In argumentation, producing understandable explanations is usually based on understandable representations such as diagrams and materials for showing the simple structure. An example is used to demonstrate a specific case, and a counter-example is used to check generality. The process of modelling usually includes problem formation, solving mathematically, and explaining the meaning. Within these three processes, three humanities such as recognising the beautifulness of patterns are learned through the appreciation of mathematical experiences; the appreciation of others' ideas also includes understanding ideas with sense such as recognising the usefulness of idea or otherwise. Under those explanations, the writing format is defined with content standards and process-humanity standards. It includes the choosing of necessary content for teaching and exploratory sequence which encompasses metacognition, critical reasoning, critical communication, and reflection. Under the selected content and the context, the three components in Figure 3 are well connected. Proficiency in skills and procedures is also developed through content and context in appropriate situations and necessary exercises.

Sample Format

SEA-BES CCRLS in Mathematics is developed under the Framework in Figure 3 and sets the following format to distinguish and relate the descriptions of standards beyond the stages. In this format, every standard embeds the ideas for what is expected to be learned, how it is related to other standards and why it is necessary for further learning.

Key Stage Number

A general explanation for every stage.

Strand: Title of Strand under the Stage

The description of each strand gives clear images of the developmental level in three Key Stages

Topics:
A set of Content standards is described under the topic. Every topic sentence begins from the gerund form of the verb that implies mathematical process-humanity. In the case of the Mathematical Process-Humanity Strand in every stage, every standard is described under the strand without categorising it into the topics.

Standard:
Every standard is described with gerund form and verb to show process and adjective to cite value and attitude as follows:

(Sample)

[K1NO1-1]

Enjoying counting orally and manipulatively with number names, without symbolic numerals

i. Develop fluency in the order of number names and use them based on situations
ii. Set the initial object for counting, the direction of counting and recognise the last object with one-to-one correspondence

Under this format, every standard is numbered to distinguish and for easy referencing as the following rule:

For example, [K1NO8-1] means K1= Key Stage 1, NO= Numbers and Operations Strand, 8-1 refers to the first standard in the 8^{th} topic.

Chapter 2

Key Stage 1

Key Stage 1 (KS1) serves as the foundation of knowledge covering the basic facts and skills developed through simple hands-on activities, manipulation of concrete objects, pictorial and symbolic representations. This stage focuses on arousing interest, enjoyment and curiosity in the subject through exploration of patterns, characterisation, identification and describing shapes, performing the four fundamental operations, identifying its algorithm, and understanding basic mathematical concepts and skills experienced in daily life. Calculations of quantities will also be established to carefully and wilfully understand the attribution of objects used to make direct and indirect comparisons.

Strand: Numbers and Operations [K1NO]

The number is introduced with situations, concrete objects, pictorial, symbolic representations and extended based on knowledge and skills learned. Ways of counting and distributions are extended to addition, subtraction, multiplication and division. The base ten number system is the key to extending the numbers and operations for standard algorithms in vertical form. Also, various procedures of calculations and algorithms are focused on. Models and diagrams are used for extensions instead of concrete materials themselves. Number sense will be developed by establishing fluency in calculations connected to situations and models. Fractions and decimals are introduced with manipulatives.

Topics:

Introducing Numbers up to 120 [K1NO1]
Introducing Addition and Subtraction [K1NO2]
Utilising Addition and Subtraction [K1NO3]
Extending Numbers with Based Ten System up to 1 000 000 Gradually [K1NO4]
Producing Vertical Forms for Addition and Subtraction and Acquiring Fluency of Standard Algorithms [K1NO5]
Introducing Multiplication and Produce Multiplication Algorithm [K1NO6]
Introducing Division and Extending it to Remainder [K1NO7]
Introducing Fractions and Extending to Addition and Subtraction of Similar Fractions [K1NO8]
Introducing Decimals and Extending to Addition and Subtraction [K1NO9]

Introducing Numbers up to 120 [K1NO1]

[K1NO1-1]
Enjoying counting orally and manipulatively with number names, without symbolic numerals

i. Develop fluency in the order of number names and use them based on situations[1]
ii. Set the initial object for counting, the direction of counting and recognise the last object with one-to-one correspondence
iii. Distinguish the original situation with concrete objects and the representation of counting chips, blocks or marbles

[K1NO1-2]
Understanding and using the cardinality and ordinality of numbers with objects and numerals[2] through grouping activities, corresponding and ordering, and developing number sense[3] and appreciating its beautifulness

i. Group objects for counting with conditions such as cups, flowers, and rabbits in situations and introduce numerals
ii. Obtain fluency in counting concrete objects and understand counting on, and recognise the necessity of zero
iii. Compare different sets by one-to-one correspondence and recognize larger, smaller or equal with appreciation in drawing paths between objects.
iv. Compose and decompose numbers to strengthen the number sense[4] and cardinality
v. Understand the difference between ordinal and cardinal numbers and use them appropriately in situations and challenge the mixed sequence.
vi. Acquire number sense[5] and appreciate the beautifulness of ordering numerals with and without concrete objects

[1] *The denomination such as 3 cups, 2 cups, and one cup is described in strands on the Quantity and Measurement.*

[2] Inclusive in reading and writing of numerals.

[3] *Units for counting that describe the Quantity and Measurement.*

[4] *Relationship of composing and decomposing numbers becomes the preparation for addition and subtraction for inverse operation.*

[5] *Number pattern is discussed under Pattern and Data Representations.*

[K1NO1-3]
Introducing the base-ten system with groupings of 10 and extending numbers up to 120[6]

i. Extend numbers to more than 10 with base-ten manipulative representing numbers in ones and tens, and appreciate the base-ten numeration system
ii. Extend the order sequence of numbers to more than 10 in relation to the size of ones and tens and compare numbers using numerals in every place value
iii. Introduce number lines to represent the order of numbers and for comparison starting from zero and counting by ones, twos, fives, and tens
iv. Enjoy various ways of the distribution of objects with counting such as playing cards, and explain it and enhance the number sense
v. Draw a diagram for representing the size of the number with base-ten blocks such as a flat (square) for a unit of hundred and a rectangular bar for a unit of ten

Introducing Addition and Subtraction [K1NO2]

[K1NO2-1]
Understanding situations for addition up to 10 and obtaining fluency in using addition in situations

i. Introduce situations (together, combine, and increase) for addition and explain it orally with manipulative to define addition for operation
ii. Develop fluency in addition expressions using a composition of numbers for easier calculation with number sense for the composition of numbers
iii. Apply addition with fluency in learners' lives

[K1NO2-2]
Extending addition to more than 10 and obtaining fluency in using addition in situations

i. Extend addition situations and think about how to answer using the idea of making 10 with decomposition and composition of numbers
ii. Explain the idea of addition with place value using base-ten blocks
iii. Develop fluency in addition expressions to more than 10 for easier calculation
iv. Apply the addition fluency in learners' lives.

[6] *For discussing the difference of hundred twenty is not twelve ten in English.*

[K1NO2-3]
Understanding situations for subtraction up to 10 and obtaining fluency in using subtraction in situations

i. Introduce subtraction situations (remaining, complement, and difference) and explain orally with manipulative to define subtraction for operation[7]
ii. Develop fluency in subtraction expressions using the decomposition of numbers for easier calculation
iii. Apply subtraction fluency in learners' lives

[K1NO2-4]
Extending subtraction to more than 10 and obtaining fluency in` using subtraction in situations

i. Extend subtraction with situations and think about how to answer using the idea of 10 with addition and subtraction of numbers (composition and decomposition of numbers)
ii. Explain the idea of subtraction in place value using base ten blocks
iii. Develop fluency in subtraction expressions to more than 10 for easier calculation
iv. Apply subtraction fluency in learners' daily life

Utilising Addition and Subtraction [K1NO3]

[K1NO3-1]
Utilising addition and subtraction in various situations and understanding their relationships

i. Understand the difference between addition and subtraction situations with tape diagrams
ii. Explain subtraction as an inverse of addition situations with tape diagrams
iii. Understand addition with three numbers, subtraction with three numbers and combination of addition and subtraction situations
iv. Apply addition and subtraction in various situations such as in ordering numbers

Extending Numbers with Base-Ten System Up to 1 000 000 Gradually [K1NO4]

[K1NO4-1]
Extending numbers using base-ten system up to 1 000

i. Experience counting 1 000 by using various units and appreciate the necessity of the base-ten system

[7] *Distinguish minuend and subtrahend.*

ii. Extend the order of numbers to more than 1 000 in relation to the size of ones, tens and hundreds
iii. Use a partial number line to compare the size of numbers through a translation of the size of every digit with the appropriate scale
iv. Represent appropriate diagram to show the size of numbers without counting such as three of hundreds mean 30 of tens and visualise the relative size of numbers
v. Represent larger or smaller numbers by the symbol of inequality

[K1NO4-2]
Extending numbers using a base-ten system up to 10 000

i. Visualise the 10 000 by using thousand, hundred, ten and one as units
ii. Extend the order sequence of numbers to more than 10 000 in relation to the size of ones, tens, hundreds and thousands
iii. Use a number line with an appropriate scale to show the size of numbers and the relative size of numbers while focusing on the scale

[K1NO4-3]
Extending numbers using base-ten system up to 1 000 000[8]

i. Extend numbers up to 1 000 000 and learn the representation of the place value for grouping every 3-digit numeral system up to a million
ii. Write large numbers using the grouping of a 3-digit numeral system[9] such as thousand as a unit and compare numbers in relation to it
iii. Develop number sense such as larger and smaller based on comparison of place values through visualisation of the relative size of numbers

Producing Vertical Form Addition and Subtraction[10] and Acquiring Fluency in Standard Algorithms [K1NO5]

[K1NO5-1]
Thinking about the easier ways for addition and subtraction and producing vertical form algorithms

i. Think about easier ways of addition or subtraction situations and use models with base-ten blocks meaningfully for representing the base-ten system
ii. Produce and elaborate efficient ways and identify the standard algorithms[11] in relation to the base-ten system with appreciation

[8] *One million is too big for counting and is introduced only for learning the three-digit system.*

[9] *3-digit numeral system such as 123 times thousand equals the same way of reading plus thousand. In the case of Chinese, the four-digit numeral system is used.*

[10] *Understanding the relationship between addition and subtraction is discussed under Pattern and Data Representations.*

[11] *Various algorithms are possible and there is no one specific form because depending on the country, the vertical form itself is not the same. Here, the standard algorithm means the selected appropriate form.*

iii. Explain the algorithms of borrowing and carrying with regrouping of base-ten models
iv. Acquire fluency in addition and subtraction algorithms

[K1NO5-2]
Acquiring fluency in standard algorithms for addition and subtraction and extending up to 4-digit numbers

i. Extend the vertical form addition and subtraction through the extension of numbers and appreciate the explanation using the base-ten block model
ii. Develop fluency in every extension up to 3-digit numbers and simple case for 4-digit numbers

[K1NO5-3]
Developing number sense[12] for estimation[13] and using a calculator judiciously for addition and subtraction

i. Develop number sense for mental arithmetic with estimation for addition or subtraction of numbers
ii. Identify necessary situations to use calculators judiciously in real life.
iii. Appreciate the use of calculators in the case of large numbers for finding the total and the difference

Introducing Multiplication and Produce Multiplication Algorithm [K1NO6]

[K1NO6-1]
Introducing multiplication and mastering multiplication table

i. Understand the meaning of multiplication[14] situations with models using the idea of addition and distinguishing from the common addition to find the total number
ii. Produce a multiplication table in the case of counting by 2 and 5 with array diagrams, pictures or block models and extend it until 9 and 1 with an appreciation of patterns[15]
iii. Develop a sense for multiplication through mental calculation with fluency
iv. Use multiplication in daily life, differentiating multiplication in various situations with the understanding that any number can be a unit for counting in multiplication

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[12] *Money system is discussed under Measurement and Relations.*
[13] *Rounding numbers are treated in key stage 2 under Measurement and Relations.*
[14] *Meaning of area is described in Measurement and Relations.*
[15] *Multiplication row of 1 is not a repeated addition.*

[K1NO6-2]
Producing multiplication in vertical form and obtaining fluency

i. Think about easier ways of multiplication in the case of numbers greater than 10 using array diagrams and block models
ii. Develop multiplication in vertical form using multiplication table, array, model, and base-ten system with appreciation
iii. Extend the multiplication algorithm to 3-digit times 2-digit numbers
iv. Obtain fluency in the standard algorithm for multiplication
v. Use estimation with the multiplication of tens or hundreds in life
vi. Compare the multiplication expressions which is larger, smaller or equivalent
vii. Appreciate the use of calculators sensibly in life in the case of large numbers

Introducing Division and Extending It to Remainder [K1NO7]

[K1NO7-1]
Introducing division with two different situations and finding the answers by multiplication

i. Understand division with quotative and partitive division for distribution situations
ii. Think about how to find the answer to division situations by distribution using diagrams, repeated subtractions and multiplication
iii. Obtain fluency to identify answers of division through the inverse operation of multiplication
iv. Appreciate the use of multiplication table for acquiring mental division

[K1NO7-2]
Extending division into the case of remainders and using division for distribution in daily situations

i. Extend division situations with remainders and understand division as a repeated subtraction with remainders
ii. Obtain fluency in the division and apply it in daily situations
iii. Understand simple cases of the division algorithm

Introducing Fractions and Extending to Addition and Subtraction of Similar Fractions [K1NO8]

[K1NO8-1]
Introducing simple fractions such as halves, quarters and so on using paper folding and drawing diagrams

i. Introduce simple fractions using paper folding and drawing diagrams in the context of part-whole relationship
ii. Use "*a half of*" and "*a quarter of*" in a daily context such as half a slice of bread
iii. Count a quarter for representing one quarter, two quarters, three quarters, and four quarters
iv. Compare and explain simple fractions in the case where the whole is the same

[K1NO8-2]
Extending fractions using tape diagram and number line to one, and think about how to add or subtract similar fractions for producing a simple algorithm

i. Extend fractions to more than one unit quantity for representing the remaining parts (unit fraction) such as measuring the length of tape, recognising the remaining parts as a unit measure of length, and understanding proper and improper fractions
ii. Appreciate fractions with quantities in two ways; firstly, the whole is a unit of quantity and secondly, based on the number of unit fraction
iii. Compare fractions in the case where the whole is the same and explain it with a tape diagram or number line, and develop fraction number sense such as $\frac{1}{10}$ with quantities and so on
iv. Think about how to add or subtract similar fractions with a tape diagram or number line and produce a simple algorithm with fluency

Introducing Decimals and Extending to Addition and Subtraction [K1NO9]

[K1NO9-1]
Introducing decimals to tenths, and extending addition and subtraction into decimals

i. Introduce simple decimals to tenths by remaining part such as using a tape diagram with appreciation
ii. Compare the size of decimal numbers on a number line with the idea of place value
iii. Extend addition and subtraction of decimals utilising the place value system in the vertical form up to tenths
iv. Think about appropriate place value for applying addition and subtraction in life

Strand: Quantity and Measurement [K1QM]

Attributes of objects are used to make direct and indirect comparisons. The non-standard and standard units are also used for comparison. Counting activities denominate units of quantities such as cups for volumes, arm-length and hand-spans for length. Standard units such as metre, centimetre, kilogram and litre are introduced. Time and durations which are not the base ten system are introduced. Money is not a complete model for a base-ten system. The concept of conservation of quantities will be established through the calculation of quantities. The sense of quantity is developed through the appropriate selection of measurement tools.

Topics:

Comparing Size, Directly and Indirectly, Using Appropriate Attributes and Non-Standard Units [K1QM1]
Introducing Quantity of Length and Expand It to Distance [K1QM2]
Introducing Quantity of Mass for its Measurement and Operation [K1QM3]
Introducing Quantity of Liquid Capacity for its Measurement and Operation [K1QM4]
Introducing Time and Duration and its Operation [K1QM5]
Introducing Money as Quantity [K1QM6]

Comparing Size, Directly and Indirectly, Using Appropriate Attributes and Non-Standard Units [K1QM1]

[K1QM1-1]
Comparing and describing quantity using appropriate expression

i. Compare two objects directly by attributes instead of stating in length and amount of water such as longer or shorter and less or more
ii. Compare two objects indirectly using non-standard units to appreciate the unification of units
iii. Use appropriate denomination[16] of quantity (such as the number of cups) for counting and appreciating the usage of units for quantity in a suitable context

[16] *Denomination is necessary for learning the group of counting. It also describes pattern and data representations and number and operation both of Key Stage 1.*

Introducing Quantity of Length and Expanding to Distance [K1QM2]

[K1QM2-1]
Introducing centimetre for length and extending to millimetres and metre

i. Compare the length of different objects and introduce centimetres with a calibrated tape[17] of one centimetre
ii. Demonstrate equivalent length with addition and subtraction such as part-part whole
iii. Extend centimetre to millimetre to represent remaining parts with ideas of equally dividing and the idea of making tens
iv. Extend centimetre to metre to measure using a metre stick
v. Estimate the length of objects and select appropriate tools or measuring units for measurement with fluency
vi. Convert mixed and common units of length for comparison[18]
vii. Convert mixed and common units of length when adding or subtracting in acquiring the sense for quantity

[K1QM2-2]
Introducing distance for the extension of length

i. Introduce kilometres to measure distance travelled using various tools and appreciate the experiences of measuring skills
ii. Distinguish the distance travelled and the distance between two places on the map
iii. Compare mixed units of length with an appropriate scale on a number line

Introducing Quantity of Mass and Its Measurement and Operation [K1QM3]

[K1QM3-1]
Introducing gram for mass and extending to kilogram and tons

i. Compare the mass of different objects directly using balance and introduce gram
ii. Demonstrate equivalent mass with addition and subtraction such as part-part whole
iii. Extend gram to kilogram, measure with a weighing scale
iv. Extend kilogram to metric ton through the relative measure (such as 25 children, each weighing 40 kilograms)

---

[17] *The plane tape can be used for direct and indirect comparison by marking. If a non-standard unit scales the tape, we can use it for measurement. If the tape is scaled by one centimetre, we can define the length of the centimetre.*

[18] *Which one is longer, 2 m 3 cm or 203 mm?*

v. Estimate the mass of objects and select appropriate tools or measuring units for measurement with fluency
vi. Convert mixed and common units of mass for comparison
vii. Convert mixed and common units of mass for addition and subtraction in acquiring the sense of quantity

Introducing Quantity of Liquid[19] Capacity and Its Measurement and Operation [K1QM4]

[K1QM4-1]
Introducing litre for capacity of liquid and extending to millilitre

i. Compare the amount of water in different containers and introduce litre with measuring cups of 1 litre
ii. Demonstrate equivalent capacity with addition and subtraction such as part-part whole
iii. Extend litre by decilitre/100-millilitre cup for representing remaining parts with ideas of equally dividing and making 10, and extend until millilitre
iv. Estimate the capacity of containers and select the appropriate measuring unit
v. Convert mixed and common units of capacity for comparison
vi. Convert mixed and common units of capacity for addition and subtraction in acquiring the sense of quantity

Introducing Time and Duration, and Its Operation [K1QM5]

[K1QM5-1]
Introducing analogue time and extending to duration

i. Tell and write analogue time of the day corresponding with different activities in daily life such as morning, noon, afternoon, day and night.
ii. Show time by using a clock face with an hour hand and a minute hand
iii. Understand the relative movement of clock hands

[K1QM5-2]
Extending clock time to a duration of one day[20]

i. Introduce duration in hours and minutes based on the beginning time and end time of activities
ii. Express time and duration on a timeline, and understand duration as the difference between two distinguished times

---

[19] *The density which explains the relationship between mass and liquid capacity is usually learned in science at a later stage. In the case of CCRLS Science, it starts in Key Stage 2 such as 1 cubic centimetre of water is equivalent to 1 gram.*
[20] *Calendar is possible in the keys stage 1 under Pattern and Data Representation.*

iii. Addition and subtraction of duration and time
iv. Extend time and duration to seconds
v. Convert mixed and common units of duration for comparison
vi. Estimate the duration of time and select an appropriate measuring unit for measurement with fluency and appreciate the significance of time and duration in life
vii. Appreciate the difference in time depending on the area (time zone) and the seasons

Introducing Money as Quantity [K1QM6]

[K1QM6-1]
Introducing money as quantity and use as the model of the base-ten system[21]

i. Introduce units of money using notes and coins and determine the correct amount of money
ii. Use counting by fives and so on for the base-10 system
iii. Appreciate the fluency in the calculation of money with all the four operations
iv. Appreciate number sense for the conversion and transaction of money in daily life

[21] *Coins and notes are dependent on the country. Some countries use currency units of twenty and twenty-five in coins or notes. These forms are not appropriate for the model of the base-10 system.*

Strand: Shapes, Figures and Solids [K1SF]

Basic skills of exploring, identifying, characterising and describing shapes, figures and solids are learned based on their features. Activities such as paper folding enable the exploration of various features of shapes. Identification of similarities and differences in shapes and solids enables classification to be done for defining figures. Using appropriate materials and tools, relationships in drawing, building and comparing the 2D shapes and 3D objects are considered. Through these activities, the skills for using the knowledge of figures and solids will be developed. The compass is introduced to draw circles and mark scales of the same length.

Topics:

Exploring shapes of objects [K1SF1]
Characterising shapes for figures and solids [K1SF2]
Explaining positions and directions [K1SF3]

Exploring Shapes of Objects [K1SF1]

[K1SF1-1]
Exploring shapes of objects to find their attributes

i. Roll, fold, stack, arrange, trace, cut, draw, and trace objects (blocks such as boxes, cans and so on) to know their attributes
ii. Use attributes of blocks for drawing the picture by tracing shapes on the paper and explain how to draw it with the shapes
iii. Create patterns of shapes (trees, rockets and so on) by using the attributes and recognise the characteristics of shapes[22]
iv. Appreciate functions of shapes of objects in learners' life
v. Appreciate the names of shapes in daily life by using one's mother tongue

Characterising the Shapes for Figures and Solids [K1SF2]

[K1SF2-1]
Describing figures with characters of shapes

i. Use characteristics of shapes for understanding figures (quadrilateral, square, rectangle and triangle, right angle, same length)
ii. Introduce line and right angle with relation to activities such as paper folding and use it for describing figures with simple properties, such as a triangle has 3 lines
iii. Classify triangles by specific components, such as side, vertex and angle (right-angled triangle, equilateral, isosceles) and then know the properties of each classification

[22] *Pattern of shapes is discussed in Key Stage 1 under Pattern and Data Representations.*

iv. Reorganising rectangular shapes and squared shapes as figures by using the right angle and length of sides

[K1SF2-2]
Describing solids with characteristics of shapes

i. Use characteristics of shapes to understand solids such as boxes can be developed by six rectangular parts with simple properties
ii. Develop boxes with the properties
iii. Appreciate solids around daily life by considering the functions of solids

[K1SF2-3]
Drawing a circle and recognising the sphere based on the circle

i. Think about how to draw a circle and find the centre and radius
ii. Draw a circle with an instrument such as a compass
iii. Enjoy drawing pictures using the function of circles such as Spirograph
iv. Find the largest circle of the sphere with a diameter and identify the sphere by its centre and radius
v. Appreciate the use of circles and spheres in daily life such as manholes, and the difference between a soccer ball and a rugby ball

Explaining Positions and Directions [K1SF3]

[K1SF3-1]
Exploring how to explain a position and direction

i. Identify simple positions and directions of an object accurately using various ways such as in my perspective, in your perspective in the classroom, and the left, right, front, back, west, east, north, south and with measurement
ii. Draw the map around the classroom with consideration of locations
iii. Design a game to appreciate the changing of positions and directions in a classroom

Strand: Pattern and Data Representations [K1PD]

Various types of patterns such as the number sequence and repetition of shapes are considered. The size of pictures can be represented by the number sequence. Tessellation of shapes and paper folding can be represented by the repetition of shapes. Exploration of patterns and features is also considered to represent the data structure using pictographs and bar graphs. Patterns and features produce the meaning of data and represent mathematical information. Patterns are represented by diagrams and mathematical sentences which are also used for communication in identifying and classifying situations to produce meaningful interpretations. In the era of Generative AI, natural language reasoning, which includes drawings, will be enhanced because it runs through using natural language.

Topics:

Using Patterns under the Number Sequence [K1PD1]
Producing Harmony of Shapes using Patterns [K1PD2]
Collecting Data and Represent the Structure [K1PD3]

Using Patterns under the Number Sequence[23] [K1PD1]

[K1PD1-1]
Arranging objects for beautiful patterns under the number sequence

- i. Know the beautifulness of patterns in cases of arranging objects based on number sequence
- ii. Arrange objects according to number sequence to find simple patterns
- iii. Arrange expressions such as addition and subtraction to find simple patterns
- iv. Express the representation of patterns using placeholders (empty box)
- v. Enjoy the arrangement of objects based on number sequence in daily life
- vi. Find patterns on number tables such as in calendars[24]

Producing harmony of shapes using patterns[25] [K1PD2]

[K1PD2-1]
Arranging tiles of different or similar shapes to create harmony

- i. Know the beautifulness of patterns in cases of arranging the objects based on shapes, colours and sizes
- ii. Arrange objects according to shapes, colours and sizes to show patterns
- iii. Arrange boxes according to shapes, colours and sizes to create a structure
- iv. Arrange circles and spheres for designing

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[23] *Number sequence is discussed in Key Stage 1 under Numbers and Operations.*

[24] *Time and duration are discussed in key stage 1 under Quantity and Measurement.*

[25] *Harmony of shapes will be discussed in Key Stage 1 under Shapes, Figures and Solids.*

v. Enjoy the creation based on different shapes, colours and sizes in daily life

Collecting data and representing the structure [K1PD3]

[K1PD3-1]
Collecting data through categorisation to get information

i. Explore the purpose of why data is being collected
ii. Grouped data by creating similar attributes on the denomination[26] of categories and counting them (check mark and count)
iii. Think about what information is obtained from the tables with categories and how to use it

[K1PD3-2]
Organising the data collected and representation using pictograms for easy visualisation

i. Produce the table and pictograms from collected data under each category
ii. Interpretation of tables and pictograms as a simple conclusion about the data being presented.
iii. Appreciate pictograms through collecting data and adding data in daily activities in learners' life

[K1PD3-3]
Representing a data structure by using a bar graph to predict the future of communities

i. Understand how to draw bar graphs from a table using data categories and sort the graph to show its structure
ii. Appreciate ways of presenting data such as using tables, pictograms and bar graphs with sorting for predicting their future communities
iii. Appreciate the use of data for making a decision

[26] *Denomination will be learned in Key Stage 1 under Quantity and Measurement.*

Strand: Mathematical Process – Humanity `[K1MH]

Enjoyable mathematical activities are designed to bridge the standards in different strands. Exploration of various number sequences, skip counting, addition and subtraction operations help to develop a number sense that is essential to support explanations of contextual scenarios and mathematical ideas. Mathematical ways of posing questions in daily life are also necessary to learn at this stage. The ability to select simple, general and reasonable ideas enables effective future learning. The application of number sense provides a facility for preparing sustainable life. The use of ICT tools and other technological tools provides convenience in daily life. At the initial stage, concrete model manipulation is enjoyable, however, drawing a diagram is most necessary for explaining complicated situations by using simple representation. This manner is necessary to develop computational and mathematical thinking.

Standards:

Enjoying problem solving through various questioning for four operations in situations [K1MH1]
Enjoying measuring through setting and using the units in various situations [K1MH2]
Using blocks as models and their diagrams for performing operations in base ten [K1MH3]
Enjoying tiling with various shapes and colours [K1MH4]
Explaining ideas using various and appropriate representations [K1MH5]
Selecting simple, general and reasonable ideas which can apply to future learning [K1MH6]
Preparing a sustainable life with a number sense [K1MH7]
Utilising ICT tools such as calculators as well as other tools such as notebooks and other instruments such as clocks [K1MH8]
Promoting creative and global citizenship for sustainable development of neighbourhood using mathematics [K1MH9]
Represent recursive process by using manipulatives and drawings for finding patterns [K1MH10]

[K1MH1]
Enjoying problem solving through various questioning for four operations in situations[27]

i. In addition, pose questions for altogether and increase situations
ii. In subtraction, pose questions for remaining and differences situations
iii. In multiplication, pose questions for the number of groups situations
iv. In division, pose questions for partition and quotation situations
v. Enjoy questioning by using a combination of operations in various situations

[27] *It is related to Numbers and Operations and Quantity and Measurement both in Key Stage 1.*

vi. In operations, pose questions to find easier ways of calculation
vii. Use posing questions for four operations on measurements in daily life

[K1MH2]
Enjoying measuring through setting and using the units in various situations[28]

i. Compare directly and indirectly
ii. Set tentative units from differences for measuring
iii. Give appropriate names (denominations) for counting units
iv. Use measurement for communication in daily life
v. Use tables and diagrams for showing the data of measures

[K1MH3]
Using blocks as models and their diagrams for performing operations in base ten[29]

i. Show increasing and decreasing patterns using blocks
ii. Show based ten system using blocks; the unit cube is 1, the bar stick is 10 and the flat block represents 100
iii. Explain the addition and subtraction algorithm in vertical form using a base-ten block model
iv. Explain multiplication table with grouped blocks
v. Explain division using equal distribution of blocks and repeated subtraction of blocks
vi. Use the number of blocks for measurement in daily life

[K1MH4]
Enjoying tiling with various shapes and colours[30]

i. Appreciate producing beautiful designs with various shapes and finding the pattern to explain it
ii. Reflect, rotate and translate to produce patterns
iii. Cut and paste various shapes and colours to form the box and ball such as developing the globe from a map

[K1MH5]
Explaining ideas using various and appropriate representations[31]

i. Explain four operations using pictures, diagrams, blocks and expressions for developing ideas
ii. Explain measurement using measuring tools, tape diagrams, containers and paper folding for sharing ideas
iii. Make a decision on how to explain the figures and the solids by using manipulative objects or diagrams or only verbal explanation

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[28] *It is related to Quantity and Measurement and Pattern and Data Representations both in Key Stage 1.*
[29] *It is related to Pattern and Data Representations and Numbers and Operations, both in Key Stage 1.*
[30] *It is related to Shapes, Figures and Solids and Pattern and Data Representations, both in Key Stage 1.*
[31] *It is related to all strands in Key Stage 1.*

iv. Explain patterns using diagrams, numbers, tables and expressions with a blank box
v. Ask questions using terms such as why, how, what, if and if not, and reply using examples and 'for example' in discussions
vi. Change the representation and translate it appropriately into daily life

[K1MH6]
Selecting simple, general and reasonable ideas which can apply to future learning[32]

i. Discuss the argument for the easier ways for addition and subtraction algorithms in vertical form
ii. Extend the algorithm to large numbers for convenience and fluency
iii. Use the pattern of increase in the multiplication table for convenience
iv. Use multiplication tables for finding the answers to division

[K1MH7]
Applying number sense[33] acquired in Key Stage 1 for preparing sustainable life[34]

i. Use mathematics for the minimum and sequential use of resources in situations
ii. Estimate for efficient use of resources in situations
iii. Maximize the use of resources through an appropriate arrangement in space
iv. Understand equally likely of resources in situations

[K1MH8]
Utilising ICT tools such as calculators as well as other tools such as notebooks and other instruments such as clocks[35]

i. Use calculators for addition of multiple numbers in situations
ii. Use mental calculations for estimations
iii. Use a balance scale to produce equality and inequality
iv. Use cups, tapes, stopwatches, and weighing scales for measuring distances and weights
v. Use calculators to explain the calculation process by solving backwards and understanding the relationship between addition and subtraction, multiplication and division
vi. Enjoy using notebooks to exchange learning with each other such as mathematics journal writing
vii. Enjoy presentations with board writing

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[32] *It is related to Numbers and Operations and Pattern and Data Representations, both in Key Stage 1.*

[33] *It is related to Numbers and Operations, Pattern and Data Representations and Quantity and Measurement, all included in Key Stage 1.*

[34] *Sustainable development goals were crafted at the 70th Session of the United Nations General Assembly and indicated as universal values in education.*

[35] *STEM education is enhanced. Mathematics is the major and base subject for STEM Education in Key Stage 1 hence, technological contents are included in Mathematics.*

viii. Use various tools for conjecturing and justifying

[K1MH9]
Promoting creative and global citizenship for sustainable development of neighbourhood using mathematics

i. Utilise notebooks and journal books to record and find good ideas and share them with others
ii. Prepare and present ideas using posters to promote good practices in the neighbourhood
iii. Listen to other's ideas and ask questions for better creation
iv. Utilise information, properties and models as a basis for reasoning
v. Utilise practical arts and outdoor studies to investigate local issues for improving the welfare of life

[K1MH10]
Represent recursive process by using manipulatives and drawings for finding patterns

i. Analysing situations by multiple action processes and alternating them with manipulatives and drawings
ii. Find the structure of repetitions which is used in repeated actions previously

Chapter 3

Key Stage 2

Key Stage 2 (KS2) is learned based on the Key Stage 1. This stage shows the extension of numbers and operations, measurement and relations, plane figures and space figures, data handling and graphs. This stage shows the extension of the four operations to daily use of numbers such as decimals and fractions, allows the use of mathematical terminologies investigations and establishing the ground for analysing, evaluating and creating in learners' lives. Appreciating the beauty of the structure of mathematics will enable them to enjoy and sustain their learning, providing the basis for Key Stage 3.

Strand: Extension of Numbers and Operations [K2NO]

Numbers are extended to multi-digits, fractions and decimals. Multiplication and division algorithms are completed with fluency. Fractions become numbers through the redefinition as a quotient instead of a part-whole relationship. Multiplication and division of decimals and fractions are also explored to develop procedures for the calculation. Various representations are used to elaborate and produce meaning for the calculation. Number sense, such as approximating numbers and the relative size of numbers and values, is enhanced for practical reasoning in the appropriate context of life.

Topics:

Extending Numbers with Base Ten Up to Billion and also to Thousandths with a Three-Digit Numeral System Gradually [K2NO1]

Making Decision of Operations on Situations with Several Steps and Integrate them in One Expression and Think About the Order of Calculations and Producing the Rule (PEMDAS) [K2NO2]

Producing the Standard Algorithm for Vertical Form Division with Whole Numbers [K2NO3]

Extending the Vertical Form Addition and Subtraction with Decimals to Hundredths [K2NO4]

Extending the Vertical Form Multiplication and Division with Decimals and Finding the Appropriate Place Value such as Product, Quotient and Remainder [K2NO5]

Using Multiples and Divisors for Convenience [K2NO6]

Introducing Improper and Mixed Fractions and Extending to Addition and Subtraction of Fractions to Dissimilar Fractions [K2NO7]

Extending Fractions as Numbers and Integrate [K2NO8]

Extending Multiplication and Division to Fractions [K2NO9]

Extending Numbers with Base Ten Up to a Billion and also to Thousandths with the Three-Digit Numeral System Gradually [K2NO1]

[K2NO1-1]
Extending numbers using the base-ten system up to a billion[36] with the three-digit numeral system[37]

i. Adopt the three-digit numeral system, extend numbers up to a billion with the idea of relative size of numbers
ii. Compare numbers such as larger, and smaller with the base-ten[38] system of place values through visualisation of the relative size of numbers using cube, plane (flat), bar (long) and unit

[K2NO1-2]
Extending decimal numbers to hundredths, and thousandths[39]

i. Use the idea of quantity and fractions, and extend decimal numbers from tenths to hundredths
ii. Compare decimal numbers such as larger, and smaller with the base-ten system of place value
iii. Adopt the ways of extension up to thousandths and so on, and compare the relative sizes

Making Decisions of Operations on Situations with Several Steps and Integrating them in One Expression, thinking about the Order of Calculations and Producing the Rule (PEMDAS) [K2NO2]

[K2NO2-1]
Finding easier ways of calculations using the idea of various rules of calculations[40] such as the associative, commutative and distributive rules

i. Find the easier ways of addition and subtraction and use them, if necessary, such as the answer is the same if add the same number to the subtrahend and minuend
ii. Find the easier ways of multiplication and division and use them in convenient ways such as 10 times multiplicand produce the product 10 times
iii. Use associative, commutative and distributive rules of addition and multiplication for easier ways of calculation, however, the commutative property does not work in subtraction and division

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[36] *Billion is too large for counting and it is introduced in the three-digit system under the relative size of number.*

[37] *In the British system, it is referred to as short scale.*

[38] *Metric system names of units are discussed under measurement and relations.*

[39] *Under the three-digit system, if we teach until thousandths, we can extend by three-digits.*

[40] *In Measurement and Relations, the use of constant sum, difference, product and quotient are described. (e.g.25 -21 = 4, 26–22 = 4, 27– 23 = 4)*

iv. Appreciate the use of simplifying rules of calculations

[K2NO2-2]
Thinking about the order of calculations in situations and producing rules and order of operations

i. Integrate several steps of calculation into one mathematical sentence
ii. Produce the rule of PEMDAS and apply it to the multi-step situation
iii. Think about the easier order of calculation and acquiring fluency in PEMDAS and rules with appreciation

Producing the Standard Algorithm Using Vertical Form Division with Whole Numbers [K2NO3]

[K2NO3-1]
Knowing the properties of division and using it for an easier way of calculation

i. Find the easier ways of division and use them, if necessary, such as the answer is the same if multiplying the same number by the dividend and divisor
ii. For confirmation of the answer of division, use the relationship among divisor, quotient and remainder and appreciate the relationship

[K2NO3-2]
Knowing the algorithm of division in vertical form and acquiring fluency

i. Know the division algorithm with tentative quotient and confirm the algorithm by the relationship among divisor, quotient and remainder
ii. Interpret the meaning of quotient and remainder in situations
iii. Acquire fluency for division algorithm in the case of up to 3-digit whole number divided by 2-digit
iv. Think about the situations with or without remainder in relation to situations for quotative and partitive division

Extending the Vertical Form Addition and Subtraction with Decimals to Hundredths [K2NO4]

[K2NO4-1]
Extending the vertical form addition and subtraction in decimals to hundredths

i. Extend the vertical form addition and subtraction to hundredths[41] place and explain it with models
ii. Appreciate the use of addition and subtraction of decimals in their life

Extending the Vertical Form Multiplication and Division with Decimals and Finding the Appropriate Place Value Such as Product, Quotient and Remainder [K2NO5]

[K2NO5-1]
Extending the multiplication from whole numbers to decimal numbers

i. Extend the meaning of multiplication with the idea of measurement by the number of unit length for multiplication of decimal numbers and use diagrams such as number lines to explain them with appreciation in situations
ii. Extend the vertical forms multiplication of decimals up to 3 digits by 2 digits with consideration of the decimal places step-by-step
iii. Obtain fluency using multiplication of decimals with sensible use of calculators in learners' life
iv. Develop number sense in the multiplication of decimals[42] such as comparing sizes of products before multiplying

[K2NO5-2]
Extending the division from whole numbers to decimal numbers

i. Understand how to represent division situations using diagrams such as number lines, and extend the diagram of decimal numbers for explaining division by decimal numbers
ii. Extend the division algorithm in the vertical form of decimal numbers and interpret the meaning of decimal places of quotient and remainder with situations
iii. Acquire fluency in the division algorithm of decimals up to 3 digits by 2 digits with consideration of decimal places step-by-step
iv. Obtain fluency using division of decimals with sensible use of calculators in learners' life
v. Develop number sense in the division of decimals such as comparing sizes of quotients before multiplying

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[41] *Discussion of decimals to hundredths is related to the use of money. It is a minimum requirement. If teaching to hundredths, further extension of place value can be understood.*
[42] *Applying the idea of multiplication into ratio, percent and proportion is discussed in Measurement and Relations.*

vi. Distinguish the situations with decimal numbers of multiplication and division

Using Multiples and Divisors for Convenience [K2NO6]

[K2NO6-1]
Using multiples and divisors for convenience with appreciation to enrich number sense

i. Understand sets of numbers by using multiples and divisors
ii. Find common multiples and appreciate their use in situations, and enrich number sense with figural representations such as an arrangement of rectangles to produce a square
iii. Find a common divisor and appreciate its use in situations, and enrich number sense with figural representations such as dividing a rectangle into pieces of square
iv. Understand numbers as a composite of multiplication of numbers as factors[43]
v. Appreciate ideas of prime, even and odd numbers in situations using multiples and divisors
vi. Acquire the sense of numbers to see the multiples and divisors for convenience

Introducing Improper and Mixed Fractions and Extending to Addition and Subtraction of Fractions to Dissimilar Fractions [K2NO7]

[K2NO7-1]
Extending fractions to improper, mixed and equivalent fractions

i. Extend fractions to improper and mixed fractions using a number line of more than one by measuring with a unit fraction[44]
ii. Find ways to determine equivalent fractions with number lines and with the idea of multiple numerators and denominators
iii. Compare fractions using number lines and the idea of multiple

[K2NO7-2]
Extending addition and subtraction of similar fractions to improper and mixed fractions, and dissimilar fractions

i. Extend addition and subtraction of similar fractions to proper and mixed fractions with explanations using models and diagrams
ii. Extend addition and subtraction into dissimilar fractions with explanations using diagrams and common divisors

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[43] *This idea is related to the strand on Measurement and Relations in Key Stage 2, for the area of a rectangle.*

[44] *Extension of a fraction to more than one is done by using the fraction with quantity for a situation such as* $\frac{4}{3}m$.

iii. Acquire fluency in addition and subtraction of fractions with appreciation of ideas to produce the same denominators

Extending Fractions as Numbers and Integrate[45] [K2NO8]

[K2NO8-1]
Seeing fractions[46] as decimals and seeing decimals as fractions

i. See fractions as decimals using division and define quotient with divisible which includes repetition of the remainder
ii. See decimals as fractions such as hundredths are per hundred
iii. Compare decimals and fractions and order them on a number line

Extending Multiplication and Division to Fractions [K2NO9]

[K2NO9-1]
Extending multiplication to fractions

i. Extend multiplication to fractions with situations using diagrams such as number lines step by step, and find the simple algorithm for the multiplication of fractions
ii. Acquire fluency in the multiplication of fractions
iii. Develop number sense[47] of multiplication of fractions such as comparing sizes of products before multiplying

[K2NO9-2]
Extending division to fractions

i. Extend division to fractions with situations using diagrams such as number lines step-by-step
ii. Acquire fluency in the division of fractions
iii. Develop number sense of division of fractions such as comparing sizes of quotients before dividing

[45] *Selecting the appropriate denomination of quantities and units for a fraction in the context* ($\frac{2}{3}$)*m is two of* $\frac{1}{3}$*m and the whole is 1m, however,* $\frac{3}{3}$*m is 3x*($\frac{1}{3}$)*m; the structure is the same as tens are ten of units and discussed under Measurement and Relations.*

[46] *Fraction as a ratio is introduced in Measurement and Relations.*

[47] *Applying the idea of the multiplication of fractions into ratio, proportion, percentage and base is discussed in Measurement and Relations.*

Strand: Measurement and Relations [K2MR]

Additive quantities, such as angles, areas, and volume, and relational quantities, such as population density and speed, are introduced. Additive quantity can be introduced by establishing the standard unit, the same as the Quantity and Measurement of Key Stage 1. Relations of quantities in situations are discussed with patterns such as sum constant, difference constant, product constant and quotient constant using tables and represented by mathematical sentences and letters. Proportion and ratio are introduced with representations of diagrams, graphs, and tables for multiplication, and connected with decimals and fractions. Percent is introduced with diagrams in relation to ratio and proportion. Relational quantity is produced by different quantities with the understanding of ratio. The area of a circle is discussed through a proportional relationship between the radius and the circumference. Ideas of ratio and proportion are fluently applied to real-world problem solving.

Topics:

Introducing Angle and Measuring it [K2MR1]
Exploring and Utilising Constant Relation [K2MR2]
Extending Measurement of Area in Relation to Perimeter [K2MR3]
Extending Measurement of Volume in Relation to Surface [K2MR4]
Approximating Quantities [K2MR5]
Extending Proportional Reasoning to Ratio and Proportion [K2MR6]
Producing New Quantities Using Measurement Per Unit [K2MR7]
Investigating the Area of a Circle [K2MR8]
Exchanging Local Currency with Currency in the ASEAN Community [K2MR9]
Extending the Relation of Time and Use of Calendar in Life [K2MR10]
Converting Quantities in Various System of Units [K2MR11]
Showing Relationships Using a Venn Diagram [K2MR12]

Introducing Angle and Measuring It[48] [K2MR1]

[K2MR1-1]
Introducing angle by rotation, enabling measure and acquire fluency using the protractor

i. Compare the extent of rotation and introduce degree as a unit for measuring an angle
ii. Recognise right angle is 90 degrees, and adjacent angle of two right angles is 180 degrees, and 4 right angles are 360 degrees
iii. Acquire fluency in measuring angles using the protractor
iv. Draw equivalent angles with addition and subtraction using multiples of 90 degrees

[48] *Right angle is learned at Key Stage 1 in Shapes, Figures and Solids for explaining the properties of figures.*

v. Appreciate measurement of angles in geometrical shapes and situations in life[49]

Exploring and Utilising Constant Relation [K2MR2]

[K2MR2-1]
Exploring equal constant relation with the utilisation of letters to represent placeholders[50]

i. Explore two possible unknown numbers such that their sum (or difference /product/quotient) is constant,[51] for example, □+Δ=12 (□ and Δ are placeholders).
ii. Use letters instead of placeholder[52] (empty box) to derive an equivalent relation
iii. Understand the laws for operations (e.g. associative, commutative and distributive etc.) to explain the simpler way of calculation
iv. Appreciate the use of diagrams such as number lines and areas to represent relations when finding solutions

Extending Measurement of Area in Relation to Perimeter [K2MR3]

[K2MR3-1]
Introducing area and producing a formula for the area of a rectangle

i. Compare the extent of an area and introduce its unit, and distinguish it from the perimeter
ii. Introduce one square centimetre (cm^2) as a unit for area and its operation using addition and subtraction
iii. Investigate areas of rectangles and squares and produce the formula of the area[53]
iv. Extend square centimetre to square metre and square kilometre for the measure of large areas
v. Convert units and use appropriate units of areas with fluency
vi. Draw the equivalent size of a rectangular area based on a given area with the factors of a whole number[54]
vii. Appreciate the use of areas in daily life such as comparing land sizes.

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[49] *Conservation of angles will be re-learnt in a triangle under Key Stage 2 Plane Figures and Space Figures.*

[50] *The idea for the use of Numbers and Operations (Key Stage 2) in finding easier ways of calculations with the idea of rules of calculations.*

[51] *Constants of multiplication and division correspond proportionality in the multiplication table at Key Stage 1 under number and operations. The constant of addition and subtraction are treated in Key Stage 1 under Pattern and Data Representation.*

[52] *Placeholder is introduced in Key Stage 1.*

[53] *Multiplications were studied in Key Stage 1 Number and Operations.*

[54] *The idea of composite numbers such as 2 times 10 equals 5 times 4 is related to factors in extending the numbers and operations at the same Key Stage 2.*

[K2MR3-2]
Extending the area of a rectangle to other figures to derive formulae

i. Explore and derive a formula for the area of a parallelogram by changing its shape to a rectangle without changing its area
ii. Explore and derive a formula for the area of a triangle by bisecting a rectangle into two triangles without changing its area
iii. Appreciate the idea of changing or dividing shapes of a rectangle, parallelogram, or/and triangle for deriving the area of other figures
iv. Use formulae to calculate areas in daily life

Extending Measurement of Volume in Relation to Surface [K2MR4]

[K2MR4-1]
Introducing volume from the area and deriving the formula for cuboid

i. Compare the extent of volume and introduce its unit, and distinguish it from the surface
ii. Introduce one cubic centimetre as the unit for volume and its addition and subtraction
iii. Investigate the volume of a cuboid and cube and produce the formulae
iv. Extend cubic centimetre to cubic metre to measure large volume
v. Convert units and use appropriate units of volume with fluency
vi. Appreciate the use of volume in life such as comparison of the capacity of containers

[K2MR4-2]
Extending the volume of a cuboid to other solid figures to derive formulae

i. Extend the formula for the volume of a cuboid as base area x height for exploring solid figures such as prism and cylinder
ii. Extend the formula for the volume of a prism and a cylinder to explore and derive the volume formula of a pyramid and cone
iii. Use the formulae to calculate volume in daily life

Approximating with Quantities [K2MR5]

[K2MR5-1]
Approximating numbers with quantities depending on the necessity of contexts

i. Understand the ways of rounding such as round up and round down
ii. Use rounding as an approximation for deciding on the quantity with related context
iii. Critique approximation beyond the context with a sense of quantity such as based on the relative size of units

Extending Proportional Reasoning to Ratio[55] and Proportion [K2MR6]

[K2MR6-1]
Extending proportional reasoning to ratio and percent for comparison

i. Understand ratio as a relationship between two same quantities or between two different quantities (the latter idea is rate)[56]
ii. Express the value of a ratio by quotient such as the rate of two different quantities[57]
iii. Understand percent as the value of a ratio with the same quantities[58] and the necessity of rounding
iv. Understand proportional reasoning for ratio as part-whole and part-whole relationships
v. Apply the rule of three[59] in using ratio

[K2MR6-2]
Extending proportional reasoning to proportion

i. Extend proportional reasoning to multiplication tables as equal ratios and understand proportions
ii. Understand proportion by multiple and a constant quotient, not changing the value of the ratio[60]
iii. Demonstrate simple inverse proportion by constant product[61]
iv. Express proportion in a mathematical sentence by letters and graph[62]
v. Use properties of proportionality to predict and explain phenomena in daily life

[55] *Band graph and pie chart for representing ratio are discussed under Key Stage 2 under the strand Data Handling and Graphs.*

[56] *Ratio of different quantities is a rate. The ratio of the same quantities has a narrow meaning of ratio.*

[57] *The value of a fraction as a ratio is not necessarily a part of a whole in situations. Fraction as a ratio is usually used in the context of multiplication situations, where the denominator is the base or a unit for comparison.*

[58] *Percent is used in Data Handling and Graphs.*

[59] *Rule of three is the method on the table to find one unknown term from the three known terms using proportional reasoning such as;*

a	c
?	d

[60] *Enlargement is discussed in Key Stage 2 under Plane Figures and Space Figures. The graph is treated at Key Stage 2 under the strand Data Handling and Graphs.*

[61] *Proportion and Inverse proportion are necessary in Key Stage 3 in science.*

[62] *This will be discussed in detail in the same Key Stage 2 under Data Handling and Graphs.*

Producing New Quantities Using Measurement Per Unit [K2MR7]

[K2MR7-1]
Producing new quantities using measurement per unit

i. Introduce average as units for distribution and comparison of different sets of values
ii. Introduce population density with the idea of average and appreciate it for comparison
iii. Introduce speed with the idea of average and appreciate it for comparison
iv. Appreciate using diagrams such as number lines and tables to decide the operations on the situations of measurement per unit quantity
v. Comparing the context of different quantities with the idea of average as rate[63]
vi. Apply the idea of measurement per unit quantity in different contexts[64]

Investigating the Area of a Circle [K2MR8]

[K2MR8-1]
Areas of a circle are discussed through the relationship between the radius and circumference

i. Investigate the relationship between the diameter of a circle and its circumference using the idea of proportion
ii. Investigate the area of a circle by transforming it into a triangle or parallelogram and find the formula of the circle
iii. Estimate the area of inscribed and circumscribed shapes based on a known formula of area[65]
iv. Enjoy to estimate the area of irregular shapes with fluency in life

Exchanging Local Currency with Currency in the ASEAN Community [K2MR9]

[K2MR9-1]
Exchanging local currency in the ASEAN community with the idea of rate

i. Extend the use of ratio for currency exchange (rate of exchange)
ii. Apply the four operations for money in appropriate notation in life
iii. Appreciate the value of money

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[63] *On Number and Operations Key Stage 2, the rate is the value of division as quotient.*

[64] *Using measurement per unit quantity with fluency to make logical judgments in daily life, refer to Key Stage 2 Data Handling and Graphs.*

[65] *Relationships on polygons and circles are discussed in Key Stage 2 under Plane Figures and Space Figures.*

Extending the Relation of Time and Use of Calendar in Life [K2MR10]

[K2MR10-1]
Extending the relation of time and use of calendar in life

i. Convert time in the 12-hour system with the abbreviation a.m. and p.m. to the 24-hour system and vice versa
ii. Investigate the numbers in a calendar to relate days, weeks, months and years using the idea of number patterns
iii. Appreciate the significance of various calendars in life

Converting Quantities in Various System of Units [K2MR11]

[K2MR11-1]
Converting measurement quantities based on international and non-international systems with the idea of base-10

i. Convert measurement system of metre and kilogram with prefixes deci-, centi-, and milli-, and with deca, hecto-, and kilo-
ii. Convert measurement system of litre with cubic centimetre
iii. Convert measurement system of area using are (a) and hectare (ha) with square meter
iv. Convert measurement of local quantities with standard quantities
v. Understand the unit system with power, such as metre, square metre and cubic metre

Showing Relationship Using a Venn Diagram [K2MR12]

[K2MR12-1]
Using a Venn diagram to show relationships between numbers and figures for making a clear logical deduction

i. Sort objects by their defining characteristics
ii. Show relationships of squares, rectangles, rhombus, parallelograms, trapeziums and quadrilaterals by using a Venn diagram
iii. Show the relationship between numbers
iv. Critique ambiguous reasoning by using a Venn diagram to make clear a definition

Strand: Plane Figures and Space Figures [K2PS]

Through tessellation, figures can be extended through plane figures. Parallelograms and perpendicular lines are tools to explain the properties of triangles and quadrilaterals as plane figures. They are also needed for identifying and recognising symmetry and congruency. Plane figures are used to produce solids in space and vice versa. Opening faces of solids would produce plane figures referred to as nets. Activities related to building solids from plane figures are emphasised and encouraged to facilitate finding the area of a circle through numerous sectors of the circle to construct a rectangle. Circles are used for explaining the nets of cylinders.

Topics:

Exploring Figures with their Components in the Plane [K2PS1]
Exploring Space Figures with their Components in Relation to the Plane [K2PS2]
Exploring Figures with Congruence, Symmetry and Enlargement [K2PS3]

Exploring Figures with their Components in the Plane [K2PS1]

[K2PS1-1]
Exploring figures with their components in the plane and using their properties

i. Examine parallel lines and perpendicular lines by drawing with instruments
ii. Examine quadrilaterals using parallel and perpendicular lines, and identify parallelogram, rhombus, and trapezium by discussion
iii. Find properties of figures through tessellations such as a triangle where the sum of the angles is 180 degrees, a straight angle
iv. Extend figures to polygons and expand them to circles by knowing and using their properties

Exploring Space Figures with their Components in Relation to the Plane [K2PS2]

[K2PS2-1]
Exploring rectangular prisms and cubes with their components

i. Identify the relationship between faces, edges and vertices for drawing sketch
ii. Explore nets of a rectangular prism and find the corresponding position between components
iii. Explore the perpendicularity and parallelism between the faces of a rectangular prism
iv. Explain positions in rectangular prisms with the idea of 3-dimensions

[K2PS2-2]
Extending rectangular prism to other solids such as prisms and cylinders

i. Extend the number of relationships between faces, edges and vertices for drawing of sketch
ii. Explore nets of prisms and cylinders, and find the corresponding position between components
iii. Distinguish prism and cylinder by the relationship of their faces

Exploring Figures with Congruence, Symmetry and Enlargement [K2PS3]

[K2PS3-1]
Exploring the properties of congruence

i. Explore properties of figures which fit when overlapped and identify conditions of congruency with corresponding points and sides
ii. Draw congruent figures using minimum conditions and confirm by measuring angles and sides
iii. Appreciate the usefulness of congruent figures by tessellation

[K2PS3-2]
Exploring the properties of symmetry

i. Explore the properties of figures which reflect and identify conditions of symmetry with line and its correspondence
ii. Draw symmetrical figures using conditions in an appropriate location
iii. Appreciate the usefulness of symmetry in designs

[K2PS3-3]
Exploring the properties of enlargement[66]

i. Explore properties of figures in finding the centre of enlargement in simple cases such as a rectangle
ii. Draw an enlargement of a rectangle using the ratio (multiplication of the value of the ratio)[67]
iii. Appreciate the usefulness of enlargement in the interpretation of a map

[66] *General cases will be discussed in Key Stage 3, Space and Geometry.*
[67] *Ratio and rate are discussed in Key Stage 2 under Measurement and Relations.*

Strand: Data Handling and Graphs [K2DG]

The process of simple data handling is introduced through data representation such as using a table, bar graph, line graph, bar chart and pie chart. Graphs are utilised depending on the qualitative and quantitative data used such as bar graphs for distinguishing and counting in every category. The discussion of producing the line graph includes taking data at specific intervals, using suitable scales, and using slopes. A histogram is necessary for interpreting the data representation of social studies and science and is also used as a special type of bar graph. Average is introduced based on the idea of ratio for making the dispersion of bar chart even, and used for summarising and comparing data on a table. *Problem-Plan–Data-Analysis–Conclusion* (PPDAC) cycles are experienced through the process of data handling by using those data representation skills. Those are necessary for using mathematical thinking and computational thinking in our lives.

Topics

Arranging Tables for Data Representations [K2DG1]
Drawing and Reading Graphs for Analysing Data [K2DG2]
Using Graphs in the PPDAC Cycle [K2DG3]
Applying Data Handling for Sustainable Living [K2DG4]

Arranging Tables for Data Representations [K2DG1]

[K2DG1-1]
Collecting and arranging data

i. Explore how to collect multi-category data based on a situation
ii. Explore how to arrange and read multi-category data on appropriate tables.
iii. Appreciate the use of multi-category tables in situations

Drawing and Reading Graphs for Analysing Data [K2DG2]

[K2DG2-1]
Drawing and reading line graphs to know the visualised pattern as the basis for the tendency of change

i. Introduce line graphs based on appropriate situations such as rainfall, temperature and others
ii. Distinguish line graph from bar graph for observation such as increase, decrease, and no-change
iii. Introduce the graph of proportion using the idea of a line graph and read the gradient by a constant ratio[68]
iv. Appreciate the line graph in various situations

[68] *Proportions are learnt in the Key Stage 2 under Measurement and Relations.*

[K2DG2-2]

Drawing and reading band graphs and pie charts for representing ratios in a whole[69]

i. Explore how to scale the band or circle to represent the ratio or percent
ii. Use the band graph and pie chart for comparison of different groups
iii. Appreciate the band graph and pie chart in a situation

[K2DG2-3]

Reading histogram[70] for analysing frequency distribution

i. Draw a simple histogram[71] from the frequency table of situations
ii. Read various histograms for analysing data distribution
iii. Use averages[72] (mean) to compare different groups in the same situation with histograms

Using Graphs in the PPDAC[73] Cycle [K2DG3]

[K2DG3-1]

Identifying appropriate graphs for problem solving in data handling using the PPDAC cycle

i. Analyse a problem situation and discuss the expected outcomes before collecting data to clarify the purpose of a survey
ii. Plan the survey for the intended purpose
iii. Collect the data based on the purpose of the survey
iv. Use appropriate graphical representation which is most suitable for the purpose
v. Appreciate the use of graphs before making the conclusion

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[69] *Ratio is learnt under Key Stage 2 Measurement and Relations.*

[70] *How to draw a histogram is discussed in Key Stage 3 under Statistics and Probability. Reading histograms is necessary in social studies and science.*

[71] *Using ICT for drawing graphs will be mentioned in mathematical activities.*

[72] *Mean is introduced as average in Key Stage 2 under Measurement and Relations.*

[73] *PPDAC itself will be described in the mathematical activities later.*

Applying Data Handling for Sustainable Living [K2DG4]

[K2DG4-1]
Applying data handling for sustainable development[74] and appreciating the power of data handling for predicting the future

i. Read data related to sustainable development on SDGs and adopting positive views for the betterment of society
ii. Understand the idea of probability as ratio and percentage in reading the data for situations related to sustainable development
iii. Experience implementing a project of reasonable size in data handling to achieve sustainable development and appreciate the power of data handling

[74] *This standard is related to SDG as inter-subject content between social studies and science.*

Strand: Mathematical Process – Humanity [K2MH]

As a follow-up of Key Stage 1, activities are designed to enable an appreciation of knowledge and skills learned and the ways of learning, such as applying knowledge of number sense to solve daily problems. Mathematical processes such as communication and reasoning explain mathematical problems and modelling. The ability to connect and reason mathematical ideas would trigger excitement among learners. Discussions of misconceptions are usually enjoyable and challenging. Mathematics learning usually begins from situations at Key Stage 1. In Key Stage 2, the development of mathematics is possible through discussions for the extension of the forms. Appreciation of ideas and representations learned becomes part of the enjoyable activities. The application of learning becomes meaningful through the consistent use of representations such as diagrams.

Standards:

Enjoying problem-solving through various questioning and conjecturing for extension of operations into decimals and fractions with proportionality and new quantities such as area and volume [K2MH1]

Enjoying measuring through settings and using the area and volume in situations [K2MH2]

Using ratio and rate in situations [K2MH3]

Using number lines, tables, and area diagrams for representing operations and relationships in situations [K2MH4]

Establishing the idea of proportion to integrate various relations with the consistency of representations [K2MH5]

Enjoying tiling with various figures and blocks [K2MH6]

Producing valuable explanations based on established knowledge, shareable representations and examples [K2MH7]

Performing activities of grouping and enjoy representing with Venn diagram [K2MH8]

Experiencing the PPDAC (Problem-Plan-Data-Analysis-Conclusion) cycle and modelling cycle in simple projects in life [K2MH9]

Preparing a sustainable life with number sense and mathematical representations [K2MH10]

Utilising ICT tools as well as notebooks and other technological tools [K2MH11]

Promoting creative and global citizenship for sustainable development of community using mathematics [K2MH12]

Represent recursive process by using manipulatives, drawings and tables for finding patterns [K2MH13]

[K2MH1]
Enjoying problem-solving through various questioning and conjecturing for extension of operations into decimals and fractions with proportionality and new quantities such as area and volume[75]

i. Pose questions to develop a division algorithm in vertical form using multiplication and subtraction
ii. Pose questions to develop multiplication and division of decimal numbers using the idea of proportionality with tables and number lines
iii. Pose questions to develop multiplication and division of fractions using the idea of proportionality with tables, area diagrams and number lines
iv. Pose questions to extend multiplication and division algorithm in vertical form to decimal numbers and discuss decimal points
v. Pose questions to use decimals and fractions in situations
vi. Pose questions to use area and volume in life
vii. Pose questions to use ratio and rate in life
viii. Pose conjectures based on ideas learned, such as when multiplying, the answer becomes larger

[K2MH2]
Enjoying measuring through settings and using the area and volume in situations

i. Compare directly and indirectly areas and volumes
ii. Set tentative units from difference for measuring area and volume[76]
iii. Give the formula for the area and volume for counting units
iv. Use measurement for communication in daily life

[K2MH3]
Using ratio and rate in situations[77]

i. Understand division as partitive (between different quantities) and quotative (between the same quantity) in situations
ii. Develop the idea of ratio and rate utilising the idea of average and per unit with tables and number lines
iii. Communicate using the idea of population density and velocity in life

[75] *This is connected to the three strands, Extension of Numbers and Operations, Measurement and Relations, and Plane Figures and Space Figures.*

[76] *Euclidean algorithm is a method of finding the largest common divisor of two numbers.*

[77] *Ratio and proportion bridge multiplication and division in a situation of two quantities with reference to the Extension of Numbers and Operations and Measurement and Relations.*

[K2MH4]
Using number lines, tables, and area diagrams for representing operations and relations in situations[78]

i. Represent proportionality on number lines with the idea of multiplication tables
ii. Use number lines, tables, and area diagrams to explain operations and relations of proportionality in situations

[K2MH5]
Establishing the idea of proportion to integrate various relations with the consistency of representations[79]

i. Use the idea of proportion as the relation of various quantities in life
ii. Identify through the idea of proportion using tables, letters, and graphs
iii. Adopt the idea of proportion to angles, arcs and areas of circles
iv. Adopt the idea of proportion to area and volume
v. Adopt the idea of proportion to enlargement
vi. Use ratio for data handling such as percent and understand the difficulties of extending it to proportion

[K2MH6]
Enjoying tiling with various figures and blocks[80]

i. Appreciate producing parallel lines with a tessellation of figures
ii. Explain the properties of figures in tessellations by reflections, rotations and translations
iii. Develop nets from solids and explain the properties of solids by each of the component figures
iv. Use the idea of tiling for calculating the area and volume

[K2MH7]
Producing valuable explanations based on established knowledge, shareable representations and examples

i. Establish the habit of explanation by referring to prior learning and ask questions using terms such as why, how, what, if and if not, and reply using examples and 'for example' in a discussion
ii. Assessing the appropriateness of explanations using representations such as generality, simplicity and clarity
iii. Use other's ideas to produce a better understanding
iv. Use inductive reasoning for extending formulae

[78] *This is a bridge to the Extension of Numbers and Operations and Measurement and Relations.*

[79] *Bridge to the three strands, Measurement and Relations, Plane Figures and Space Figures and Data Handling and Graphs.*

[80] *Connected to the two strands, Measurement and Relations, and Plane Figures and Space Figures.*

[K2MH8]
Performing activities of grouping and enjoying representing with Venn diagram

i. Use the idea of the Venn diagram for social study
ii. Understand classifications based on characteristics and represent them by using Venn diagrams

[K2MH9]
Experiencing the PPDAC (Problem-Plan-Data-Analysis-Conclusion) cycle and modelling cycle in simple projects in life

i. Understand the problem of context
ii. Plan appropriate strategies to solve the problem
iii. Gather data and analyse using appropriate methods and tools
iv. Draw a conclusion with justification based on data analysis

[K2MH10]
Preparing a sustainable life with number sense and mathematical representations[81]

i. Use minimum and sequential use of resources in situations
ii. Use data with number sense, such as order of quantity and percentage, for the discussion of matters related to sustainable development
iii. Estimate the efficient use of resources in situations
iv. Maximise the use of resources through an appropriate arrangement in a space such as a room
v. Understand "equally likely" of resources in situations

[K2MH11]
Utilising ICT tools as well as notebooks and other technological tools

i. Use internet data for the discussion of matters related to sustainable development
ii. Distinguish appropriate or inappropriate qualitative and quantitative data for using ICT
iii. Use calculators for organising data such as average
iv. Use calculators for operations in necessary context
v. Use projectors for sharing ideas as well as board writing
vi. Enjoy using notebooks to exchange learning experiences with each other such as in mathematics journal writing
vii. Use protractors, triangular compasses, straight edges, and clinometers for drawing and measuring
viii. Use the idea of proportionality to use mechanisms such as rotating once and moving twice (wheels, gears)
ix. Use various tools for conjecturing and justifying

[81] *It is related to Numbers and Operations, Pattern and Data Representations and Quantity and Measurement all under Key Stage 1.*

[K2MH12]
Promoting creative and global citizenship for sustainable development of community using mathematics

i. Utilise notebooks, journal books and appropriate ICT tools to record and find good ideas and share them with others
ii. Prepare and present ideas using posters and projectors to promote good practices in the community
iii. Listen to others' ideas and ask questions for producing better designs
iv. Utilise information, properties, models and visible representations as the basis for reasoning
v. Utilise practical arts, home economics and outdoor studies to investigate local issues for improving the welfare of life

[K2MH13]
Represent recursive process by using manipulatives, drawings and tables for finding patterns

i. Analysing situations by multiple action processes and alternating them with manipulatives, drawings and tables
ii. Find the structure of repetitions which is used in repeated actions previously
iii. Confirm recursive structures by using manipulatives, drawings and tables

Chapter 4

Key Stage 3

Key Stage 3 (KS3) is developed based on Key Stage 2. This stage focuses on four strands: numbers and algebra, relations and functions, space and geometry, as well as statistics and probability. Symbolic representations allow the dealing of abstract ideas and concepts that enhance critical and creative thinking through the application of knowledge. Understanding and using mathematical concepts and principles in this stage through discussions, dialogue, and arguments enable learners to participate in contemporary societal, economic, technological, political, environmental and mathematical issues. This stage is the basis for creating a better future with predictions. It bridges further mathematics learning in various job demands.

Strand: Numbers and Algebra [K3NA]

Numbers are extended to positive and negative numbers and square roots. Algebraic expressions are already introduced by the mathematical sentences and symbols at Key Stage 2. At Key Stage 3, algebra is operated by expressions and equations until the second degree. On the extension from numbers to symbolic algebra, various possible ways of calculations are explored until their appropriateness is established. Like and unlike terms are introduced in algebraic sentences and in simplifying expressions. Properties of equations are introduced to find simple equivalents and solve equations with fluency. Substitution, addition and subtraction of equations enable further operations of simultaneous equations. Expansion and factorisation enable further operations of the polynomials. Finally, quadratic equations can be solved using various operations.

Topics:

Extending Numbers to Positive and Negative Numbers [K3NA1]
Utilising Letters for Algebraic Expressions and Equations [K3NA2]
Algebraic Expressions, Monomials and Polynomials and Simultaneous Equations [K3NA3]
Expansion and Factorisation of Polynomials [K3NA4]
Extending Numbers with Square Roots [K3NA5]
Solving Quadratic Equations [K3NA6]

Extending Numbers to Positive and Negative Numbers [K3NA1]

[K3NA1-1]
Extending numbers to positive and negative numbers and integrating four operations into addition and multiplication

i. Understand the necessity and significance of extending numbers to positive and negative numbers in relation to directed numbers with quantity
ii. Compare numbers which are greater or less than on the extended number line and use absolute value for the distance from zero
iii. Extend operations to positive and negative numbers and explain the reason
iv. Get efficiency on a calculation about the algebraic sum

Utilising Letters for Algebraic Expressions and Equations [K3NA2]

[K3NA2-1]
Extending the utilisation of letters for a general representation of situations and finding ways to simplify algebraic expressions

i. Appreciate the utilisation of letters for a general representation of situations to see the expression as a process and value
ii. Find ways to simplify expressions using distributive law and figural explanations, establish the calculation with like and unlike letters
iii. Acquire fluency in simplifying an expression and appreciate it for representing the pattern of the situation

[K3NA2-2]
Thinking about a set of numbers in algebraic expression with letters as variables and representing them with equality and inequality

i. Recognise numbers as positive and negative numbers, and explain integers as a part of numbers
ii. Represent a set of numbers using variables with equality and inequality
iii. Translate given sets of numbers on the number lines using interval and inequality notations
iv. Appreciate redefining even and odd numbers using letters to represent different sets of variables

[K3NA2-3]
Thinking about how to solve simple linear equation

i. Review the answers to equations from the set of numbers and think backwards
ii. Know the properties of equations which keep the set of answers of the equation

iii. Appreciate the efficient use of properties of equations to solve linear equation
iv. Use equations based on life situations to develop fluency in solving equations, and interpreting the solutions

Algebraic Expressions, Monomials and Polynomials and Simultaneous Equations [K3NA3]

[K3NA3-1]
Thinking about the calculations of monomials and polynomials for simple case[82]

i. Introduce terms, monomials and polynomials
ii. Introduce a number raised to the power of two as a square and a number raised to the third power as a cube
iii. Get fluency in the calculation of polynomials such as combining like terms and the use of the four operations in simple cases

[K3NA3-2]
Thinking about how to solve simultaneous equations in the case of linear equations

i. Understand the meaning of solutions of linear equations and simultaneous linear equations as a pair of numbers
ii. Know the substitution and elimination methods of solving simultaneous linear equations
iii. Get fluency in selecting the methods from the form of the simultaneous linear equations
iv. Appreciate simultaneous linear equations in situations

Expansion and Factorisation of Polynomials [K3NA4]

[K3NA4-1]
Acquisition to see the polynomials in the second degree with expansion and factorisation and use it

i. Use the distributive law to explain the formulae for expansion and explain them on diagrams
ii. Acquire proficiency in selecting and using the appropriate formulae
iii. Use the expansion formulae to factorise the second-degree expression and recognise both formulae with the inverse operation
iv. Solve simple second-degree equations using factorisation and apply it in life situations

[82] *Simple cases may vary from and depending on the countries based on the mapping of the curriculum.*

Extending Numbers with Square Roots [K3NA5]

[K3NA5-1]
Extending numbers with square roots and calculating the square roots algebraically

i. Define square root and discuss ways to estimate the nearest value of a square root by the Sandwich Theorem
ii. Understand that some square roots cannot be represented as fractions
iii. Compare square roots using a number line and understand that the order does not change but the differences between two consecutive square roots varied
iv. Think about multiplication and addition of square roots and understand the algebraic way of calculation which is similar to polynomial
v. Appreciate square roots in applying to situations in life[83]

Solving Quadratic Equations [K3NA6]

[K3NA6-1]
Solving simple second-degree equations using factorisation and applying the situation

i. Find the answers to simple second-degree quadratic equations by substitution and explore by completing the square, quadratic formula, and factorisation[84]
ii. Get fluency in selecting the appropriate ways to solve quadratic equations
iii. Apply quadratic equations in life situations

[83] *Pythagorean Theorem is discussed in Space and Geometry strand.*
[84] *The graph of the quadratic equation will be treated in Relations and Functions strand.*

Strand: Relations and Functions [K3RF]

Relationships are represented by equations and a system of equations. Functional relations are treated amongst situations, tables, and equations of function are introduced based on patterns and relations with algebraic representation in Key Stage 2 and Stage 3. The solution of a simple equation is done by equivalence deduction based on algebra learnt earlier. Two variables in simultaneous equations as a simple system of equations are solved by substitution and additive-subtractive methods. Three representations, the table, equation and graph, are used as methods to analyse the properties of every function. Proportion and inverse proportions are redefined with those representations mentioned. The proportional function is extended to line functions. The comparisons of inverse proportion and line functions are explained clearly by using the property of linearity with a 'constant ratio of change'. The concept of proportion is extended to the function of $y = ax^2$. Ways of translations between table and equation, equation and graph, and graph and table are specific skills for every function to be developed with fluency.

Topics:

Extending Proportion and Inverse Proportion to Functions with Variables [K3RF1]
Exploring Linear Function in Relation to Proportions [K3RF2]
Exploring Simple Quadratic Function [K3RF3]
Generalising Functions [K3RF4]

Extending Proportion and Inverse Proportion to Functions with Variables [K3RF1]

[K3RF1-1]
Extending proportion and inverse proportion to functions with variables on positive and negative numbers

i. Extend proportions to positive and negative numbers, using tables and equations on situations
ii. Plot a set of points as a graph for proportions defined in ordered pairs (x, y) in the coordinate plane using appropriate scales precisely[85]
iii. Introduce inverse proportion using tables, equations and graphs
iv. Introduce function as correspondences of two variables in situations
v. Explore the property of proportional function with a comparison of inverse proportional function
vi. Appreciate proportion and inverse proportion functions in life

[85] *Utilising ICT is recommended in mathematical activities.*

Exploring Linear Function in Relation to Proportions [K3RF2]

[K3RF2-1]
Exploring linear functions in relation to proportion and inverse proportions

i. Identify linear functions based on situations represented by tables and compare them with proportional functions
ii. Explore properties of the linear function represented by tables, equations and graphs and compare it with direct and inverse proportional functions
iii. Acquire fluency to translate the rate of change of a linear function represented in the table as a coefficient in an expression and gradient in a graph[86]
iv. Acquire fluency to translate y values of x = 0 in a table, constant in an expression, and *y-intercept* in a graph
v. Apply the graphs of linear functions to solve simultaneous equations
vi. Apply the linear function for data representation on situations to determine the best-fit line

Exploring Simple Quadratic Function [K3RF3]

[K3RF3-1]
Exploring quadratic function $y = ax^2$ in relation to a linear function

i. Identify the quadratic function on situations using tables and comparing it with a linear function
ii. Explore properties of a quadratic function using tables, equations as well as graphs and compare it with a linear function
iii. Apply the quadratic function to situations in daily life and appreciate it

Generalising Functions [K3RF4]

[K3RF4-1]
Generalising functions with various representations[87] of situations

i. Distinguish domain, range and intervals and is appropriately used for explaining the function
ii. Use various situations for generalising ideas of functions such as moving point A and moving point B with time
iii. Compose a graph as a function of two or more graphs with different domains in a situation
iv. Introduce situations of step-functions[88] using a graph to generalise the idea of a function in which an equation cannot be represented.

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[86] *Family of linear functions are recommended to use ICT tool under mathematical activities strand.*
[87] *Utilising functions as models in daily life which is necessary for STEM education.*
[88] *Intervals are taught in the Key Stage 3 on Numbers and Algebra.*

Strand: Space and Geometry [K3SG]

Space and Geometry provide ways of reasoning for exploring properties in geometry and produce ways of argument to explain justifications of visual reasoning. The calculations of angles are not just simple calculations but also ways of using geometric propositions to justify answers by explaining why it is correct based on fundamental properties. The properties of congruency and similarity are identified and described by explaining the relationship of figures using transformation. Finding the value of angles and building arguments to prove them are ways to develop the habit of reasoning in the properties of plane figures. The conditions of congruency, similarities, and properties of circles are also used to explain and prove the appropriateness of geometric conjectures about triangles, quadrilaterals, and circles. Dynamic geometric software, a simple compass and a ruler are used for conjecturing. It shows general ideas from consistency in variations. The counterexample is also found as a special case from variations.

Topics

Exploring Angles, Construction and Designs in Geometry [K3SG1]
Exploring Space with its Components [K3SG2]
Exploring Ways of Argument for Proving and Its Application in Geometry [K3SG3]

Exploring Angles, Construction and Designs in Geometry [K3SG1]

[K3SG1-1]
Exploring angles to explain simple properties on the plane geometry and doing simple geometrical Construction[89]

i. Explain how to determine the value of angles using the geometrical properties of parallel lines, intersecting lines, and properties of figures
ii. Use a ruler and compass to construct a simple figure such as perpendicular lines and bisectors
iii. Appreciate the process of reasoning that utilises the properties of angles and their congruency in simple geometrical constructions

[K3SG1-2]
Exploring the relationship of figures using congruency and enlargement for designs

i. Explore the congruence of figures through reflection, rotation and translation and explain the congruency using a line of symmetry, point of symmetry and parallel lines

[89] *Simple geometric construction is discussed by the ruler and compass with reasoning. Dynamic Geometric software usually draws entire circles. For knowing invariants, dynamic geometric software is useful.*

ii. Explore the similarity of figures with enlargement using points, ratios, and correspondences
iii. Enjoy using transformations in creating designs

Exploring Space with its Components [K3SG2]

[K3SG2-1]
Exploring space by using the properties of planes, lines and their combinations to form solids

i. Explore the properties produced by planes, lines and their combinations, such as parallel lines created by the intersection of parallel planes with another plane
ii. Produce solids by combining planes such as nets and motion such as rotation, reflection and translation
iii. Recognise the space of an object based on its properties and projection

Exploring Ways of Argument for Proving and its Application in Geometry[90] [K3SG3]

[K3SG3-1]
Exploring properties of congruency and similarity on plane geometry

i. Explore ways of arguments using the congruence of two triangles and appreciate the logic of the argument in simple proving
ii. Explore ways of arguments using the similarity of two triangles based on ratio and angles and appreciate the logic of arguments in simple proving
iii. Explore the proof of the properties of circles, such as inscribed angles and intercepted arcs
iv. Appreciate proving through making the order of proven propositions to find new propositions

[K3SG3-2]
Exploring Pythagorean theorem in solving problems in plane geometry and spaces

i. Explore the proving of the Pythagorean theorem using a diagram and use it in solving problems involving plane figures
ii. Apply the Pythagorean theorem on the prism by viewing the figures through faces.
iii. Explore the situations for simple trigonometry using special angles with the Pythagorean theorem
iv. Appreciate the use of the Pythagorean theorem in life[91]

[90] *Dynamic Geometry software is useful for finding the invariant properties which is discussed in Mathematical Processes and Humanity strand in Key Stage 3.*

[91] *Pythagorean theorem is used for re-understanding the topic of square root under Key Stage 3 Numbers and Algebra.*

Strand: Statistics and Probability [K3SP]

Data handling is extended to explore the dispersion of histogram with mean, median, mode and range. Exploratory data analysis (EDA) attempting to represent and visualise the structure from the given data using Information Communication and Technology (ICT) is enhanced. The histogram shows different dispersion if we change the class. Probability is introduced as a ratio with the law of large numbers. Sample space with the assumption of equal probability becomes the point of discussion. Logical analysis to understand possible cases, such as a tree diagram, is introduced for knowing how to represent logical reasoning. Histogram can be seen relatively and produce frequency distribution polygon. The difference between the sample and the population is discussed. Box plots with quartiles are an extension of the median, and the range is used to compare distributions. Using skills in statistics and probability makes problem solving in situations possible. Analysing and identifying the trends in situations and making decisions is necessary, such as issues for sustainable living.

Topics:

Exploring Distribution with the Understanding of Variability [K3SP1]
Exploring Probability with the Law of Large Numbers and Sample Space [K3SP2]
Exploring Statistics with Sampling [K3SP3]

Exploring Distribution with the Understanding of Variability [K3SP1]

[K3SP1-1]
Exploring distribution with histograms, central tendency and variability

i. Use histograms with different class intervals to show a different distribution of the same set of data
ii. Identify alternative ways to show distribution, such as dot plots, box plots and frequency polygons
iii. Investigate central tendencies such as mean, median, mode[92] and their relationships in a distribution
iv. Investigate dispersion such as range and inter-quartile range in a distribution
v. Appreciate the analysis of variability through the finding of the hidden structure of distribution on situations using the measure of central tendency and dispersion

[92] *Mean, median and quartile are fixed depending on the data. However, the mode changes depending on the class.*

Exploring Probability with the Law of Large Numbers and Sample Space [K3SP2]

[K3SP2-1]
Exploring probability with descriptive statistics, the law of large numbers and sample space

i. Experiment with tossing coins and dice to explore the distribution of the relative frequency and understand the law of large numbers
ii. Use the idea of equally likely outcomes to infer the value of the probability[93]
iii. Analyse sample space of situations represented by a table to determine the probability and use it for predicting the occurrence
iv. Use various representations such as tables, tree diagrams, histograms and frequency polygons to find the probability
v. Analyse data related to issues on sustainable development and use probability to infer and predict future events

Exploring Statistics with Sampling [K3SP3]

[K3SP3-1]
Exploring sampling with the understanding of randomness

i. Discuss the hidden hypothesis behind the sample and population
ii. Use randomness to explain sampling
iii. Analyse the data exploratory, such as dividing the original into two to know better data representations and discuss appropriateness, such as regrouping
iv. Appreciate data sampling in a situation with sustainable development

[93] *The probability here is called equiprobability when all possible cases are equally likely. In the upper-grade level, probability will be redefined based on distributions.*

Strand: Mathematical Process – Humanity [K3MH]

In the era of Generative AI, the critical argument in mathematics is enhanced through communication with others beyond Key Stage 2. These proposed challenging activities will promote metacognitive thinking at different levels of arguments to make sense of mathematics. Translating real-life activities into mathematical models and solving problems using appropriate strategies are emphasised in functional situations. The process of doing mathematical activities involves patience, which develops perseverance in learners and helps them take responsibility for their learning. At this stage, the habitual practice of self-learning will eventually build confidence. Thus, the opportunity for challenges to extend mathematics and the ability to plan sequences of future learning is also enhanced.

Standards:

Enjoying problem solving through various questioning and conjecturing for extension of operations into algebra, space and geometry, relationship and functions, as well as statistics and probability [K3MH1]
Enjoying measuring space using calculations with various formulae [K3MH2]
Producing proof in geometry and algebra [K3MH3]
Utilising tables, graphs and expressions in situations [K3MH4]
Using diagrams for exploring possible and various cases logically [K3MH5]
Exploring graphs of functions by rotation, by symmetry and by translation of proportional function [K3MH6]
Understanding ways for extension of numbers [K3MH7]
Designing a sustainable life with mathematics [K3MH8]
Utilising ICT tools as well as other technological tools [K3MH9]
Promoting creative and global citizenship for sustainable development of society in mathematics [K3MH10]
Represent recursive process by using manipulatives, drawings and tables for finding patterns [K3MH11]

[K3MH1]
Enjoying problem solving through various questioning and conjecturing for extension of operations into algebra, space and geometry, relationship and functions, as well as statistics and probability[94]

i. Pose questions to extend numbers and operations into positive and negative numbers, algebraic operations, and further extension into polynomial operations, and numbers with square roots
ii. Pose questions to solve linear equations, simultaneous equations and simple quadratic equations
iii. Pose assumptions in geometry as objects of argument and proof

[94] *Connected to the three strands, namely Numbers and Algebra, Relations and Functions, and Space and Geometry.*

iv. Pose questions to transform three-dimensional objects into two-dimensional shapes and vice versa
v. Pose questions in relations and functions for knowing properties of different types of function
vi. Pose questions to explore data handling and to know the structure of distributions
vii. Pose questions that apply PPDAC in relation to statistical problem solving
viii. Pose questions in relation to "equally likely" events
ix. Pose assumptions to discuss the hypothesis based on the sample and population
x. Pose conjectures such as if x increases and y decreases, then it is an inverse proportion

[K3MH2]
Enjoying measuring space using calculations with various formulae[95]

i. Extend the number line to positive and negative numbers and compare the size of numbers with the idea of absolute value
ii. Derive the square root using unit squared paper through the idea of the area
iii. Explain the expansion of polynomials using area diagrams
iv. Use the projection of space figures to plane figures using the Pythagorean theorem
v. Apply similarity and simple trigonometry for measurement
vi. Use common factors to explain factorisation of the area of a rectangle based on the area of a square

[K3MH3]
Producing proof in geometry and algebra

i. Have an assumption through exploration and produce propositions
ii. Justify the proposition using examples and counter-examples to achieve understanding
iii. Rewrite propositions from sentences to mathematical expressions by using letters and diagrams
iv. Search the ways of proving by thinking backwards from the conclusion and thinking forward from the given
v. Show the proof and critique for the shareable and reasonableness
vi. Deduce other propositions in the process of proving and after proving using what if and what if not
vii. Adapt ways of proving to other similar propositions of proof
viii. Explain the written proof in geometry and algebra by the known
ix. Revise others' explanations meaningfully

[95] *Connected to the three strands, namely Numbers and Algebra, Relations and Functions, and Space and Geometry.*

[K3MH4]
Utilising tables, graphs and expressions in situations[96]

i. Explore the properties of functions by using tables, graphs and expressions and establish the fluency of connections among them for interpreting functions in the context
ii. Analyse the distribution of raw data by using tables, graphs and expressions in daily life

[K3MH5]
Using diagrams for exploring possible and various cases logically[97]

i. Use number line with inequality to identify range and set
ii. Use a circle to identify the relationship between the circumference and the central angle (acute, obtuse and right)
iii. Use a rectangle and rotate a point on the side rectangle to draw the graph of the area
iv. Use a tree diagram for thinking about all possible cases sequentially

[K3MH6]
Exploring graph of functions by rotation, by symmetry and by translation of proportional function[98]

i. Use the slope of a graph for the proportional function to rotate the graph or to determine the point of intersection
ii. Explore to know the nature of two simultaneous equations by using translation
iii. Use the y-axis, x-axis and $y = x$ as the line of symmetry to explore the proportional function
iv. Explain the linear function graph by translating the proportional function.

[K3MH7]
Understanding ways for extension of numbers[99]

i. Extend the numbers based on the necessity of solving equations such as $x + 5 = 3$ and $x^2 = 2$, and show examples for demonstrating the existence, such as on the number lines, and understand it as a set
ii. Compare the size of numbers and identify how to explain the order of numbers and their equivalence

[96] *Connected to the strand on Relations and Functions.*
[97] *Connected to the two strands of Relations and Functions and Space and Geometry.*
[98] *Connected to the two strands of Numbers and Algebra and Relations and Functions.*
[99] *Connected to the strand on Numbers and Algebra.*

iii. Extend operations to keep the form [100] beyond the limitations of meaning[101]

[K3MH8]
Designing models for sustainability using mathematics[102]

i. Discuss and utilise probabilities in life, such as weather forecasting, for planning
ii. Design cost-saving lifestyle models through comparison of data such as cost of electricity, water consumption, and survey
iii. Plan emergency evacuation such as heavy rain and landslide where the calculations on the amount of water in barrel per minute exceed the maximum standards
iv. Forecast the future with mathematics

[K3MH9]
Utilising ICT tools as well as other technological tools

i. Use dynamic geometry software for assumption, specialisation and generalisation
ii. Use a graphing tool for comparison of the graph and knowing the properties of a function
iii. Use data to analyse statistics with software
iv. Use internet data for the discussion of sustainable development
v. Use calculators for operations in the necessary context
vi. Use a projector for sharing ideas such as project surveys, reporting and presentation
vii. Use the idea of function to control a mechanism
viii. Use ICT tools for conjecturing and justifying to produce the object of proving.

[K3MH10]
Promoting creative and global citizenship for the sustainable development of society using mathematics

i. Utilise notebooks, journal books and appropriate ICT tools to wisely record and produce good ideas for sharing with others
ii. Prepare and present ideas using posters, projectors, pamphlets and social media to promote good practices in society

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[100] *There are three meanings of the form: (1) Permanence of form means "keep the pattern of operation" such as (-3)x(+2)=-6, (-3)x(+1)=-3, (-3)x0=0, and (-3)x(-1)=+3, and (-3)x(-2)=+6. Here, the product of the pattern increases by 3; (2) The form means "Principle of the permanence of equivalence form" which means to keep the law of commutativity, associativity and distributivity; and (3) The form means the axiom of field in Algebra. Normally, in education, we only treat (1) and(2).*

[101] *For the extension of numbers to positive and negative numbers, beyond the limitations of meaning such as subtracting smaller numbers from larger numbers. For the extension of numbers to irrational numbers, beyond the limitation of meaning such as rational numbers is quotient (value of division).*

[102] *Connected to the three strands, Numbers and Algebra, Relations and Functions and Space and Geometry.*

iii. Promote the beautifulness, reasonableness and simplicity of mathematics through contextual situations in the society
iv. Listen to other's ideas and ask questions for better designs, craftsmanship and innovations
v. Utilise information, properties, models and visible representations as the basis for making intelligent decisions
vi. Utilise practical arts, home economics, financial mathematics and outdoor studies to investigate local issues for improving the welfare of life

[K3MH11]
Represent recursive process by using manipulatives, drawings and tables for finding patterns

i. Analysing situations by multiple action processes and alternating them with manipulatives, drawings and tables
ii. Find the structure of repetitions which is used in repeated actions previously
iii. Confirm recursive structures by using manipulatives, drawings and tables
iv. Use the established recursive structures to produce new situations

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Appendix A

Terms of the Revised CCRLS Framework in Mathematics

Higher-order thinking is the curriculum terminology, but it is not specified in mathematics. Here, it is explained generally as acceptable terms from the perspective of mathematics in education. The following terms are samples that appear in SEA-BES: CCRLS (Figure 1) on Mathematical Thinking and Processes and Values and Attitudes. These terms are explained to make clear descriptions of the objectives of teaching. If you use these terminologies for writing the teaching objectives, you will be able to consider how you teach them in the process. Mathematical Thinking and Processes can be explained through (i) Mathematical Ideas, (ii) Mathematical Ways of Thinking, and (iii) Mathematical Activities. Mathematical values and attitudes for Human Character Formation are explained through (iv)Value and (v)Attitude. Mathematical Ideas and Ways of Thinking can be developed through reflection on the processes, while Values and Attitudes are developed through appreciation.

Mathematical Thinking and Processes

(i) Mathematical Ideas

Although every mathematics content embeds some necessary ideas, essential mathematical ideas are used in various situations/contexts. Mathematical ideas are not exclusive but function as complementary. The following are samples of crucial mathematical ideas.

Set

A set is a collection of elements based on certain conditions. When the condition of the set changes, the result of reasonings related to the set may change too. By the definition of figures, we can compare the inclusion relationship between figures if every figure is defined by the conditions. If it is the number of elements, the sets are compared by one-to-one correspondence. The idea of a set is reflected through activities that require us to think about the membership (by elements or conditions) of a set. In addition, activities involving subsets, cardinality and power are extended ideas of a set. The number of elements is called a cardinal or cardinal number (or set number). An ordinal number does not imply the number of elements. Other ideas include operations of sets such as union, intersection, complement, the ordered pair/combination of elements such as Cartesian products and dimension mapping. A number system is a set with structures of equality, order (greater, less than), and operations, which are developed and extended throughout the curriculum from natural numbers to complex numbers: the set of complex numbers does not have the structure of order.

Unit

Units are necessary for counting, measurement, number lines, operations, and transformation. It is represented as “denomination’ for discrete quantity, such as 1 “apple’ for situations involving counting, or continuous quantity, such as 1 gram for situations involving measurement. Mathematically, a unit indicates a number by mapping it with the quantity in a situation.

In a situation, it can be fixed based on the context of comparison, which can be a direct or indirect comparison. In this context, a remainder or a difference from a comparison can be used to fix a new arbitrary unit for measurement that is a fraction of the original unit. This process of determining a new unit is the application of the Euclidean algorithm for finding the greatest common divisor.

For the base-10 place value number system, every column is defined by units such as ones, tens, hundreds, etc. However, in other place value number systems such as the binary system, every column is defined by units such as ones, twos, fours and so on. Therefore, in a place value number system, the unit is not always multiple to the power of ten.

In multiplication, any number can be seen as a unit for counting. Other number systems are made up of different units. For the calendar system, the lunar calendar is based on 30 (29.5) days, while the solar calendar is based on 365 (365.25) days. On the other hand, the imperial and U.S. customary measurements include units in the base-12 and base-16 systems. In the ancient Chinese and Japanese systems, there were units in the base-4, including the base-16. On the other hand, the units used in different currency systems depend on various cultures and countries. However, many countries had lost the unit of 1/100 on their currency systems, which originated from "per centos", which means percent. Even though the based-10 place value system represents the value of money, many currency systems use the units for 2, 5, and 25 in their denominations instead.

Unit for a new quantity can be derived from the ratio of different amounts. For example, the unit for speed (km/h) is the ratio of distance (km) to time (h), which cannot be added directly. A car moves at 30km/h and then at 20km/h does not mean the car moves at 50km/h.

The identity element for multiplication is one, but the additive identity is zero. Identity for multiplication is the base for multiplicative and proportional reasoning. The inverse element for multiplication is defined by using one.

Comparison

In any mathematical investigation, particularly in the mathematics classroom, problem-solving approaches and comparisons of various ideas, representations and solutions are key activities for discussion and appreciation. This comparison is the nature of the mathematical activity to find better ideas.

When comparing amounts, concrete objects can be compared directly or indirectly without measurement units. As mentioned in the Unit, a **direct comparison** can be used to fix a new unit of measurement. In contrast, **indirect comparison** can promote logic for transitivity, including syllogism.

Comparison of multiple denominated numbers with different unit quantities on the same magnitude, such as 5.2 m and 5 m 12 cm, can be done if they are represented by a single denominated number by the unified unit quantity, such as 520 cm and 512 cm. Furthermore, comparing expressions with the same answers on the same operation, such as 2+4, 3+3 and 4+2, can be used to find rules and patterns. For example, 2+4 = 4+2 shows a commutative rule for addition, whereas 2+4 = 3+3 can show a pattern when 1 is added to 2 and subtracted from 4; the sum is still the same.

Comparison of fractions is an activity to find the unit fraction. For comparison of fractions, such as $\frac{1}{2}$ and $\frac{1}{3}$, we have to find the unit fraction $\frac{1}{6}$ which can measure $\frac{1}{2}$ and $\frac{1}{3}$. $\frac{1}{6}$ is the common denominator for $\frac{1}{2}$ and $\frac{1}{3}$. The algorithm to find the unit fraction as the common denominator is called 'reduction of fraction'.

On numbers, the relationship between two numbers can be equal, greater or less. The number set up to a real number is a total/linear order set; thus, two numbers can be compared to a real number set. However, a complex number as an extension of a real number cannot be compared directly because it has two dimensions.

On the number line as a real number, the size of the number (distance) is defined by the difference, 1=2-1=3-2=4-3=… Here, the difference is the subtraction value as a binary operation, which can be seen as the equivalence class. Regarding an equivalence class, the value of operations can be compared. On a plane such as a complex plane, even though the number is not simply ordered, the size of the number (distance) is defined by the Pythagorean theorem. By using this definition, I1I= I$(\frac{\sqrt{2}}{2})(1 + i)$I = I$i$I = I$(\frac{\sqrt{2}}{2})(1 - i)$I=... the theorem produces the distance on the plane, and the distance can be compared.

As explained in the Unit of measurement, the magnitude is given by defining the unit of magnitude of measurement. One of the ways to produce the unit magnitude is a direct comparison, which provides the difference, and the Euclidean Algorithm produces the unit of measurement as the greatest common divisor originating from the difference.

Another way to produce the unit is by using the ratio and multiplication. Such a newly produced magnitude lost linearity. In Physics, 'dB' is the size of volume produced by the common logarithm of sound pressure. 'dB' fits well with a human's impression of the size of sounds on its linearity. It is known as Weber-Fechner's law that human senses are proportional to the logarithm of the stimulus. In science, a logarithmic scale is used for a semi-log graph and a log-log graph for demonstrating extended linearity, even if it is an exponential phenomenon. Logarithms produce the scale to illustrate multiplicative phenomena as additive phenomena.

Operations

Addition, Subtraction, multiplication and division are four basic arithmetic operations. These binary operations involve any two numbers with symbols of operations +, –, × and ÷. Polynomial expressions such as $2x + 3$ are seen as a combination of binary operations. Mental arithmetic may be used in the column method with the base-10 place value system. An operation is not just a rule but can be demonstrated using various representations. For example, an operation can be represented by the manipulation of concrete objects as well as expressions. However, the actual manipulating process is different from the operation process. To show it the same way as the operation, we cut off some parts of the actual process and represented them as diagrams. If students can omit such parts, they can think of it as an operation.

From Key Stage 3 up to the field theory at the university level, arithmetic operations are only expressed as addition and multiplication. The negative vector is represented using the minus symbol on the axiom of vector space.

Although arithmetic operations are well known, limited operations exist for each specific set of numbers, such as modulus. Depending on the context, it is called a function or algorithm.

Algorithm

An algorithm is a set of sequential activities in a unique situation to produce a particular solution to a set of tasks. The column method is based on the base-10 place value system. When using this method, 200 + 300 is done by just 2 + 3 on the hundreds place value. This algorithm is adapted for mental calculation. Representation of the column method is not universal like an expression as algebraic representation. The algorithm for the column method is fixed as a formal form based on every culture. However, it can be created by manipulating the base ten blocks. On the other hand, an algebraic form of operation such as 2+3=5 is the universal form.

A formula also functions like an algorithm. It can be applied without understanding its meaning. However, it can only be created with understanding and recognising its underlying structure or meaning. If we know the structure or meaning, such as ratio and proportionality, memorising it is unnecessary.

Fundamental Principles

Fundamental principles are rules related to mathematical structures and forms in general.

Commutativity, Associativity and Distributivity are three fundamental principles for arithmetic operations. Commutativity does not work on subtraction and division. On the discussion of Distributivity, if a, b, and c are positive numbers, then the expressions $a(b + c)$, $(b + c)a$, $a(b - c)$, and $(b - c)a$ are different. However, if a, b, and c are positive and negative numbers, the four expressions can be seen as the same. Before the appearance of Algebraic Axiomatic Systems such as Group, Ring and Field, these principles existed as the principles. There are also other fundamental principles for arithmetic operations to simplify the operations at the elementary level, such as the following:

$1 + 9 = 10$	$2 \times 3 = 6$	$8.1 \div 9 = 0.9$	$8.1 \div 9 = 0.9$
$\downarrow{+1}\quad\downarrow{-1}$	$\downarrow{\times 10}\qquad\downarrow{\times 10}$	$\downarrow{\times 10}\qquad\uparrow{\div 10}$	$\downarrow{\times 10}\quad\downarrow{\div 10}$
$2 + 8 = 10$	$20 \times 3 = 60$	$81 \div 9 = 9$	$8.1 \div 90 = 0.09$

Most of them are related to proportionality used by the Ancients.

Principles can be identified through a comparison of equations. They are necessary for explaining algorithms and thinking about how to calculate by using models and other representations. On the extension of numbers and operations, principles are used to discuss the permanence of form (see the permanence of form).

In geometry, the extendable nature of a line changes its functions in the curriculum. For example, the shape is extended to the figure; the edge, which may include the inner part of a shape, is extended to the side, which may not include the inner part of a figure. Then, the side is extended to a line, which enables the discussion of the possibility of escribed circles. In addition, parallel lines are necessary to derive the area formula for triangles with various heights.

Permanence of Form

The Principle of the Permanence of the Equivalence of Form, Hankel's Principle, is known as Commutativity, Associativity, and Distributivity for algebra for the field theory. It is a fundamental principle.

On the other hand, the permanence of form appeared in the history of mathematics in the 16th century and functioned to shift from arithmetic algebra to symbolic algebra. Peacock's Permanence of Form is not only the limited three rules like Hankel's but also applies to any algebraic symbolic form.

In Education, the form is not limited to fundamental principles but includes patterns, and the permanence of form can be used in various situations. It is especially used for the extension of numbers and operations from elementary level to secondary level education like the following:

(+3) + (+2) = +5	(+3) – (+2) = (+1)
↓-1 ↓-1	↓–1 ↓+1
(+3) + (+1) = +4	(+3) – (+1) = (+2)
↓-1 ↓-1	↓–1 ↓+1
(+3) + 0 = +3	(+3) – 0 = (+3)
↓ ↓	↓ ↓
(+3) + (-1) = ?	(+3) – (-1) = ?

The '?' are unknown and have not yet been learned, but other parts are known. However, people could imagine the '?' by analogical reasoning with the idea of the permanence of the patterns. Here, the permanence of patterns is used as a hypothesis, making it possible to apply it to unknown cases. And it provides the necessary explanation for unknown '–(-1)' as known '+(+1)'.

The permanence of form appears at the initiation of numbers in Key Stage 1 and can be seen at all other key stages. For example, it explains the necessity of introducing zero (0), which does not have any necessity to represent it to count the existing objects.

Various Representations and Translations

Every specified representation provides some meanings based on its essential nature of representation, which can be produced by specified symbols and operations. Different representations have different natures and use different symbols and operations. Every representation has the limitation of interpretation on its nature. Thus, thinking by using only one specified representation provides the limitation of reasoning and understanding. If one type of representation is translated into another type of representation, then the representation can be interpreted in other ways. If the idea of specified representation with a certain embedded nature is translated into various representations, a rich and comprehensive meaning and use will be produced. However, to make the translation meaningful, it is necessary to know the way of translations, which consist of correspondences between symbols and operations on different representations.

For example, proportional number lines only function for teachers and students who know well how to represent the proportionality on the tape diagrams and number lines, and so on. If they know what it is, they can use it to explain and produce an expression.

Comprehensive learning of mathematics using various representations and translations is necessary; however, students have to learn how to represent it in other representations and translate, at first, between expressions and situations.

According to Isoda (2016, 2018), representation in mathematics is specified by context/objective, symbols, and operations of symbols, even if they are figural representations. A proportional number line is the symbol. Operations are multiple under proportionality. The objective is to find the answer to multiplication and division. The proportional number line just looks like teachers' explanation tool if they do not teach it as a way of representation. However, if teachers teach it as a way of representation to students, they can use it as a tool for thinking by and for themselves. For teaching proportional number lines, teachers must teach how to scale the lines using the idea of multiple because it is the way of operation for them.

Pattern, Recursion and Invariant

Mathematics is the science of patterns. In other words, the science of finding invariants. A pattern means the existence of an invariant, something with no change in repetition. In the case of numbers, they are usually related to natural number sequences and something constant. For example, sequence patterns such as the arithmetic sequence have the same difference, and the geometric sequence has the same ratio. In a table, if the first difference is constant, it is an arithmetic sequence, but if the second sequence is constant, it is a quadratic sequence. The table also shows some functional relationships between x and y. The pattern is usually found on the table when given pairs of numbers are lined in an appropriate order. Lining ordered pairs under natural numbers appropriately is necessary to find the pattern.

Recursion on natural number sequence is usually represented by the recurrence formula. In mathematics, it is used for complete induction. In the programming, recursion means just repetition in a limited set on the natural number for the recurrence formula, and it focuses on the part of complete induction; if p(k) is true, then p(k+1) is true. Instead of algebraic form, such recursive process on the equations is usually represented by the repetition of figural diagrams (representation), which use the previous drawing at the next step in a consistent manner.

In the case of a figure, a pattern is also found on the tessellation of the congruence triangle. It is an appropriate repletion of the same triangle. If we tessellate it, we can find the invariant properties of angles relating to parallel lines. On the tessellated design as a whole, we can also find translation, rotation and symmetry. We can also find the enlarged figure, which shows the similarity of the figures. Mandelbrot set on infinite geometry is the recursion of figures with the natural number sequence. In the programming, finite cases can be identified on the screen.

An invariant is a stronger word in mathematics if we compare it with the usage pattern, which means repetition in informatics, such as programming because it is necessary to be proved in the mathematics system. Dynamic Geometry Software (DGS) provides ways to find something invariant which should be proved. It is normal, nothing strange, that two lines meet at one point if it is not parallel. However, it is exceptional and strange if three lines meet at one point because three lines usually produce a triangle. Thus, constructing the circumcenter, incenter and centroid on any triangle is amazing and should be proven.

Graphing Tools (GT) for functions are the tools that represent invariant properties in Algebra and Analysis. On f(x)=ax2+bx+c, if it is drawn by GT with parameters a, b, c, and fix b and c as any constant and changes a real number, the graph of f(x) produces the family of functions on the screen. Any graph of the f(x) family intersects with the y-axis on (0, c), and y=bx+c is the tangent line for any f(x). We should prove these findings as invariant in mathematics based on the chosen mathematical system. Various proofs are possible if we change the theory in mathematics.

Ordering

Ordering is a way of comparison by using ordered numbers such as Natural Numbers, which have total/linear order, and ordering activities are bases for finding patterns and invariants. Natural Numbers are produced by Peano's Axiom, which includes the first number (1), the next number (+1) and mathematical induction. The number sequence, such as 2, 5, 8, 11, 14, …, n+3, … is already ordered by the first number (2), the next number (+3), and the mathematical induction $a_{k+1} = a_k + 3$. It might be difficult to find the pattern if it is ill-ordered, such as 11, 8, 2, 14, and 5. If the order is given as 2, 5, 8, 11, 14, we can find the difference of 3 as an invariant and guess the next number as 17.

Ordering is the activity of arranging objects from an ill-ordered situation to a well-ordered situation. When a table is used to find the property of a function, the top row of the table is usually given by integers. Otherwise, we could not find any pattern. Thus, to teach the necessity of order, we should begin from an ill-ordered situation to find the appropriate order to show patterns and invariants.

Symmetry

Symmetry is used for reasoning on patterns. It is known in geometric reasoning by point symmetry and line symmetry. Mirror images in 3D usually provide symmetry between left and right but do not change the top and bottom because the mirror plane is fixed on our standing plane to our viewpoint.

Symmetry is also used for algebraic reasoning. For example, in the symmetrical equation $(x+y)^2 = 1$, and it means $x^2 + 2xy + y^2 - 1 = 0$, then it is $x^2 + 2xy + (y+1)(y-1) = 0$. Because of the symmetry, $y^2 + 2yx + (x+1)(x-1) = 0$ is also true. In an algebraic explanation for truth, we consider that the given equation means $x + y = \pm 1$ and substitute $y = -x \pm 1$ to the first result. However, it is unnecessary to do these operations because the given equation is symmetrical.

Maxima and Minima

When we have various possible solutions to real-world problems, special cases such as maxima and minima are usually focused. It is valuable depending on the issues of the real world. In computer programming, the algorithm to produce minimum steps which produce the answer in the shortest time has a significance for preferable. Some companies make their decisions to maximise profit with minimum efforts, such as cost cuts, even though it will reduce employment. For making decisions mathematically, we usually produce/apply functions (mathematical models) to represent the order and find maximum or minimum solutions. For example, to decide where we should open a new restaurant, the population density, defined by the function (population)/(km^2) is used as an indicator to select the area. Costs for renting the place, employment and raw food materials are also considered. However, the humanity of the workforce is a necessary variable to success.

(ii) Mathematical Ways of Thinking

Mathematical ways of thinking support the students' thinking process by and for themselves when they appreciate their values. The following are well-known samples in Mathematics.

Generalisation and Specialisation

Generalisation is to consider the general under the given conditions of situations. Any task in mathematics textbooks is usually explained by using some examples, known as special cases; however, it is usually preferred to discuss the general ideas. Given that conditions are usually unclear in textbooks, the conditions become clear if students are asked to consider other cases, such as by saying 'for example'. In the upper grades, variable, domain, range, parameter, discreet, continuous, zero, finite and infinite become the terms of a set to consider the conditions for general.

Specialisation is to consider the thinkable example of situations and necessary to find the hidden conditions. Considering general with thinkable example is called generalisable-special example. In mathematics, the general theory is usually stronger than the local theory. It is one of the objectives for generalisation and specialisation.

In mathematical inquiry, the process usually goes from special to general cases to establish a stronger theory. Thus, the task sequence of every unit in textbooks usually progressed from special cases to general for generating simple procedures, exceptionally the tasks for exercise and training procedures, which usually progress from general to special because students already learned the general.

For students engaging in generalisation and specialisation by and for themselves, students have to produce examples. Thus, developing students who say 'for example' by and for themselves is a minimum requirement for teaching mathematics.

Extension and Integration

The extension means extending the structure beyond the known set. The product of multiplication is usually increased if both factors are natural numbers. However, if we extend it into fractions and decimals, there are cases where the products decrease.

On the extension of structure beyond set, learned knowledge produces the misconception explained by the over-generalisation of learned knowledge. In the mathematics curriculum, we cannot initiate numbers from fractions and decimals instead of natural numbers. Thus, producing misconceptions is inevitable in the mathematics curriculum and its learning. Even though it is a source of difficulty for students to learn mathematics, it is the most necessary opportunity to think mathematically and to justify the permanent ideas to be extended. After experiencing the extension with misconceptions, it is a moment of integration if students learn what ideas can be extended.

Extension and Integration is a mathematical process for mathematisation to reorganise mathematics. It implies that the school mathematics curriculum is a kind of net which connects the local theory of mathematics as a knot even though their strings/paths include various inconsistencies with contradictions. It is a long-term principle for task sequence to establish relearning on the spiral curriculum.

To develop a way of thinking about extension and integration, teachers need to make efforts to clarify the repetition of both the same patterns for extensions and the reflections after every extension in the classroom. For example, there are repetitions for the extensions of numbers from Key Stage 1 to 3. On every extension of numbers, there are discussions of the existence of numbers with quantity, the comparison with equality, greater and lesser, and constructing operations of numbers with the permanence of form. By reflecting on every repetition, students can learn what should be done on the extension of numbers.

The permanence of Form also extends known ideas for overcoming inconsistency, such as misconceptions as an over-generalisation.

Inductive, Analogical and Deductive Reasoning

The three reasoning are general ways of reasoning, as for the components of logical reasoning in any subject in our life. However, mathematics is the best subject in schools to teach them.

Inductive reasoning is the reasoning that generalises a limited number of cases to the whole set of situations. It is usually discussed in natural number sequential situations. Mathematical/complete induction is deductive reasoning as the form of proof.

Three considerable cases or more will be the minimum to be considered inductive reasoning. To promote inductive reasoning, teachers usually provide a table for finding the patterns; however, it is not the way to develop inductive reasoning. To promote inductive reasoning, students must consider various possible parameters in a situation and choose two or more parameters by fixing other parameters. Then, consider the relationship in the situation to know the cause and effect, and subsequently, set the ways to check the cases by well-ordering the cause and get the effect, followed by recording the data in a table. Teachers usually provide the table first to find the patterns that promote inductive reasoning, but it is not the way to develop inductive reasoning. It is the concern of students to develop inductive reasoning by learning the ordering of natural numbers because it is the cause of effect and beautifulness of patterns that originated from the order of natural numbers. A table is a tool for finding patterns but students cannot produce their inductive reasoning by and for themselves if it is given by teachers. Students have to learn how to order to find the pattern inductively.

Analogical reasoning is the reasoning to apply known ideas to the unknown set or situations when recognising similarities with the known set or situations. It is the most popular reasoning in our life. Most of the reasoning to find ways of solution for unknown-problem solving by using what we already know is analogical reasoning. Depending on the case, it is called abduction. Analogical reasoning is to recognise similarities between the unknown problem and the known problems.

Even though the rule of translations between different representations is not well established, analogical reasoning may function as a metaphor for understanding. Many teachers explain operations by using diagrams. It appears meaningful to provide a hint for solving. Still, most students cannot use the hint by and for themselves because they do not recognise the similarity in their analogy. To develop analogical reasoning, the most necessary way is to develop the habit of using what students learned before, by and for themselves. Providing assisting tasks before posing the unknown problem is also used as a strategy to find similarities.

Deductive reasoning is the reasoning to establish systems with components of already approved notions and given by using 'if… then' and logic for propositions such as the transitivity rule. 'If not' also functions for proof by contradiction as well as counter-examples. In cases where the rules of translations are well established, the translations of various representations still function under their limitations too. Various methods for proving such as a complete induction are also done by deductive reasoning.

Inductive and analogical reasoning are necessary to find ways of explaining and proving. Analytical reasoning, thinking backwards from the conclusion to the given, is also used, but it does not allow writing as a part of the formal proof done by deductive reasoning. Arithmetic and algebraic operations can be seen as automatised deductive reasoning. Most students do it just by recognising the structure of expression intuitively without explaining why. To clarify the reason, teachers need to ask why.

Knowing the objectives of reasoning is necessary to develop the three reasonings. Inductive reasoning is applied to find general hypotheses. Analogical reasoning is applied to see the unknown as known for problem solving. Deductive reasoning is applied to explain or prove the local system in general.

Abstracting, Concretising and Embodiment

Abstracting and **concretising** are changing perspectives relatively by changing representations such as expressions. Abstracting is usually done to make a structure clearer. Concretising is usually done to make ideas meaningful by concrete objects. For numerical expressions, manipulatives and diagrams function as concrete. For algebraic expressions, numerical expressions function as concrete objects. For both examples, abstract and concrete representations do not correspond one-to-one in translation because concrete representations usually have some limitations as models. However, concrete representations function as metaphors for abstract ideas.

Embodiment functions in both abstracting and concretising. When abstract ideas can be concretised, it implies those abstract ideas are embedded with some specified concrete ideas. When concrete ideas can be abstracted, it implies that some ideas in concrete are embedded into abstract ideas, but other ideas in concrete do not represent the abstract ideas. Both embodiments function for understanding ideas as metaphors, but their translations are limited only to corresponding contents. Thus, the embodiment of abstract ideas changes how to see concrete objects before and after the embodiment.

Objectifying by Representation and Symbolising

A mathematical representation can be characterised by its symbols and operations with specified purpose and context. In the process of mathematisation, lower-level operational matters are usually objectified as for new symbolising and its operations, likely arithmetic algebra such as simplify $a \times x = ax$.

Until Key Stage 2, (whole) numbers do not mean positive and negative numbers. The number in red on the financial matter is large if the number is 'large'. The number in black on the financial matter is large if the number is 'large'. Here, the meaning of 'large' is defined as the opposite of the number rays; thus, it cannot easily be compared to the numbers in red and black.

At Key Stage 3, as for integration, we have to alternate new symbols and operations. We represent the red number by the negative symbol '-' and the black number by the positive symbol '+' and integrate the one direction for comparison into a one-dimensional number line. Here, in Key Stage 3, 'larger' for comparison (an operational matter on number rays) on the lower level becomes the object of a higher level to produce the comparison (an operational matter on a number line) for new number symbols with positive and negative as a directed number. Later, by the algebraic sum, positive '+', addition '+', and negative '-' and subtraction '-' symbols are integrated as 'plus' and 'minus'.

It is the process of mathematisation that objectifies the operational matter to establish new symbols and operations. This process of abstracting from concrete can be seen as the process of mathematisation.

Relational and Functional Thinking

Relational and functional thinking are ways of thinking that can be represented by relation and function if we need to describe them by mathematical notation. It was used as significant terminology to explain mathematical thinking in the Klein Movement 100 years ago. Relation in pure mathematics is a fundamental axiom. Relating to functional thinking, it is known as the ordered pairs between two sets. Mapping is a relationship, and ordering is also a relationship if both support a structure. An equivalence class is given as a relationship in which the axiom is defined by reflective ($x = x$), symmetry ($x = y \rightarrow y = x$) and transitive ($x = y,\ y = z \rightarrow x = z$) properties, or ($x >= y$) is given as a relation in which the axiom is defined by reflective ($x >= x$), antisymmetric ($x >= y,\ y >= x \rightarrow x = y$), and transitive ($x >= y,\ y >= z \rightarrow x >= z$) properties.

A function is a binary relation between two sets that associates every element of the first set to exactly one element of the second set. In education, the second set usually consists of numbers. Relational thinking can be represented by various representations, such as graphs. It is an activity for students who do not know such representations, and teachers have to teach such representations on the necessity of students to engage in their activity. Students are welcome to create their necessary informal representations, such as graphs and diagrams, by themselves.

Historically, Hamley described the characteristics of functional thinking (1934). He defined functional thinking as having four components: Class, Order, Variable and Correspondence. Class is a set that can include equivalence in operations. For example, 5 can be seen as the value of 1+4, 2+3, 3+2, 4+1. Order is discussed in a set and between sets. Variables are domain and range, which are different from the original set and the destination set. Correspondence is discussed between domain and range. When the teacher provides the table, a variable and between variables are already ordered. To find the functional relationship with a pattern in a situation with various variables, we can change the selected variable in order and other variables fixed as constant and try to find the influence of selected variables. To find influential variables and to remove no influential variable, it has to continue to find functional relationships.

Functional thinking is useful to predict and control a situation. At the elementary level, proportionality and operations are used for functional thinking on this objective. Tables and graphs are helpful for knowing the changing properties of each function. Teachers usually provide tables and graph papers at the beginning of the experiment to find patterns and properties of the function. However, this is not the way to develop functional thinking because teachers take over the opportunities from students to consider and fix the sets, orders, variables, correspondences and so on by themselves.

The rate of change is used to check if a function has linearity. In this case, if it is constant, it is a linear function. If not, otherwise. The limit of the rate of change for finding the tangent is the definition of differentiation. Correlation in statistics is a relation but does not necessarily function the relation as a causal relation.

Thinking Forward and Backwards

Thinking forward and backwards are the terminologies of Polya. In Pappus of Alexandria (4th Century A. D.), **thinking forward corresponds to synthesis** and **thinking backward corresponds to analysis**. Synthesis is the deductive reasoning (proving) from the given and known. In contrast, analysis is the reasoning from the conclusion to find the possible ways of reasoning from the given and known.

In Ancient Greece, analysis was a method of heuristics, which was the way to find adjoining lines on construction problems, and valance for area and volume problems. It hypothetically uses a conclusion to find the solutions. Ancient Egyptians set a tentative number for a solution and got a tentative answer; they compared it with the necessary answer and then adjusted the tentative number to produce an appropriate solution. In Descartes's era, they used an unknown x for an algebraic problem instead of a tentative number. Leibniz used an unknown limit x of the function for calculus. This hypothetic-heuristic reasoning begins from the conclusion, such as if the construction is achieved if the valance is kept, if the unknown x is given and if the limit of x exists. Since the analysis began with a hypothesis, without proving the given, people used to believe that analysis produced tautology, which is not allowed to be written in the system. In Basian probability, we assume unknown probability as p(x) and begin the reasoning. It is an analysis of Statistics and Probability.

On the other hand, the modern mathematics system itself begins from the axiom as a presupposition. Thus, if analytic reasoning becomes a part of presupposition, it is allowed in a written form. For this reason, the unknown x can be written in Algebra, and the limit of x can be written in Calculus. However, until such reformations of mathematics, analysis, and a method of heuristics, it is not allowed to be written as a part of a theory. This is why some mathematics textbooks look very difficult to understand; they have to be written in the form of deductive reasoning for constructing the system from an axiom that does not include heuristics and ways of finding. As a consequence, there are old-fashioned textbooks that are just oriented to exercise the procedures as rules without explaining why.

Current school textbooks are oriented to writing the problem-solving process with various solutions and misconceptions using what is already learned, including heuristics such as thinking backwards and so on. In Key Stages 1 and 2 standards, addition and subtraction are inverse operations, and multiplication and division are inverse operations to verify answers on operations. Such ideas are reformulated at the algebra in Key Stage 3.

The case where 'If your saying (conclusion) is true, it produces contradiction which we already knew' is known as a dialectic in communication. It was formalised as the proof by contradiction in mathematics. It is also a way of analysis for thinking backwards. In mathematical communication, thinking backwards is a part. Without the preparation of a lesson plan, which includes thinking backwards, teachers cannot realise the classroom communication, including misconceptions, because the counter-example is the component for the proof by contradiction. Through the communication of objectives and ways of reasoning, such as thinking backwards by using what students already learned, we can develop students' mathematical thinking.

(iii) Mathematical Activities

Mathematical Activities usually explain the teaching and learning process and embed mathematical ideas and thinking. As for the style of teaching approach, they are usually enhanced. However, the style of teaching itself is not made clear in them.

Problem Solving

Pure mathematicians inquire about problems that have never been solved yet, and they develop new theorems to solve their problems. It produces a part of the system. Such authentic activity is the model of problem solving in education because it usually embeds rich ideas, ways of thinking and values in mathematics. As mathematicians usually pose problems for themselves, the activity includes problem posing and reflection, which is necessary for establishing new theories.

In education, there are two major approaches for embedding them in learning.

The first is setting the opportunity to implement problem solving such as the unit or project. Here, solving the problem itself is an objective for students. It focuses on heuristics: it is usually observed unexpectedly, and planning it is inevitably not easy. Due to this difficulty, the problem-solving tasks are usually provided in two types. The first type focuses on mathematical modelling from the real world. The second type focuses on open-ended tasks for students because it provides the opportunity for various solutions.

The second one, the problem-solving approach, tries teaching content through problem solving in classes. The content here includes mathematical ideas and so on. This case is only possible if teachers prepare a task sequence that enables students to challenge the unknown task by using what students have already learned (Zone of Proximal Development, ZPD). In the problem-solving approach, the tasks given by teachers are planned so that students can learn the content, mathematical ideas and ways of thinking. For teachers, solving the tasks themselves is not the objective of their classes, but students reveal the objectives of teaching by teachers by recognising problems as problematic of an unknown and finding solutions. From students' perspective, by solving problems which appeared from the task as problematic, they can learn by and for themselves. Even if the teacher gives a task, the teacher has to focus on the problem which originated from the given task. If students feel the problematic, they can begin to think for themselves. If not, they must wait for teachers' explanations on solving the given task because they do not recognise it as being solved by and for themselves.

For the problem-solving approach, it is necessary to plan the class in preparation for future learning and use the learned knowledge. Textbooks such as Japanese textbooks are equipped with task sequences for this purpose. In such textbooks, heuristics is not accidental but purposeful because every task in the textbook will be solved using the already learned representation. It is called a 'guided discovery' under ZPD because it expects well-learned students on the learning trajectory and never expects genius students to produce unknown ideas. Thus, in Japan, the most necessary problem-solving approach to using well-configured textbooks is to establish the custom of using what is already learned and acquiring the necessary representations to represent ideas in future learning.

For observers, the method of teaching using the problem-solving approach in a class cannot be distinguished from the open-ended approach. The open approach is not necessary to prepare the task sequence because it is characterised by an independent open-ended task. If teachers set the open-ended task independently, it is the first approach. If teachers set the task sequence of open-ended tasks for learning mathematics, it is the second problem-solving approach. Both approaches have been known as Japanese innovation in textbooks since 1934 for the elementary level and since 1943 for the secondary level. Japanese textbooks until the middle school level currently equipped the task sequence for problem-solving approaches.

Even though problem solving in education resembles the activities of authentic mathematicians, it is not the same because the problems of mathematicians are usually unsolvable beyond decades. At the same time, tasks in the classroom can be explained by teachers who pose the problems. When teachers refer to problem solving in education, it includes various objectives such as developing mathematical ideas, thinking, values and attitudes. These terminologies are used in education to develop students who learn mathematics by and for themselves, while mathematicians only use some of them. In teacher education, if teachers only learned the mathematics content, they may lack the opportunity to learn the necessary terminology. If teachers do not know it, the higher order thinking becomes a black box which cannot be explained.

Exploration and Inquiry

Exploration has been enhanced in finding hypotheses by using technology such as Dynamic Geometry Software and Graphing Software. The software provides an environment for students to explore easily. Exploration of the environment produces a hypothesis.

The inquiry includes exploration as a part but orients to the justification and proving through reflections.

Students' questioning enhances both exploration and inquiry. Thus, the process and finding will depend on students' questioning sequence, not likely on problem solving by the task, which teachers can design before the class.

Mathematical Modelling, Mathematisation and Systematisation

Mathematical modelling is a necessary way to solve real-world problems. A mathematical model is hypothetically set by using mathematics to represent the situation of the problem. Mathematical answers based on the model are confirmed by interpretation. Modelling enhanced various possibilities for applying multiple representations in mathematics.

In education, modelling has been enhanced for problem solving after the new math movement, which recognised school mathematics with set and structure and enhanced numerical solutions by using computers, which became current mathematical science tools. In this era of Artificial Intelligence (AI) and Big Data, computational modelling is done through programming with algorithms, and it is also a part of mathematics.

Before mathematical modelling, mathematics functioned as a metaphor for exact science, theory and language for nature. In Ancient Greece, music and astronomy were exactly theorised under geometric representations. Today, mathematics has various theories with algebraic representations and numerical solutions, which need to consider the limited round number for the possible number of digits. Thus, like computer simulation, modelling means a hypothetical approach to the real world using universal mathematical language instead of exact science.

Mathematisation has two usages: The first is for science and engineering in establishing mathematical models and producing new mathematics theories based on the model. In physics, mathematical problem solving of nature has produced various theorisations in mathematics for solving problems in general. The second usage is used in education as the mathematics curriculum sequence principle, which enhances the reorganisation of mathematical experiences. Freudenthal (1973) explained that the means for organising at a lower level becomes the subject matter for reorganising, which is done by the new means of reorganising. It includes the process of extension and integration based on the prior learned knowledge. The second usage of mathematisation includes the establishment of local theory after the integration.

Systematisation means the establishment of a local theory. In pure mathematics systems, if the conclusion of the proposition is proven as correct, all propositions, definitions and axioms used previously can be seen as necessary conditions for the conclusion. However, in school mathematics, there are so many hidden conditions which support the conclusion and the propositions are usually demonstrated locally. From the perspective of pure mathematics systems, reorganisation of local theory can be said in the process of systematisation.

Programming

Programming activity is the activity to represent the procedure for computer by programming language. Programming language is a kind of mathematical language for using computers. In programming, a situation is analysed and divided into parts for finding the convenient solution by using a computer. Then, excerpting the parts to be interpreted and its process is represented by a programming language to make it operational with a computer. When we compared programming language with other mathematical language that produce mathematical system, it has physical limitations such as the limitation of memory and limitation of time in the computer system. However, if we ignore such limitations, it is the mathematical language. Thus, programming activity can be seen as a kind of mathematical activity.

Conjecturing, Justifying and Proving

Conjecturing, justifying and proving have been used in the context of proof and refutation. In education, students conjecture hypotheses with reasoning on exemplars. Conjecture is conceived through generalisation and justifying with appropriate conditions. Proving includes not only the formal proof in a local system but also the various ways of explanations. A counterexample is a way of refutation. A counterexample is meaningful because it sets off the reasoning from which the conclusion is true.

In the dialectic discussion on the proof and refutation, a counterexample is produced by saying, "'If your saying is true 'or 'if your conclusion is true,' 'what will happen?'" This manner of discussion enhances hypothetical reasoning by getting others' perspectives, which promotes mutual understanding.

Conceptualisation and Proceduralisation

Conceptual knowledge is the knowledge to explain the meaning and is used for conscious reasoning, while procedural knowledge is the skillful knowledge used for unconscious-automatised reasoning. A unit of mathematics textbook usually begins with the initiation of new conceptual knowledge by using learned procedural-conceptual knowledge called conceptualisation. After the initiations, new conceptual knowledge formulates the new procedural knowledge for convenience, and the exercises produce proficiency in proceduralisation.

In mathematics, the proposition format 'if…, then…' is the basic format to represent knowledge; however, in school mathematics, it is impossible to make clear the proposition from the beginning because the 'if' part can be clarified later. For example, 'number' changes the meaning several times in the school curriculum. In multiplication, products become large if it is […] number. Until numbers are extended to decimal, […] part cannot be learned. On this problem, it is normal that students encounter difficulty in their learning and are challenged to produce appropriate knowledge if they are provided a chance to over-generalise knowledge for knowing […] part.

For this reason, to produce exact knowledge in mathematics, conceptualisation and proceduralisation are a journey that continues recursively in mathematics learning to change the view of mathematics. It is the opportunity to learn necessary mathematical ideas and ways of thinking and to develop value and attitude. The process of mathematisation can also be seen from the perspectives of conceptualisation and proceduralisation in the context of numbers and algebra.

From conceptual and procedural knowledge perspectives, conceptual change/progress is the nature of the mathematics curriculum. Even though you can suppose a concept like Plato's idea, you cannot configure the curriculum to acquire the concept directly in school because it continuously progresses in the mathematics curriculum. Even if we target it with pure mathematics, using the pure mathematics textbook for school students might be challenging. For example, many primary school teachers write lesson plans to develop the number concept even though they cannot explain what the number concept is. Such a discussion produces illusions about what they are teaching. The possible way for the school curriculum is to describe clearly how the conceptual and procedural knowledge progress in each grade with tentative meanings and procedures.

Representations and Sharing

A mathematical representation comprises symbols, operations, and objectives (context) (Isoda, 2018). Solving algebra equations can be seen as specific ordered elements of every equation that has the same answer. The order of equations shows the context of the process of solving. The property of equality can explain the operation of equations between one equation to another equation. Each equation is a symbolic sentence.

If the operations of representation are missing, it is not a mathematical representation even though it has some artistic images, such as diagrams. When students draw diagrams, it is sharable if the rule of the drawing (operation) is shared. In a classroom, students produce their images in diagrams. It is helpful to encourage their explanation by themselves, but every answer is independent until knowing the hidden ideas, as in the comparison. Other students cannot re-present it until the ways of drawing (operation) are shared. Thus, comparing various representations is part of the process of recognising ideas for producing symbols and operations, and translations as in mathematical representations.

In mathematics, different representations use different symbols and operations. If there is a rule for correspondence between symbols and operations on different representations, it can be translated and produce rich meanings. It is the way to produce a mathematics system.

In mathematics, a representation system can be defined universally, which is the product of convention by mathematicians. For teachers, it looks far for students' activity in the classroom. However, it is the opportunity for students to reinvent the representation and its system by considering the why and how. For example, producing a metre as the measurement quantity includes such activities: There are historical episodes on why and how 'm' was defined by Condorcet and others in the middle of the French Revolution. In Engineering, Informatics and Science, applied mathematicians usually try to produce new measurements based on the necessity of research to conceptualise the idea mathematically and operationally. Setting the measurable quantity is a part of mathematical modelling for real-world problem solving because if it is measurable, we can apply known mathematics.

From the perspective of Radical Constructivism (Glasersfeld, 1995), another function of representation is 're-present' for understanding others and sharing. To understand what others are saying, we must re-present other's ideas in our minds. Mathematics is a possible subject for constructing others' ideas in each of our notebooks with reasoning from everyone. If not, students say that it is difficult to understand. In this meaning, re-present in one's mind what others are saying is the most necessary activity to share ideas in the mathematics classroom. The mathematical community becomes as enjoyable as long as everyone tries to re-present what others say. Thinking hypothetically, like 'if your saying is true', is also necessary to get others' perspectives.

Mathematical Values and Attitudes

(i) Mathematical Values Seeking for:

Values indicate the direction that we seek. Thus, they set the direction of thinking. In mathematical values, generalisable and expandable ideas are usually recognised as strong ideas. Proving is necessary in mathematics to seek reasonableness. Harmony and beautifulness are described not only as relating to mathematical arts but also in the science of patterns and the basic structure of the system of mathematics. Usefulness and simplicity are used in the selection of mathematical ideas and procedures. Here, each term does not mean an independent category but is related to a feeling, such as 'Simple is beautiful in Mathematics.' Values can be learned through reflection and appreciation. The moment to appreciate the values of ideas and the ways of thinking can be set when the various ideas and ways of thinking of the past and the present are compared. When students are making comparisons, various ideas and ways of thinking should be shown on the board. Teachers verbalise these values when students say 'Aha' and so on. To make clear the verbalisation, the uncomfortable situation, such as feeling problematic, is also meaningful because it provides the object for comparison on the later situation.

Generality and Expandability

School mathematics seeks to establish general theory as well as university mathematics. For example, in the case of multiplication, the following learning process proceeds to seek for general operation with extension. At the initiation stage of multiplication, if students acquire the multiplication table from Row 1 to Row 9, they are released from accumulation (repeated addition) and begin to learn multiplication as the binary operation. However, without the extension of numbers for multiplication, such as in Row 10, Row 11, and so on, students still have to do multiplication as a repeated addition. Should they have to memorise all of them? It is impossible! If they acquire column multiplication, they can multiply any digits with their acquired knowledge of the multiplication table. The column multiplication can extend to decimals if they can manage the position of the decimal point on the base-ten place value notation. In these extensions and generalisation of multiplication, the accumulation form functioned to learn multiplication as a repeated addition at the initiation stage. However, in the next stage, the column form under the place value system functioned to alternate the accumulation to the column multiplication as a binary operation. These are the processes for seeking generality and expandability in the case of multiplication.

In learning the value of seeking generality and expandability, teachers must begin from the un-general and before-expansion cases, even though teachers know the general and the final extended forms. As long as teachers try to teach their general knowledge, students will never get the opportunity to learn these values. A good teacher can design the process of generalisation and expansion and seeks to teach the values through reflection and appreciation. If students appreciate, they can learn the values and seek them by and for themselves.

Reasonableness and Harmony

Mathematics is reasonable. Good teachers usually ask why for developing students who will explain the reason by and for themselves by using what students have already learned. Here, a logical-reasonable explanation for using learned knowledge is not always deductive but more analogical and inductive and is related to numbers at the primary level. Significance and objective are also considered as the reason why. In teaching reasonableness, students may have the intuition to feel strange or good first and explain it by using the learned knowledge or something sharable. An illogical and unreasonable situation is necessary to create the opportunity to learn the reasons itself. This means good teachers know what is unreasonable for students by comparing what they already learned and plan the process to recognise the unreasonableness and reach appropriate reasons by using what they have already learned. In the problem-solving approach, the unknown problem itself is set by such a teacher to provide students with the feeling of unknown or unreasonableness. However, in the given appropriate task sequencing designed by good teachers, students can apply what they have learned to the unknown problem. Thus, on a well-designed task sequence, teachers can ask students why.

Mathematics is a harmonious/harmonised subject: how do you explain it? In Ancient Greece, harmony is a name for the subject of mathematics, as represented by the Pythagorean music scale developed by the ratio 1 to 2, 2 to 3 and 3 to 4. Buildings were also constructed using the special ratio in Ancient Greece. Some ratios were used to represent beautifulness (Eros) in the Era.

In the case of music, we have to consider the overtone for the development of scale and code. The height of scale (sound) is developed by a multiple; however, dividing real strings (code) itself for the development of the musical instruments themselves is division as the multiple of reciprocal. It was the origin of music as a subject of education in Europe. In current mathematics, harmony has limited use, such as the harmonic series for the extension of overtone. On another story, the current music scale itself is defined as equal temperaments, which are the products of multiple $2^{\frac{1}{12}}$ to produce a harmonised code in orchestration is not the ratio on the Pythagorean scale because it produces a unique growl in a special case.
Here, as for mathematics education, harmony as a seeking value is metaphorically used, such as the harmony produced by the current music orchestration, even if it is not the usage of current mathematics. Firstly, when mathematical notions are proved on a system, it is recognised that they already existed from the beginning in a system: It is called pre-supposed/established harmony, likely the providence such as the god already planned before. Up to the final moment to be proved in a system, the notions do not have a clear position; however, at the proved moment, it is a part of the system based on mathematical structure. Even though they are not proved in one system (theory), they can be used hypothetically as true. Each of them is proved on different systems (theories), not one system, and used at the same time as long as it works. However, it looks strange until the harmonised position in the system is reset, like the history of negative and imaginary numbers.

In music, different scales/temperament systems produce different sounds. However, the music appears the same. For example, Current Buch players use instruments with equal temperaments even if they use different cords. We are listening to and playing different Bach music; however, we still recognise it as Bach. Metaphorically, it implies that we recognise mathematical notions on the different systems and use them harmoniously through proven under different systems/theories, even though they should be proved in a system later. Indeed, the school mathematics curriculum is the possible sequence of local theories, which includes several contradictions. However, we can accept them harmoniously. In this metaphorical usage, a teacher has eds to prepare the class based on various mathematical systems/theories that students have already learned.

Secondly, with more metaphorical usage in a mathematics classroom, each student looks at a player to produce mathematical notions harmoniously with collaboration. Current orchestration is possible through each player and conductor's contributions and collaborations. Each player is a necessary component to make a resonance of code. Metaphorically, it is a form of verbal and non-verbal communication in the mathematics classroom where each student's idea produces harmonised reasoning as one whole and appreciates the likely resonance of code. Although there may be students who do not talk verbally, they may contribute by non-verbal manners such as attitude and eye line and so on. A well-harmonised lesson can be seen through such attitudes and so on.

Usefulness and Efficiency

Mathematics usually alternates the meanings that are necessary for reasoning to the procedure that automates and compresses the reasoning. Students can learn the efficiency of the procedure if this alternation is recognised and done by students. If teachers only teach the final procedure from the beginning without any meaning, students lose the opportunity to learn efficiently. Even though teachers take over the opportunity from students, teachers can teach the usefulness of the application problem after they teach the procedure. In the previous example of multiplication, accumulation is necessary at the beginning. However, if we memorise the multiplication table at once and use it for column multiplication, it is unnecessary to consider the accumulation. What good things you learned about column multiplication can be explained if they learned the accumulation before column multiplication. What are the good things you learned about positive and negative numbers? These questions are first answered by usefulness and efficiency, even though there are other values. To answer these questions, students might be able to compare before and after.

Simpler and Easier

These values are similar to the values of usefulness and efficiency; however, simpler values are usually used for selecting procedures and setting the mathematical form. In the development of the procedure, we chose simpler forms and so on. For example, numbers are read from the largest place value. Thus, children usually try to add from the largest place initially. However, in the column addition, if there is regrouping, they have to rewrite the number several times in the process. It becomes simple if they decide to add from the lowest place value. To explain the procedure, we use a diagram such as a proportional number line to make it easier to produce the expression and interpret the meaning. It would be easier to understand if students could draw a diagram to explain the meaning.

Beautifulness

In mathematics teaching, beautifulness is used on several occasions with various values such as simple reasoning and thus beautiful. If we find a structure/pattern/invariant, it is beautiful. If we make a line with equal signs in the operation of the equation easier to see, then it is beautiful. For students to feel the beautifulness, teachers need to set students to feel the ugly/not beautiful situation and change it to feel beautiful. For example, it is just a heap of coins before the arrangement of the same coins. However, if the coins are arranged in vertical bars as in ordering natural numbers, we can alternate the numbers to the height. If the coins are regrouped by the height of 10 coins, we can easily count by 10. Comparisons before and after are necessary to discuss how beautiful the order is.

Beautifulness is also discussed in specific mathematical ideas. In number and operation, the answer of operation should be a number, then $1 \div 3 = \frac{1}{3}$, thus $\frac{1}{3}$ is a number even if it is $0.\dot{3}$. $\frac{1}{3}$ looks more beautiful than $0.\dot{3}$. In algebra, the general form of the quadratic function is $y = ax^2 + bx + c$ and the standard form of the quadratic function is $y=a(x-\alpha)^2+\beta$. If we do not know how these two forms are beautiful, we cannot operate a quadratic function. In a figure, symmetry is usually found in the tessellation of a figure. Symmetry is the word to represent the beautifulness of geometry and algebra. On the sequence, the recursion on the numbered sequence is beautiful because it is a representation of an invariant pattern.

(ii) Mathematical Attitude Attempting to:

Here, attitude means one's mindset. Similar to mathematical value, it will be learned through reflection and appreciation in a social context. From a social perspective, it can be developed as a part of human character in mathematics classrooms through competitive and sympathetic experiences for excellent ideas. Excellence is usually related to representing mathematical ideas and ways of thinking and qualified by mathematical value through comparison. Like mathematics and seeking values, mathematics is a competitive subject that tries to produce innovative ideas faster than others, but it should generally be appreciable and useful ideas under specified conditions. Such an idea demonstrates one's excellence in the community. Secondly, mathematics is a sympathetic subject that allows us to appreciate others through re-presenting others' ideas. If others re-present one's ideas and share, and the others would like to use them, they all become the owners of excellent ideas. If there are no reactions from others for one's sharing action, one's excellence may not survive in their community. Thus, to show excellence, one has to make an effort to explain one's idea as simple and reasonable as possible so that it will be re-presented by others.

See and Think Mathematically

Every content in mathematics provides the ways to see. If students learned multiplication as any number can be seen as a unit for counting, they try to array the objects for counting by the unit for each. If each box has the same number of objects and the number of the boxes is known, we can apply multiplication. Such a thing is the way to see what should be learned when they learn multiplication. If students learn the significance of some representations such as proportional number lines and if they would like to use it, they may use it at the next task. If students appreciate the way of thinking on some representations such as generalisation by using permanence of form in a sequential pattern and if they would like to use it, they may use it at the next opportunity. Thus, to develop an attitude of attempting to see and think mathematically, students are necessary to have the opportunity to discuss the ways to see and think in every learning content.

Pose Questions and Develop Explanations

In the beginning, if students cannot pose questions by and for themselves, teachers have to pose the questions. However, through certain repetitions of similar questioning by teachers, teachers must develop students who pose similar questions by and for themselves. In the beginning, teachers have to say 'for example' by themselves if students never say it. However, they shift it to a question 'Tell me other examples.' If students can produce their examples, and teachers recognise students can say 'for example' in their explanation, then teachers challenge the question 'What do you want to do, next?' If students can show their example for this question, it means they internalised the way of problem posing by using the word 'for example' because it asks to produce different situations or representations for a given original task. Such questions are posed when students feel problematic or uncomfortable for others in context.

Mathematical explanations are done using necessary representations. The necessity of explanations originated from problematic, however, to produce possible explanations for problematics, it is necessary to utilise or find necessary representations.

Generalise and Extend

Without the experience of generalising and extending, reflecting and appreciating, students could not learn the value of generalisation and extension. The attitude of attempting to generalise and extend will be possible as long as such an experienced student would like to do. We must use what has already been learned for generalising and extending. However, students are not sure which learned knowledge should be generalised for what objective. For the unknown situation, they have to use something learned. Misconception is known as over-generalisation. To develop an attitude of attempting to generalize and extend, teachers and other students have to accept it and consider the learned which was overgeneralized. To grow the attitude to generalize and extend, we need to accept any possibility of generalization because it is evidence to be generalized and extended by oneself.

Appreciate others' Ideas and Change Representations for Meaningful Elaborations with Dialectic

Even though misconceptions, from the perspective of the dialectic approach, a necessary idea is to implement dialectic discussions such as 'If your saying is true what will happen?' For example, to know what A is, we should know what non-A is. If two ideas, A and B, appear, even if B is an overgeneralised idea, through the discussions of A and B, the original idea A evolves to idea A' with a comparison of B. For meaningful elaboration on the other side, A has to get the perspective of B, and B has to get the perspective of A. If each side understands the perspectives of the other, it is possible to discuss with each other the question 'If your saying is true, what will happen?' If a counterexample is necessary, we may need to develop different examples of the extended situation and hear what others say. In these discussions, to change the representation to be understandable for another side will be necessary to implement meaningful elaborations for each other. The generalisable idea will be chosen as the situation for dialectic discussion.

Analyse Process and Plan

For planning a problem solving, we need to find ways to reach a conclusion or solution. When the conclusion is given in the problem that seek proving, the reasoning from the conclusion which try to revert to the given is an *analysis* (Thinking backwards). On the other hand, in problems in our lives such as carpenters who use civil engineering, the final deadline and the conclusion such as the product are usually given at the beginning. In this case, *analysis* means breaking the process to possible parts, planning the process in each part and integrating them within the schedule is planning. In STEAM education, it is a necessary part of *designing*. In mathematics, even though an unknown problem, we try to solve it analogically or inductively. These are also analyses. In the case of analogical analysis, we try to compare it with known problems and adapt the known ways. In the case of inductive analysis, we select one parameter in various parameters and change the only selected parameter, step by step. It is a kind of simulation when we already established the computational model in the computer system by mathematical modeling. In experimental science and engineering, the inductive analysis is necessary for planning experiments.

Appendix B

(Extracted from the First Edition, 2017)

Vision of SEAMEO and Purpose of CCRLS

The goal of regional integration in the development of an 'ASEAN Community' provides the opportunity for the development of an educational policy framework for all SEAMEO Member Countries to enhance access to educational opportunities, support the development of quality basic education and encourage regional mobility. Such a framework will support all Governments as the main providers of basic education to meet the learning needs of all students.

Indeed, the SEAMEO Education Agenda #7 "*Adopting a 21st Century Curriculum*" states to pursue a radical reform through systematic analysis of knowledge, skills, and values needed to effectively respond to changing global contexts, particularly to the ever-increasing complexity of the Southeast Asian economic, socio-cultural, and political environment, developing teacher imbued with ASEAN ideals in building ASEAN community within 20 years (2015-2035). SEAMEO established the Action Agenda from 2016 to 2020 under the 7 priority areas. SEA-BES is one of the projects for the action agenda under #7.

SEAMEO Council consisting of the Ministers of Education (2015) enhanced the linking of the seven priority areas with the curriculum and moving towards global citizenship. Inevitably, the region needs common grounds that allow the stakeholders to develop the fullest potential and capabilities of its citizens. In SEAMEO Member Countries, mathematics and science are major component subjects to be learned. Mathematics can be known as part of the most basic literacy for learning other subjects as well as to think and reason as an independent citizen. Science is known as part of the most necessary literacies for technological and environmental concerns of society as well as the scientific exploration of nature. This document is envisioned to serve as a tool for analysing curriculum for 21st-century skills, the collaboration of ASEAN for changing global context, to produce best practices for reform, systematic discussion for further integration, a platform for curriculum and professional development, and assessment[103].

The 21st-century curriculum encompasses learning, literacy and life skills. Competent learners should be able to use tools such as language and technology to convey ideas and thoughts, act autonomously based on rational decisions and ability to interact well with others in the community. In that context, learners will grow and develop knowledge and skills that enable them to find jobs, be responsible, self-reliant and contribute to society.

OECD clarified the 21st-century skills in terms of competency (OECD, 2005). It defined competency for a successful life and a well-functioning society. Societies are continuously

[103] Common Core Regional Learning Standards(CCRLS) is a document for the development of regional curriculum standards which is not the curriculum itself. It is a key reference for further collaboration of the curriculum development, assessments, and professional development on the demands of 21st century which is related with SEAMEO Priorites No.5 and 7. National Curriculum Standards in every SEAMEO Member Countries which are established under every country are respected. The document will be fuctioning for progressive syncronization of curriculum developers and teacher educators in SEAMEO member countries directed for ASEAN Community and revised for these demands. For these demands, it is described for curriculum developers and teachers educators.

changing to seek success and welfare development. United Nations sets sustainable development goals (UN, 2015) under the necessity of the development of every society as well as the sustainability of social welfare. In this context, Mathematics and Science are necessary subjects in education for success in various fields as well as welfare in our life based on mutual understanding. Mathematics and Science are the tools for overcoming the challenges of diversities in Southeast Asia through developing the competency for competitiveness and understanding others to create a harmonious society.

The purposes of the Common Core Regional Learning Standards for Mathematics and Science are to develop basic human characters, creative human capital, and well-qualified citizens in Southeast Asia for a harmonious society through mathematics and science education. The purposes can be developed and achieved through three major components. Firstly, cultivating basic human characters, values, attitudes and habits of mind is essential to be developed through mathematics and science. Values are bases for setting objectives of undertakings and making decisions for future directions. Attitudes are mindsets for attempting to pursue endeavours. Habits of mind develop soft skills which are necessary for living harmoniously in society. Secondly, for developing creative human capital, process skills need to be developed. Thirdly, knowledge of mathematics and science is essential for cultivating well-qualified citizens.

Developing the competency under the three components includes the selection of necessary content for teaching and context for exploration which encompasses metacognition, critical reasoning and communication. Under the selected content and the context, the three components are well connected and the competency developed through the reflection of the processes. Proficiency in applying competency is also developed through content and context in appropriate situations. Those three components under the context will be clarified by the descriptions of the standard in later chapters.

Based on the preceding background, the purposes of the SEA-BES project were conceptualised in 2015 stated as follows:

Purposes of CCRLS under the SEA-BES Project

The SEAMEO Basic Education Standards initiative would support SEAMEO Member Countries in the following respects:

a) to use it as an analytical tool to support the future development of a regional integrated curriculum necessary for ASEAN integration with emphasis on 21st-century skills;
b) to strengthen ASEAN collaboration on curriculum standards and learning assessment across different educational systems to effectively respond to the changing global context and complexity of ASEAN;
c) to promote in every member country the establishment of best practices to overcome differences in curriculum;
d) to produce a systematic discussion process for the establishment of the regional integrated curriculum and assessment;
e) to use as a platform for curriculum development and professional development for all stakeholders developing teachers imbued with ASEAN ideals in building the ASEAN community; and
f) to serve as a platform for assessment such as the Southeast Asia Primary Learning Metrics (SEA-PLM).

For these purposes of the SEA-BES project, CCRLS defines the standards of mathematics and science based on the following principles:

i. The standards are the common ground to develop the fullest potential and capabilities to acquire competency in the 21st century.
ii. The standards are presumed for competitiveness in this globalization era and understanding others in creating the ASEAN harmonious society under global citizenship.
iii. The standards serve as tools for analysing the curriculum for the project as stated in (a) to (f).

Those principles are elaborated for well-functioning ASEAN societies taking into account the cultural differences and development disparity. The standards for mathematics and science will be discussed separately in the later chapters. (Publish as Mathematics and Science volume in 2024)

Development Process of CCRLS

SEA-BES Project Framework in 2014

SEA-BES Project was initiated in 2014 before the establishment of the 7 Priority Areas and as the response to the SEAMEO College Project for preparation of the ASEAN Community. The original conceptual framework is shown in Figure 1.

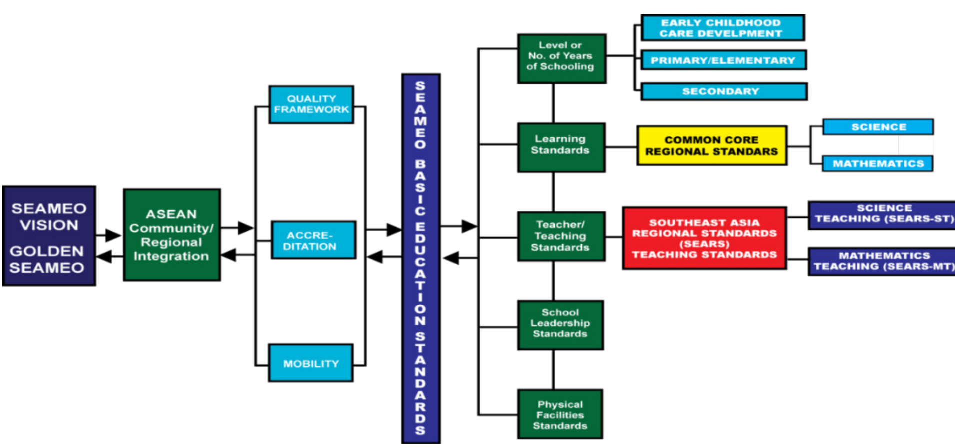

Figure 1. The position of the CCRLS under the SEA-BES project framework.

Methodology to Develop CCRLS

Figure 2 shows the original flow process of the development of the common core regional learning standards in mathematics and science.

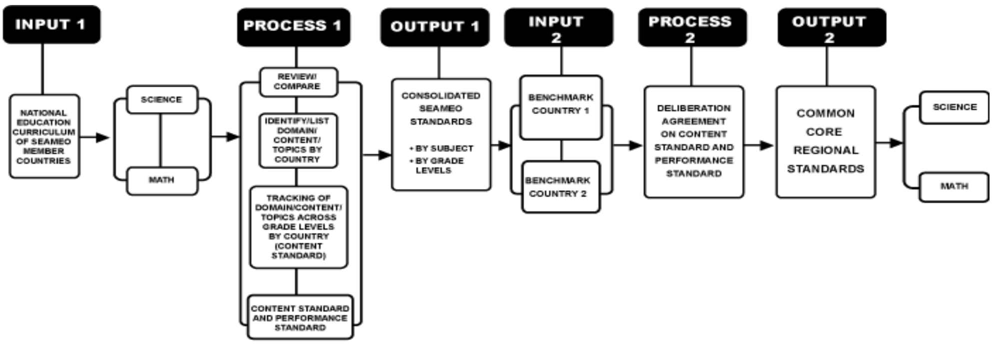

Figure 2. Framework in developing the CCRLS in Mathematics and Science for SEA-BES (Mangao, Tahir, and Zakaria, 2015).

The CCRLS was developed based on the strengths of the existing national education standards of the SEAMEO Member Countries. The various activities in the development of the CCRLS document for Mathematics and Science undertook the following processes:

- curriculum review and comparison of the national education curriculum of the seven SEAMEO Member Countries in Mathematics and Science, namely; Brunei Darussalam, Cambodia, Indonesia, Malaysia, Philippines, Singapore and Thailand;
- identification of similarities and differences in terms of content/domain/topics/strand/year level by country; and
- tracking of content/domain/topics/strands across grade levels from the primary to secondary level.

There are diversities among SEAMEO Member Countries. The year levels are not always the same. Thus, the CCRLS in Mathematics and Science documents are categorised into three key stages: Key Stage 1 covers Grades 1 to 3, Key Stage 2 covers Grades 4 to 6 and Key Stage 3 covers Grades 7 to 9.

Other activities undertaken include the development of the appropriate strands for every stage and the topics under the strand and the elaboration of every standard description under the participation of representatives from Member Countries (including Myanmar, Vietnam, Lao PDR and Timor Leste) and other leading researchers from non-SEAMEO Member Countries such as APEC Economies.

The consolidated SEAMEO Mathematics and Science Standards were benchmarked with the standards of highly qualified education countries /economies such as Hong Kong, Japan, Australia, United Kingdom and the U.S.; reviewed documents such as Trends in International Mathematics and Science Study (TIMSS), Programme for International Student Assessment (PISA), and National Council of Teachers of Mathematics (NCTM) as well as research studies and literature available related to curriculum development.

Standards found in every stage are useful for every country to engage in their activity on purposes (a) to (f), especially for considering the experiment and challenges for every country's reform.

Series of SEA-BES Workshops

A series of workshops which aimed to develop the Common Core Regional Learning Standards (CCRS) in Mathematics and Science were conducted on different dates, levels and venues as follows:

A. National Level

- 2 April 2015 (In-house/Local Level, SEAMEO RECSAM, Malaysia)
- 11 May 2015 (In-house/Local Level, SEAMEO RECSAM, Malaysia)
- 21-22 May 2015 (In-house/Local Level, SEAMEO RECSAM, Malaysia)
- 23 July 2015 (In-house/Local Level, SEAMEO RECSAM, Malaysia)
- 27 August 2015 (In-house/Local Level, SEAMEO RECSAM, Malaysia)
- 17 September 2015 (In-house/Local Level, SEAMEO RECSAM, Malaysia)
- 27-28 January 2016 (In-house/Local Level, SEAMEO RECSAM, Malaysia
- 24-25 March 2016 (In-house/Local Level, SEAMEO RECSAM, Malaysia)

B. Regional Level

- 4-5 November 2014 (Regional Level, SEAMEO RECSAM, Malaysia)
- 20-22 October 2015 (Regional Level, SEAMEO RECSAM, Malaysia)
- 15-18 February 2016 (Regional Level, University of Tsukuba, Japan – Only for Mathematics Standards)
- 9-12 February 2017 (Regional Level, University of Tsukuba, Japan) – Only for Mathematics Standards)
- 28-30 March 2017 (Regional Level, SEAMEO RECSAM, Malaysia)

Participation and Involvement of Experts, Educators and Institutions

Maximum participation and involvement of experts and teachers across the Southeast Asian region and beyond were solicited in the development of the CCRLS document. Their tasks include being a member of the curriculum working group giving inputs and providing specific, constructive feedback on the draft Standards. The following groups were involved:

- Consultants (Professor Masami Isoda, Centre of Research on International Cooperation in Educational Development, University of Tsukuba, Japan; Dr. Mark Windale, Centre for Science Education, Sheffield Hallam University, United Kingdom; and Professor Kerry J. Kennedy, Hong Kong Institute of Education, Hong Kong)
- Curriculum experts in science and mathematics from the 11 Ministries of Education of SEAMEO Member Countries
- SEAMEO Secretariat
- Science and mathematics specialists from SEAMEO centres (i.e. RECSAM, QITEP in Science, QITEP in Mathematics, SEAMOLEC)
- Mathematics professors and experts from APEC economy members who were present during APEC-Tsukuba International Conference X and APEC-Tsukuba International Conference XI (Grant No.26245082 and APEC HRD 03-2015A)

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